# Germs of bifurcation diagrams and SN - SN families

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**Abstract**

We study the geometry of the bifurcation diagrams of the families of vector fields in the plane. Countable number of pairwise non-equivalent germs of bifurcation diagrams in the two parameter families is constructed. Before this effect was discovered for three parameters only. Our example is related with so called SN - SN families: unfoldings of vector fields with one saddle-node singular point and one saddle-node cycle. We prove structural stability of this family. By the way, the tools that may be helpful in the proof of structural stability of other generic two-parameter families are developed. One of these tools is the embedding theorem for saddle-node families depending on the parameter. It is proved at the end of the paper.

# 1 Introduction: definitions and main results

## 1.1 Arnold's conjecture

*"For a generic $l$-parameter family of vector fields on $S^2$:*

$$<\cdots>$$

*Any bifurcation diagram is (locally) homeomorphic to one of a finite number (depending only upon $l$) of generic examples $<\cdots>$",* [AAIS], Chapter 3, Section 2.8.

The definition of bifurcation diagram is not specified here; [AAIS] and [A] contain two definitions that are not equivalent. We call these two kinds of bifurcation diagrams *coloured* and *simple*.

## 1.2 Coloured bifurcation diagrams

*"Consider a family of vector fields $v(\cdot,\varepsilon)$. Topological orbital equivalence (or weak equivalence) defines a partition of the parameter space into classes. This partition is called the bifurcation diagram of the family"* [AAIS], Chapter 1, Section 1.7.

These classes include open subsets that correspond to structurally stable vector fields. To make the definition of bifurcation diagram closer to the next one, we delete these classes, and come to the following definition.

**Definition 1.** *Coloured bifurcation diagram of a local family of vector fields (whose base is a germ $(\mathbb{R}^k, 0)$) is the set of all the parameter values that correspond to structurally unstable vector fields in this family, together with the partition of this set into classes; each class consists of parameter values corresponding to orbitally topological equivalent vector fields.*

One may imagine that each class has its own colour; from here the name.

## 1.3 Simple bifurcation diagrams

*" ··· in sufficiently good cases the bifurcation set in function space has the structure of a local direct product of its section by a finite dimensional subspace (transversal to the stratum of the bifurcation set to which the point being studied belongs) and an infinite dimensional manifold of finite codimension (equal to the codimension of the stratum and the dimension of the cross-section), along which "nothing essential changes.*

*In this case the generic family with finitely many parameters is also a transversal section of the indicated stratum. The bifurcation set in the function space leaves a trace (the primage) in the parameter space called the* bifurcation diagram *of the family."* [A], Section 4.5.

This gives rise to a definition:

**Definition 2.** *Bifurcation diagram in a generic finite-parameter family is the set of all parameter values that correspond to structurally unstable vector fields.*

This definition is obviously formally non-equivalent to the previous one. The following theorem shows that non-equivalence is not only formal.

## 1.4 Non-equivalence

Families that we consider are semi-local and semi-global. They are local in the parameter: the base is a germ of $\mathbb{R}^k$ at zero. They are global in the phase variable. For such families we use a term *glocal.*

**Theorem 1.** *There exists an open set in the space of pairs of one-parameter glocal families of vector fields on $S^2$ such that two families from any pair have the same simple but different coloured bifurcation diagrams.*

This means that there exists a homeomorphism of $(\mathbb{R}, 0)$ to $(\mathbb{R}, 0)$ that brings the simple bifurcation diagram of one family from the pair to that of another one, but there is no homeomorphism that moreover respects the partition into the equivalence classes.

## 1.5 Previously known counterexample to Arnolds' conjecture: ensemble "lips"

At the beginning of 90's my student Anna Kotova listed all the polycycles that may occur in 2 and 3-parameter glocal families. This resulted in "Kotova zoo" [KS]. One of the species in the zoo was an ensemble "lips". It is formed by two saddle-nodes of multiplicity 2 whose parabolic sectors are turned towards each other, and whose separatrices of the hyperbolic sectors coincide. This ensemble consists of a continual set of polycycles.

In [KS] it is proved that for any $k$ there exists a generic 3-parameter glocal family that unfolds the ensemble lips such that under the bifurcation of the ensemble at least $k$ limit cycles may be born. As a consequence, there exists a countable set of three-parameter glocal families whose germs of coloured bifurcation diagrams are pairwise not topologically equivalent.

## 1.6 A counterexample to Arnold's conjecture for two-parameter families.

The first main result of this paper is

**Theorem 2.** *There exists a countable number of two-parameter glocal families whose germs of simple bifurcation diagrams are pairwise topologically non-equivalent.*

Two parameter families in the theorem are of class SN-SN described in the following way. These families are unfoldings of codimension two degeneracies; namely, of vector fields that have one saddle-node singular point and one saddle-node limit cycle, both of multiplicity two; from here name. The set of all generic unfoldings of such vector fields in denoted by $SN^2$. For any $n$ let us now define a vector field of class $SN^2_n$. This is a vector field with the following properties:

1. It is of class $SN^2$; let $N$ be the saddle-node singular point, and $\gamma$ be the saddle-node limit cycle of this field.

2. The unstable separatrise of the hyperbolic sectors of $P$ winds towards $\gamma$ from outside in the positive time, see Fig. 1.

3. There are exactly $n$ hyperbolic saddles, such that exactly one separatrix of each saddle enters the parabolic sector of $N$.

4. There is exactly one hyperbolic saddle inside $\gamma$, whose one stable separatrix winds to $\gamma$ in the negative time.

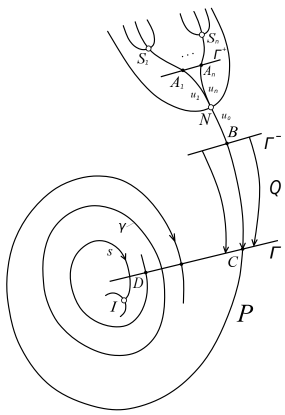


Figure 1: Unpeturbed SN - SN vector field

5. The vector field has no other non-hyperbolic singular points or limit cycles, neither separatix connections.

Yu. Kudryshov noticed that a simpler class of families that demonstrates Theorem 2 is a class of unfoldings of vector fields $v$ with the following properties:

- $v$ has one saddle - node and a mutual separatrix of the hyperbolic sectors of this saddle-node and a huperbolic saddle;
- $n$ separatrices of hyperbolic saddles enter the parabolic sector of this saddle-node.

The bifurcation diagram of this unfolding is a graph with one vertex and $n+1$ edges. The justification of this description would prove Theorem 2. We consider yet the $SN^2$-families. They are more interesting, and their study provides tools for future investigations.

## 1.7 Structural stability

The second main result of this paper is

**Theorem 3.** *Generic families of class* $SN^2_n$ *are structurally stable for any* $n$.

This theorem heavily relies upon the concept of the *large bifurcation support* (LBS) introduced in [GI], and on the main result of that paper.

## 1.8 Continuum of pairwise different coloured bifurcation diagrams

Not only countable, but a continual set of pairwise non-homeomorphic coloured bifurcation diagrams exists; it occurs not in two, but in three-parameter families. These families are introduced in [IKS] and called families of class $TH$. Their construction is recalled below in Section 4. Here we recall that vector fields of class $TH$ have a polycycle with two vertices, both hyperbolic saddles, with the characteristic values $\lambda$ and $\mu$.

**Theorem 4.** *Generic unfoldings of two vector fields of class $TH$ with different values of the ratio $\frac{\ln \lambda}{\ln \mu}$ have non-equivalent germs of colored bifurcation diagrams.*

As the set of values of the ratio $\frac{\ln \lambda}{\ln \mu}$ is continual, this immediately implies that a continual set of pairwise non-homeomorphic colored bifurcation diagrams exists.

## 1.9 Embedding theorem for parabolic germs with a parameter

One of the main tools of our investigation is a parameter version of the embedding theorem from [IYa].

**Theorem 5.** *Consider a $C^\infty$ unfolding $F$ of a parabolic germ of a diffeomorpism of $(\mathbb{R}, 0)$ of multiplicity 2 depending on $k$ parameters. It is smooth equivalent to a $C^\infty$ local family*

$$x \to x + (x^2 + \varepsilon)(1 + f(x, \varepsilon, \lambda)) \tag{1}$$

*where $\varepsilon \in (\mathbb{R}, 0), \lambda \in (\mathbb{R}^{k-1}, 0), f(0) = 0$. Then in the domain $\varepsilon \geq 0$ this family is $C^\infty$ equivalent to an embeddable one, namely to a time one phase flow transformation $g_w^1$ of a vector field $w = w_{\varepsilon,\lambda}$ that determines an equation on $(\mathbb{R}, 0)$.*

$$\dot{X} = w_{\varepsilon,\lambda}(X), \ w_{\varepsilon,\lambda}(X) = \frac{X^2 + \varepsilon}{1 + a(\varepsilon, \lambda)X}. \tag{2}$$

This theorem for $k = 1$ (no $\lambda$ at all) is proved in [IYa]. For arbitrary $k$ it is proved in Section 6. $X$ is called the *normalizing coordinate* for $F$.

## 1.10 Conjectures

**Conjecture 1.** *There exists an open set of 5-parameter families whose simple bifurcation diagrams have numeric moduli.*

**Conjecture 2.** *If a coloured bifurcation diagram of a generic family is structurally stable (in an obvious sense) then the family itself is structurally stable.*

## 1.11 Plan of the paper

The paper is organized as follows. In Section 2 we prove Theorem 2. In Section 3 we prove Theorem 1. In Section 5 we prove Theorem 3. In Section 4 we prove Theorem 4. In Section 6 we prove Theorem 5. In Section 7 we give a sketch of the proof of Conjecture 1.

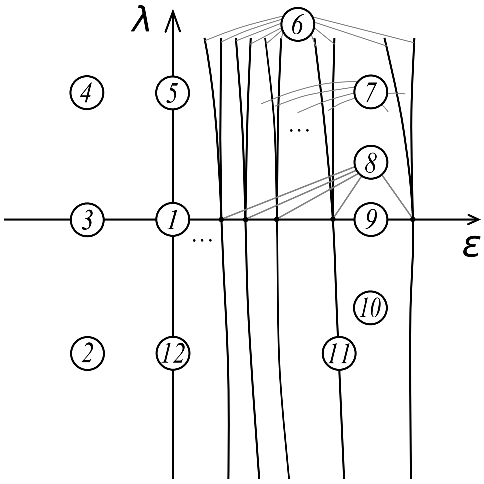


Figure 2: Simple bifurcation diagram for an $SN_2^2$ family

# 2 Simple bifurcation diagrams of SN-SN families

In this section Theorem 2 is proved.

## 2.1 The simple bifurcation diagram of the $SN_n^2$ family

This diagram is shown on Fig 2. The corresponding phase portraits are shown on Fig. 3.

The coordinates in the parameter space are chosen in a special way.

First, $\varepsilon$ is responsible for the bifurcation of the non-hyperbolic cycle: for $\varepsilon = 0$ the semistable cycle persists, for $\varepsilon > 0$ it vanishes; for $\varepsilon < 0$ it splits into two hyperbolic cycles.

Second, $\lambda$ is responsible for the bifurcation of the saddle-node singular point: for $\lambda = 0$ the saddle-node point persists, for $\lambda > 0$ it vanishes, for $\lambda < 0$ it splits into two hyperbolic singular points: a saddle and a node.

Let $V$ be an $SN^2$ family.

By Theorem 5, the family $F$ of the parameter depending Poincare maps of the semistable cycle $\gamma$ for the family $V$ is smooth equivalent to a family (2) for $k = 2$. Analogously, the family $V$ near the point $N$ is orbitally smooth equivalent to a family that generates an equation

$$\dot{x} = \frac{x^2 + \lambda}{1 + a(\varepsilon, \lambda)x} \tag{3}$$
$$\dot{y} = -y.$$

For $\varepsilon = \lambda = 0,\ x(P) = y(N) = 0$

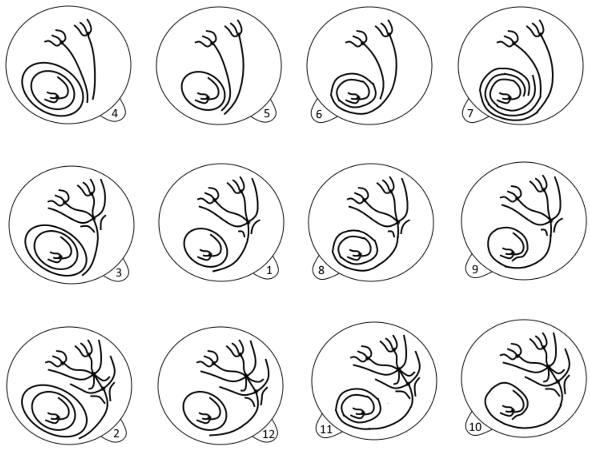


Figure 3: Phase portraits of the vector fields of the family $SN_2^2$

When $\lambda = 0$, the vector field (3) has a separatrix $s_0(\varepsilon, 0)$ of the hyperbolic sectors: $y = 0,\ x \geq 0$. In the original coordinates, this separatrix depends smoothly on the parameter.

Consider a subfamily $W$ of $V$ corresponding to $\lambda = 0$ (the saddle-node $N = N(\varepsilon, 0)$ persists), $\varepsilon > 0$ (the semistable cycle vanishes). Vector fields of subfamily $W$ have sparking saddle connections between the saddle-node $N(\varepsilon_m, 0)$ and the saddle $I(\varepsilon_m, 0)$. These connections correspond to a sequence $(\varepsilon_m, 0)$. This and the following statements is proved in Section 2.2.

When $\lambda < 0$, the saddle-node $N(\varepsilon, 0)$ is split into a saddle $S_0(\varepsilon, \lambda)$ and a node $N_0(\varepsilon, \lambda)$. The saddle $S_0(\varepsilon, \lambda)$ has an unstable separatrix $s_0(\varepsilon, \lambda)$ continuous in $\varepsilon, \lambda$ that tends to $s_0(\varepsilon, 0)$ as $\lambda \to 0$. This separatrix forms sparkling saddle connections with the saddle $I(\varepsilon, \lambda)$. For any $m$, these connections correspond to parameter values on a graph of a function $\varepsilon = \varphi_m(\lambda),\ \lambda \leq 0,\ \varphi_m(0) = \varepsilon_m$.

When $\lambda > 0$, the saddle-node $N(\varepsilon_m, 0)$ vanishes, and the separatrices of the saddles $S_1, \ldots, S_n$ break through to the cross-section $\Gamma$. These separatrices form sparkling saddle connections with the saddle $I(\varepsilon, \lambda)$. For any $m$ and $j \in \{1, \ldots, m\}$ connection between $S_j$ and $I$ corresponds to parameter values on a graph of a function $\varepsilon = \varphi_{m,j}(\lambda)$; $\varphi_{m,j}(\lambda) \to \varepsilon_m$ as $\lambda \to 0$.

Thus the bifurcation diagram of $V$ near the point $(\varepsilon_m, 0)$ is homeomorphic to a graph with one vertex and $n + 1$ edges. Thus the number $n$ is a topological invariant of the simple bifurcation diagram of the family $V$. This proves Theorem 2 modulo justification below.

## 2.2 Justifications

### 2.2.1 Sparkling saddle connections between the saddle-node and the interior saddle

Consider the subfamily $W$ introduced above: $\lambda = 0$, $\varepsilon > 0$. Vector fields of this family have a saddle-node $N(\varepsilon, 0)$ with a separatrix of its hyperbolic sectors denoted by $s_0(\varepsilon, 0)$.

Let us choose some cross-sections of the vector field $v_{0,0}$ and analyse some holonomy maps between them.

Let $\Gamma^+$ and $\Gamma^-$ be two segments given in coordinates $x, y$ near $N$, see (3), by the equations:

$$\Gamma^- = \{(-\delta, y) | |y| \leq \delta\},$$
$$\Gamma^+ = \{(\delta, y) | |y| \leq \delta\},$$

$\delta > 0$ small. The map from $\Gamma^+$ to $\Gamma^-$ along the orbits of the vector field $v_{\varepsilon,\lambda}$ is defined for $\lambda > 0$ (when the saddle-node vanishes) and has the form:

$$\Delta_{\varepsilon,\lambda} : y \mapsto C(\varepsilon, \lambda)y,$$

$C(\varepsilon, \lambda) \to 0$ as $\lambda \searrow 0$ uniformly in $\varepsilon$.

Let $\Gamma$ be a cross-section to the cycle $\gamma$, and $P_{\varepsilon,\lambda}$ be the Poincare map of $\gamma$ defined near the point $O = \gamma \cap \Gamma$ for all the small values of $\varepsilon, \lambda$. By Theorem 5, there exists a normalizing coordinate $X = X_{\varepsilon,\lambda}$ on $\Gamma$ (depending on the parameters $\varepsilon, \lambda$) such that for $\varepsilon \geq 0$, when the cycle $\gamma$ persists or vanishes,

$$P_{\varepsilon,\lambda} = g^1_{w_{\varepsilon,\lambda}}, \ w_{\varepsilon,\lambda} = \frac{X^2 + \varepsilon}{1 + b(\varepsilon, \lambda)X}.$$

Let $T_{\varepsilon,\lambda}$ be the "time distance" on $\Gamma$ determined by the vector field $w_{\varepsilon,\lambda}$:

$$T_{\varepsilon,\lambda}(a, b) = \int_{X(a)}^{X(b)} \frac{dX}{w_{\varepsilon,\lambda}(X)}. \tag{4}$$

For $a, b$ on one and the same side of $O$, this function is defined for all $\varepsilon \leq 0$; for $a, b$ on different sides of 0, this function is well defined for $\varepsilon > 0$, and tends to infinity as $\varepsilon \searrow 0$ uniformly in $\lambda$.

Let $B(\varepsilon, 0)$ be the intersection of the separatrix $s_0(\varepsilon, 0)$ with $\Gamma^-$. Let $Q_{\varepsilon,\lambda} : \Gamma^- \to \Gamma$ be the holonomy map along the orbits of $v_{\varepsilon,\lambda}$, defined in a neighborhood of $B(0, 0)$. let $C(\varepsilon, 0) = Q_{\varepsilon,0}(B(\varepsilon, 0))$.

Let $D(\varepsilon, \lambda)$ be a smoothly depending on the parameters intersection point of $\Gamma$ with the separatrix of the saddle $I(\varepsilon, \lambda)$ that winds towards $\gamma$ in the negative time. There is countably many such points; we choose and fix one, no matter what.

Sparkling saddle connections between $N(\varepsilon, 0)$ and $I(\varepsilon, 0)$ occur when

$$D(\varepsilon, 0) = P^m_{\varepsilon,0}(C(\varepsilon, 0)). \tag{5}$$

Let

$$\xi(\varepsilon, 0) = X_{\varepsilon,0}(D(\varepsilon, 0)),$$
$$\psi(\varepsilon, 0) = X_{\varepsilon,0}(C(\varepsilon, 0)).$$

Equation (5) is equivalent to

$$t(\varepsilon) := T(\xi(\varepsilon, 0),\ \psi(\varepsilon, 0), \varepsilon, 0) = m. \tag{6}$$

The function $t$ monotonically tends to plus infinity as $\varepsilon \searrow 0$. Then equation (6) has exactly one solution $\varepsilon_m$ for any $m$ sufficiently large. This implies existence of sparkling saddle connections between $N(\varepsilon_m, 0)$ and $I(\varepsilon_m, 0)$.

### 2.2.2 Sparkling saddle connections between the saddle generated by the split saddle-node and the interior saddle

For $\lambda < 0$, the saddle-node $N$ is split to a saddle $S_0(\varepsilon, \lambda)$ and a node, still defined by $N(\varepsilon, \lambda)$. Note that for $\lambda \nearrow 0$, $N(\varepsilon, \lambda) \to N(\varepsilon, 0)$, $S_0(\varepsilon, \lambda) \to N(\varepsilon, 0)$. The saddle $S_0(\varepsilon, \lambda)$ has a separatrix $s_0(\varepsilon, \lambda)$, defined for $\lambda > 0$. In the normalizing coordinates $(x, y)$ it is given by the equation $y = 0$.

Let

$$B(\varepsilon, \lambda) = s_0(\varepsilon, \lambda) \cap \Gamma^-,\ C(\varepsilon, \lambda) = Q(B(\varepsilon, \lambda)).$$

Sparkling saddle connections between the saddles $S_0(\varepsilon, \lambda)$ and $I(\varepsilon, \lambda)$, $\lambda < 0$, occur when

$$D(\varepsilon, \lambda) = P^m_{\varepsilon,\lambda}(C(\varepsilon, \lambda)). \tag{7}$$

Let

$$\psi(\varepsilon, \lambda) = X(C(\varepsilon, \lambda)),\ \lambda > 0.$$

Note that $\psi\varepsilon, \lambda \to \psi(\varepsilon, 0)$ when $\lambda \nearrow 0$. Equation (7) is equivalent to

$$t(\varepsilon, \lambda) := T(\xi(\varepsilon, \lambda),\ \psi(\varepsilon, \lambda),\ \varepsilon, \lambda) = m. \tag{8}$$

The same arguments as before prove that the equation (8) for a fixed $\lambda$ has exactly one solution $\varepsilon = \varphi_m(\lambda)$. it is continuous in $\lambda$, because $t$ is, and $\varphi_m(\lambda) \to \varepsilon_m$ as $\lambda \nearrow 0$.

### 2.2.3 Sparkling saddle connections between the interior and exterior saddles

For $\lambda > 0$, the saddle-node $N$ vanishes, and the separatices $s_1, \ldots, s_n$ of the exterior saddles $S_1, \ldots, S_n$, that entered the saddle-node for $\lambda = 0$, now reach $\Gamma^-$. Let

$$A_j(\varepsilon, \lambda) = s_j(\varepsilon, \lambda) \cap \Gamma^+.$$

$$B_j(\varepsilon, \lambda) = \Delta_{\varepsilon,\lambda}(A_j(\varepsilon, \lambda)) \in \Gamma^-.$$

We have:

$$y(B_j(\varepsilon, \lambda)) = c(\varepsilon, \lambda) y(A_j(\varepsilon, \lambda)), \; c(\varepsilon, \lambda) \to 0 \text{ as } \lambda \searrow 0.$$

The points $A_j$ and $B_j$ depend smoothly on $\varepsilon, \lambda$, where defined.

Sparkling saddle connections between the saddles $S_j \varepsilon, \lambda$ and $I \varepsilon, \lambda, \; \lambda > 0$, occur when

$$D(\varepsilon, \lambda) = P^m_{\varepsilon,\lambda}(C_j(\varepsilon, \lambda)). \tag{9}$$

The same arguments as before prove that for $m$ large enough and for any small $\lambda$ fixed, this equation has a unique solution

$$\varepsilon = \varphi_{m,j}(\lambda),$$

and

$$\varphi_{m,\lambda}(\lambda) \to \varepsilon_m \text{ as } \lambda \searrow 0.$$

This completes the justification of the description of the bifurcation diagram for the family $V_n$, and proves Theorem 2.

# 3 Nonequivalence

Here the simplest example of a class of one-parameter families with homeomorphic simple and non-homeomorphic coloured bifurcation diagrams is given.

This section heavily relies upon the papers [MP] and [GIS].

Recall that vector fields of class $PC$ are vector fields with one parabolic cycle of multiplicity two and no other degeneracies. That is, the vector field has only hyperbolic singular points, only hyperbolic limit cycles except for one, and no saddle connections.

Consider a class of vector fields $v \in PC$ such that any vector field from this class has a semistable (parabolic) cycle $\gamma$, two saddles $E_1$ and $E_2$ outside $\gamma$ and two saddle $I_1, I_2$ inside $\gamma$. Suppose that one separatrix of each saddle $E_1, E_2$ winds towards $\gamma$ in the positive time (denote these separaterices by $u_1, u_2$ as they are unstable). Suppose that one separatrix of each saddle $I_1, I_2$ winds toward $\gamma$ in the negative time (denote these separatices by $s_1, s_2$ as they are stable).

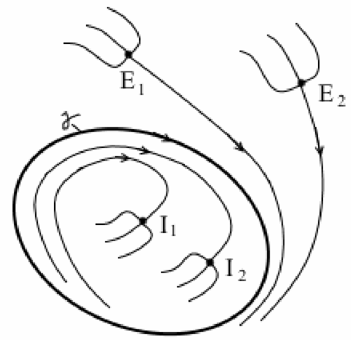


Figure 4: The unperturbed vector field of a $PC$ family

Let $V$ be a one-parameter unfolding of $v \in PC$, $\varepsilon \in (\mathbb{R}, 0)$ be the parameter of the family such that 0 corresponds to $v$. Suppose that for $\varepsilon > 0$ $\gamma$ vanishes.

Let $\Gamma$ be a cross-section to $\gamma$. Let $U_j \in u_j \cap \Gamma$ (choose and fix one intersection point out of a countable number), $S_j = s_j \cap \Gamma$, $j = 1, 2$, see Fig 4.

Suppose that the other unstable separatix of the saddles $E_1, I_1$ that does not wind to $\gamma$ tends to an attracting point, and the other unstable separatix of the saddles $E_2, I_2$ tends to limit cycles. Then vector fields of the family having saddle connections $E_i I_j$ and $E_{i'} I_{j'}$ with $(i, j) \neq (i', j')$ are orbitally topologically non-equivalent.

In [MP] and [GIS] it is proved that as $\varepsilon \searrow 0$, sparkling saddle connections occur in the family $V$. The simple bifurcation diagram of this family is a sequence convergent to 0 together with the limit point 0. The coloured bifurcation diagram adds colours to the points of the simple bifurcation diagram:

the points corresponding to connection $E_i I_j$ are:

$$\text{red for } (i, j) = (1, 1)$$

$$\text{blue for } (i, j) = (1, 2)$$

$$\text{yellow for } (i, j) = (2, 1)$$

$$\text{green for } (i, j) = (2, 2)$$

In [MP], [GIS] it is proved that for $\varepsilon$ small the colours go in a cyclic order. If we glue together two subsequent red points then a segment between them will become a circle with four points painted in different colours. There is no orientation preserving homeomorphism of a circle that changes the order of four coloured points on it. All the possible orders of the four coloured points on a circle may be realized in a generic $PC$ family. The corresponding coloured bifurcation diagrams are not homeomorphic. This proves Theorem 1.

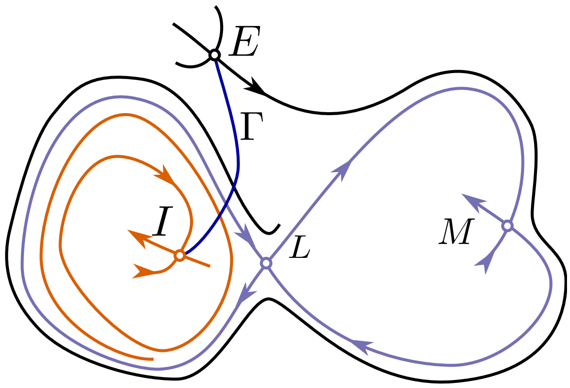


Figure 5: Unpeturbed vector field of class **T**

# 4 Continuum of pairwise different coloured bifurcation diagrams

Here we prove Theorem 4.

Vector fields $v$ of class **T** are defined as follows [IKS].

The vector field $v$ has a polycycle $\gamma$ formed by two saddles $L$ and $M$ and their saddle connections. Two connections between $L$ and $M$ for a polycycle "heart", and two separatrices of $L$ from a saddle loop $l$. The characteristic numbers of $L$ and $M$ are denoted by $\lambda$ and $\mu$ respectively and satisfy the following conditions: $\lambda < 1$, $\lambda^2\mu > 1$. There are two saddles: $E$ outside $\gamma$ and $I$ inside $\gamma$. One separatix of $E$ winds towards $\gamma$, and one separatix of $I$ winds towards $l$ in the negative time, see Fig. 5. This and the next figure are borrowed from [IKS]. Other assumptions on the class **T** that are not essential for now are listed in [IKS].

Generic 3-parameter unfoldings of the vector fields of class **T** are families of class $TH$.

Any family $V$ of class $TH$ has a subfamily $\mathcal{E}$ for which the polycycle "heart" is preserved, and the loop $l$ is broken for $\varepsilon \neq 0$, where $\varepsilon$ is a parameter of the subfamily $\mathcal{E}$. Denote by $v_\varepsilon$ vector field of the family $\mathcal{E}$ corresponding to the parameter value $\varepsilon$. There are two sequences $\varepsilon_n$ and $i_m$ that monotonically tend to 0 and have the following property: vector field $v_{\varepsilon_n}$ has a sparkling saddle connection between $E$ and $L$; vector field $v_{i_m}$ has a sparkling saddle connection between $L$ and $I$. The relative density of two sequences $(\varepsilon_n)$ and $(i_m)$ equals $\nu = \frac{\ln \lambda^{-1}}{\ln \lambda^2 \mu}$.

The sequence of points $(\varepsilon_n) \subset \mathcal{E}$ corresponds to a separate class of orbitally topologically equivalent vector fields of the family $V$.

Indeed, vector fields of this class are not orbitally topologically equivalent to any other field of the family: only vector fields of this class have the following three saddle

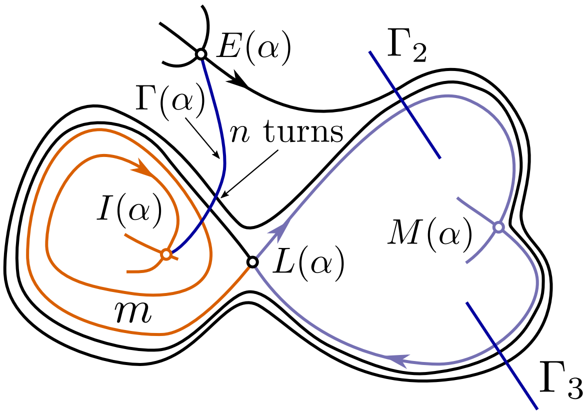


Figure 6: Perturbed vector field of class **T** that belongs to the subfamily $\mathcal{E}$

connections: two between $L$ and $M$, and one between $E$ and $L$. On the other hand, any two vector fields of this class are topologically equivalent to each other: the equivalence is produced by an appropriate Dhen twist.

Moreover, no other vector field from $V$ is equivalent to a vector field from $\mathcal{E}$: the latter fields are the only ones that have the polycycle "heart" unbroken.

So, the homeomorphism between two coloured bifurcation diagrams of the families of class $TH$ should respect these classes; it exists therefore only if the ratio $\nu$ is the same for both families. This proves Theorem 4.

These arguments fail when we replace coloured bifurcation diagrams by simple ones. It is unclear whether the families of class $TH$ give rise to a continual set of pairwise non-equivalent simple bifurcation diagrams.

# 5 Structural stability of SN - SN families

In [IKS] structurally unstable three-parameter families of vector fields on $S^2$ where discovered. A conjecture from [IKS] claims that all the generic 2-parameter families are structurally stable. The way to prove this conjecture is to classify generic two parameter families and to prove structural stability for every class. Theorem 3 is a first step in this direction.

## 5.1 Moderate equivalence

Families of class $SN_n^2$ may have zillions of different phase portraits of the unperturbed vector field $v_0$: many hyperbolic limit cycles and singular points may be added to the elements described in the definition of the $SN-SN$ families.

The main theorem of [GI] excluded the influence of these additional points and cycles. In order to state it, we need some definitions.

**Definition 3.** *Two families $V = \{v_\varepsilon\}$ and $W = \{w_\delta\}$ defined on $B \times S^2$ and $B' \times S^2$ respectively, $B$ and $B'$ are $k$-balls with a marked point* 0 *are moderate equivalent provided that there exists a map*

$$H : B \times S^2 \to B' \times S^2, \ (\varepsilon, x) \mapsto (h(\varepsilon), H_\varepsilon(x)),$$

*where $h : B \to B'$ is a homeomorphism , $h(0) = 0$, $H_\varepsilon$ is a homeomorphism $S^2 \to S^2$ that brings the phase portrait of $v_\varepsilon$ to that of $w_{h(\varepsilon)}$, preserves orientation, and is continuous in $\varepsilon, x$ on the set*

$$S(v_0) \cup \partial(\overline{PerV} \cup \overline{SepV}) \cap \{\varepsilon = 0\}. \qquad (10)$$

*Moreover, $H^{-1}$ is continuous on the set*

$$S(w_0) \cup \partial(\overline{PerW} \cup \overline{SepW}) \cap \{\delta = 0\} \qquad (11)$$

*Here $S(v)$ is the separatrix sceleton of a vector field $v$; Per $V$ and Sep $V$ are unions of all the cycles and separatrices of the vector fields of the family $V$.*

A local version of moderate equivalence (a moderate equivalence in neighborhoods of given closed invariant subsets) is needed. We will apply this version to neighborhoods of large bifurcation supports.

**Definition 4.** *Two local families of vector fields on $S^2$, $V = \{v_\varepsilon, \varepsilon \in (B, 0)\}$ and $W = \{w_\varepsilon, \varepsilon \in (B', 0)\}$, are* moderately equivalent in neighborhoods of closed sets *$Z_1, Z_2 \subset S^2$ if*

1. *$Z_1$ is $v_0$-invariant, and $Z_2$ is $w_0$-invariant;*
2. *There exists a neighborhood $U \supset Z_1$ and a map*
$$\mathbf{H} : (B, 0) \times U \to (B', 0) \times S^2, \ (\varepsilon, x) \mapsto (h(\varepsilon), H_\varepsilon(x)), \qquad (12)$$
*such that $h$ is a homeomorphism, $h(0) = 0$, and for each $\varepsilon \in (B, 0)$ the map $H_\varepsilon \colon U \to S^2$ conjugates vector fields $(v_\varepsilon)|_U$ and $(w_{h(\varepsilon)})|_{H_\varepsilon(U)}$;*
3. *$H_0(Z_1) = Z_2$, and moreover,*
4. *For each neighborhood $V$ of $\{\varepsilon = 0\} \times Z_1$, its image $\mathbf{H}(V)$ contains some neighborhood of $\{\varepsilon = 0\} \times Z_2$. The same holds for the inverse map $\mathbf{H}^{-1}$;*
5. *The map $\mathbf{H}$ is continuous with respect to $(\varepsilon, x)$ on the intersection of its domain with* (10).

   *The map $\boldsymbol{H}^{-1}$ is continuous with respect to $(\varepsilon, x)$ on the intersection of its domain with* (11).

## 5.2 Large bifurcation supports

**Definition 5.** *The* extra large bifurcation support $ELBS(v_0)$ *of a vector field* $v_0$ *on the sphere is the union of all non-hyperbolic singular points and non-hyperbolic limit cycles of* $v_0$*, plus the closure of the set of all nonsingular points for which both* $\alpha$*- and* $\omega$*-limit sets are interesting.*

We do not need here the general definition of the interesting $\alpha$ and $\omega$-limit sets. It is sufficient to say that for the SN - SN families interesting $\alpha$ and $\omega$-limit sets are saddles, the saddle-node and the parabolic limit cycle. So

$$ELBS(v_0) = (\cup_0^n S_j) \cup (\cup_0^n u_j) \cup N \cup u_0 \cup \gamma \cup s \cup I. \tag{13}$$

**Definition 6.** *The* large bifurcation support *of a local family* $V$ *of vector fields on the sphere is* $LBS(V) = ELBS(v_0) \cap \big(Sing\ v_0 \cup (\overline{Per\ V} \cup \overline{Sep\ V}) \cap \{\varepsilon = 0\}\big)$.

The set $(\overline{\text{Per } V} \cup \overline{\text{Sep } V}) \cap \{\varepsilon = 0\}$ coincides with the ELBS$(v_0)$. Hence,

$$LBS(V) = ELBS(v_0).$$

## 5.3 Main theorem about the LBS

**Theorem 6.** *[GI] Let two vector fields* $v_0$ *and* $w_0$ *be orbitally topologically equivalent on* $S^2$*; denote the corresponding homeomorphism by* $\hat{H}$*. Let* $V = \{v_\varepsilon, \varepsilon \in (B, 0)\} \subset Vect^* S^2, W = \{w_\varepsilon, \varepsilon \in (B', 0)\} \subset Vect^* S^2$ *be smooth families unfolding these fields. Suppose that there exists a neighborhood* $U$ *of* $LBS(V)$ *and a map*

$$\mathbf{H} : (B, 0) \times U \to (B', 0) \times S^2,\ \mathbf{H}(\varepsilon, x) = (h(\varepsilon), H_\varepsilon(x)),$$

$h(0) = 0$*, which is a moderate equivalence of* $V, W$ *in neighborhoods of* $LBS(V), LBS(W)$ *in the sense of Definition 4. Suppose moreover that* $\hat{H}|_U = H_0$*.*

*Then the families* $V$ *and* $W$ *are weakly equivalent on the whole sphere; namely there exists a map*

$$\hat{\mathbf{H}} : (B, 0) \times S^2 \to (B', 0) \times S^2,\ \hat{\mathbf{H}}(\varepsilon, x) = (h(\varepsilon), \hat{H}_\varepsilon(x))$$

*that provides a weak equivalence of the families* $V$ *and* $W$*.*

So, in order to prove structural stability of a family $V \in SN_N^2$, it is sufficient to prove that it is moderate equivalent to any nearby family in some neighborhood of its large bifurcation support only.

## 5.4 Structural stability of coloured bifurcation diagrams

**Definition 7.** *Two coloured germs of bifurcation diagrams are equivalent if there exists a germ of a homeomorphism of the parameter space that brings one diagram to another and respects the equivalence classes painted in the same colour.*

**Lemma 1.** *For a generic glocal family of class $SN_n^2$ its colour bifurcation diagram is structurally stable.*

**Proof** This follows from the description of the bifurcation diagrams in Section 2.1; we skip the routine details. □

**Lemma 2.** *Generic $SN-SN$ glocal family is moderate equivalent to any nearby family of the same class near its large bifurcation support.*

We will construct the required moderate equivalence "by hands", with the use of a general lemma 3 below.

## 5.5 Marked saddle-node families

### 5.5.1 Statement of the conjugacy lemma

The following lemma is stated in a more general setting than necessary for the proof of Lemma 2.

Consider a map with a parabolic fixed point of multiplicity two on $(\mathbb{R}, 0)$ and its generic unfolding (1) (a saddle-node local family). Let $C$ and $D$ be two finite sets of points on the rays $(\varepsilon, \lambda) = 0,\ x < 0$ and $x > 0$ respectively. Suppose that they satisfy non-synchronization condition [MP]:

time distance between the points of $C$ (in sense of (4)) are pairwise different from those for the set $D$.

**Lemma 3.** *Consider two saddle-node families*

$$F = \{f_{\varepsilon,\lambda}\} \text{ and } \tilde{F} = \{\tilde{f}_{\varepsilon,\lambda}\}$$

*with the same number of parameters of the form (1). Let $C = \{C_1 \ldots, \mathcal{C}_n\}$ be a set of points on $[C_1, f_{0,0}(C_1)] \times \{0, 0\}$ with $0 < x(C_j) < x(C_{j+1})$. Let $D = \{D_1, \ldots, D_k\}$ be a similar set with $x(D_j) < x(D_{j+1}) < 0$. Let $\Gamma_C$ be a set of smooth hypersurfaces $\Gamma_{C_1}, \ldots, \Gamma_{C_m}$ in the total space $x, \varepsilon, \lambda$ transversal to the line $(\varepsilon, \lambda) = 0,\ \Gamma_{C_j} \ni C_j$ and $\Gamma_D$ be a similar set, $\Gamma_{D_j} \ni D_j$. Let $C$ and $D$ satisfy the non-synchronization condition. This family $F$ with the subsets $\Gamma_D, \Gamma_C$ of the total space distinguished is called the marked saddle-node family. Let $\tilde{F}$ be another marked saddle-node family, finite sets and sets of the hypersurfaces $\tilde{C}, \tilde{D}, \Gamma_{\tilde{C}}, \Gamma_{\tilde{D}}$ having the same property.*

*Then there exists a weak equivalence $H$ of the families $F$ and $\tilde{F}$ that brings $C_j$ to $\tilde{C}_j$, $(\Gamma_{C_j}, C_j)$ to $(\Gamma_{\tilde{C}_j}, \tilde{C}_j)$, $D_j$ to $\tilde{D}_j$, $(\Gamma_{D_j}, D_j)$ to $(\Gamma_{\tilde{D}_j}, \tilde{D}_j)$. Moreover, $H$ is continuous at the sets $C, D$ and at the point $0 = (0, 0, 0)$.*

### 5.5.2 Intersection points of $\Gamma_C$ and iterated $\Gamma_D$

**Proof** The total space of the family $F$ is fibered over the base $B = (\mathbb{R}^k, 0)$:

$$(\mathbb{R}^{k+1}, 0) = (\mathbb{R}^k, 0) \times (\mathbb{R}, 0),$$

$(\mathbb{R}, 0)$ is the $x$-fiber. In turn,

$$B = (\mathbb{R}^k, 0) = (\mathbb{R}^{k-1}, 0) \times (\mathbb{R}, 0),$$

$(\mathbb{R}, 0)$ is the $\varepsilon$-fiber. Let $D_i(\varepsilon, \lambda)$ ($C_i(\varepsilon, \lambda)$ ) be the intersection point of the hypersurface $\Gamma_{D_i}$ ($\Gamma_{C_i}$) with the $x$-fiber over the base point $\varepsilon, \lambda$. For any $l, i, j$ and any $\lambda$ sufficiently small there exists $\varepsilon = \varepsilon_{l.i.j}(\lambda)$ such that

$$f^l_{\varepsilon,\lambda}(D_i(\varepsilon, \lambda) = C_j(\varepsilon, \lambda). \tag{14}$$

This means that the points $D_j(\varepsilon, \lambda)$ and $C_j(\varepsilon, \lambda)$, $\varepsilon = e_{l.i.j}(\lambda)$, belong to the same orbit of $F$.

The non-synchronization condition stated below implies that for fixed $(\varepsilon, \lambda)$ there exists no more than one value of $l$ (may be none) such that relation (14) holds.

Let us now start to construct the conjugacy $H = (h, H_\varepsilon)$ mentioned in the lemma. Let us first construct the parameter change $h$ that preserves $\lambda$:

$$h : (\varepsilon, \lambda) \to (\tilde{\varepsilon}, \lambda), \ \tilde{\varepsilon} = h_1(\varepsilon).$$

For any fixed $\lambda, i, j$ the sequence $\varepsilon_{l,i,j}(\lambda)$ is monotonic in $l$ and decreasing. The same holds for $\tilde{\varepsilon}_{l,i,j}(\lambda)$. Let us take $h_1$ to be a $\lambda$-depending homeomorphism of the $\varepsilon$-fiber of the base such that

$$h_1(\varepsilon_{l,i,j}(\lambda)) = \tilde{\varepsilon}_{l,i,j}(\lambda).$$

More requirements on $h_1$ are stated below.

### 5.5.3 Some geometry

For the further construction of $h_1$ we need the following geometric consideration. Let $v = v_\lambda$ be the generator of $F_\lambda$. By definition, the (oriented) $v-$length of an arc on the circle $\varepsilon = \text{const}$ with the endpoints $a$ and $b$ oriented from $a$ to $b$ is the time needed for $a$ to come to $b$ with the velocity $v$, that is

$$\text{dist}_v(a, b) = \int_a^b \frac{dx}{v}.$$

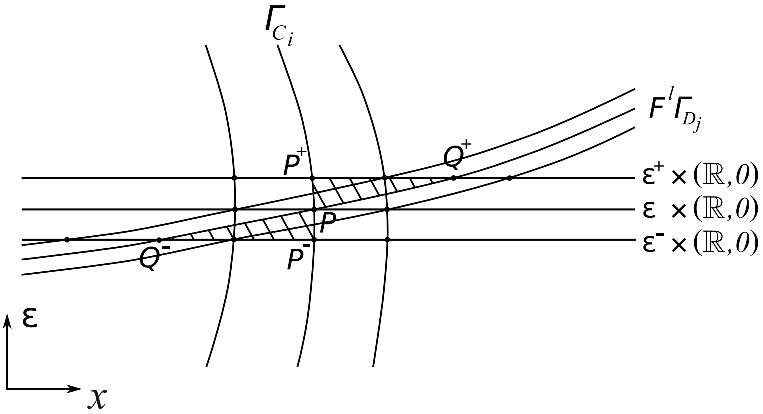


Figure 7: A neighborhood of an intersection point of $\Gamma_{C_i}$ and $F^l\Gamma_{D_j}$

For any $\delta > 0$ and any set $S \subset \{\varepsilon > 0\}$ let $S^\delta$ be a "tubular $\delta$-neighborhood of $S$":

$$S^\delta = \{(x, \varepsilon) | \text{dist}_{v_\varepsilon}(x, S \cap (\{\varepsilon\} \times (\mathbb{R}, 0))) \leq \delta\},$$

where $v_\varepsilon = v|_{\{\varepsilon\}\times(\mathbb{R},0)}$. Let

$$\Sigma = \cup_i \Gamma_{C_i} \cup_{l,j} F^l \Gamma_{D_j}.$$

For a small $\delta$ to be chosen later let $\Sigma^\delta$ be the tubular $\delta$-neighborhood of $\Sigma$. The set $\Sigma$ is a union of smooth curves. Some of them do intersect, see Figure 7. The intersection points satisfy (14). Denote the point $C_j(\varepsilon, \lambda)$ from (14) by $P = P_{l,i,j}$. Let $P^+, Q^+, P^-, Q^-, \varepsilon^+$ and $\varepsilon^-$, ($P^+ = P^+_{l,i,j}, Q^+ = Q^+_{l,i,j}$ and so on) be defined in the following way:

$$P^+ = \Gamma_{C_j} \cap \{\varepsilon^+\} \times (\mathbb{R}, 0), Q^+ = F^l\Gamma_{D_j} \cap \{\varepsilon^+\} \times (\mathbb{R}, 0), \text{dist}_v(P^+, Q^+) = 2\delta,$$

$$P^- = \Gamma_{C_j} \cap \{\varepsilon^-\} \times (\mathbb{R}, 0), Q^- = F^l\Gamma_{D_j} \cap \{\varepsilon^-\} \times (\mathbb{R}, 0), \text{dist}_v(Q^-, P^-) = 2\delta.$$

For any $\varepsilon \in [\varepsilon^-, \varepsilon^+]$ let

$$P_\varepsilon = C_i(\varepsilon),\ Q(\varepsilon) = F^l D_j(\varepsilon),$$

$$\sigma(\varepsilon) = |dist_v(Q(\varepsilon), P(\varepsilon))|.$$

This completes the geometric constructions that we need.

**Proposition 1.** *For large $l, l'$ and small $\delta$, the intervals $[\varepsilon^-, \varepsilon^+]$ with different indexes $l, i, j; l', i', j'$ are pairwise disjoint.*

This proposition will be proved later.

### 5.5.4 Construction of $h$

Let us now turn back to the construction of $h : (\varepsilon, \lambda) \mapsto (h_1(\varepsilon, \lambda), \lambda)$. Let $\tilde{F}$ be another marked saddle-node family with the generator $\tilde{v}$ defined for $\varepsilon > 0$, and the sets $\tilde{\Sigma}, \tilde{\Sigma}^\delta$ ($\delta$ the same as for $F$!), $\tilde{P}$ and so on be analogous objects for $\tilde{F}$; their notations are obtained from those for objects defined for $F$ by adding a tilde.

Let us now define $h_1$. By Proposition 1, any $\varepsilon$ belongs to no more than one interval $[\varepsilon^-, \varepsilon^+] = [\varepsilon^-_{l,i,j}, \varepsilon^+_{l,i,j}]$. Let $h_1([\varepsilon^-, \varepsilon^+]) = [\tilde{\varepsilon}^-, \tilde{\varepsilon}^+]$. More precisely

$$h_1([\varepsilon^-_{l,i,j}, \varepsilon^+_{l,i,j}]) = [\tilde{\varepsilon}^-_{l,i,j}, \tilde{\varepsilon}^+_{l,i,j}].$$

Let $h_1$ on $[\varepsilon^-, \varepsilon^+]$ be such that

$$\sigma(\varepsilon) = \tilde{\sigma}(h_1(\varepsilon)).$$

By Proposition 1, $h_1$ is well defined. This completes the construction of $h_1$ on the (countable) union of segments $[\varepsilon^-, \varepsilon^+]$. Let us extend $h_1$ to the complement of this union by the linear interpolation. This completes the construction of $h_1$ for $\varepsilon > 0$. Let $h_1(0) = 0$.

### 5.5.5 Construction of $H_\varepsilon$ for $\varepsilon > 0$

As $h$ preserves $\lambda$, we now define the map $H = (h, H_\varepsilon)$ on each $(x, \varepsilon)$-plane over fixed $\lambda$. The restriction of $F$ to this plane is denoted by $F_\lambda$. The dependence on $\lambda$ is mentioned no more. The map $h$ is already defined. Consider a domain $\Pi$ which is a fundamental domain for $F_\lambda$ restricted to

$$\Omega = \{\varepsilon > 0\} \cup \{\varepsilon \leq 0,\ x > \sqrt{-\varepsilon}\}.$$

The domain $\Pi$ is a curvilinear strip bounded by a segment $\sigma : x = x_0 > 0,\ |\varepsilon| \leq \varepsilon_0,\ x_0, \varepsilon_0$ small, and $F_\lambda(\sigma)$. Let us glue together the points $(x_0, \varepsilon) \in \sigma$ and $F_\lambda(x_0, \varepsilon) \in F_\lambda(\sigma)$. We will get a cylinder $Z$ and a projection $\pi : \Omega \to Z = \sigma \times S^1$, that brings an orbit of $F_\lambda$ into its intersection with $\Pi$, together with subsequent gluing $\Pi \to Z$. The images $\pi\Gamma_{C_j} \subset Z$ are graphs of smooth functions $\sigma \to S^1$, see Fig 8. The field $\pi_* v$ on $Z$ tangent to the circular fibers $\{\varepsilon\} \times S^1$ is well defined for $\varepsilon \geq 0$, because $v$ is $F$-invariant. Let $\tilde{Z}, \tilde{\pi}$ be the same objects for $\tilde{F}$. The projection $\pi\Gamma_{D_j}$ is a curve that winds around $Z \cap \{\varepsilon > 0\}$ infinitely many times and approaches the cycle $\{0\} \times S^1$. The two curves $\pi\Gamma_{D_j}$ and $\pi C_j$ and the fibers $\varepsilon = \varepsilon_{l,i,j}$ intersect at exactly one point for all natural $l$ sufficiently large.

Let us first construct $H_\varepsilon$ for $\varepsilon > 0$.

**Definition 8.** *Consider two copies of $S^1$ with a vector fields $v$ and $\tilde{v}$ on them. A map of an arc of the first circle to an arc of the second one is called a $(v, \tilde{v})$-isometry if it brings $v$ to $\tilde{v}$. It is called $(v, \tilde{v})$-affine if it is onto and brings the vector field $v$ to $c\tilde{v}$ for some $c$.*

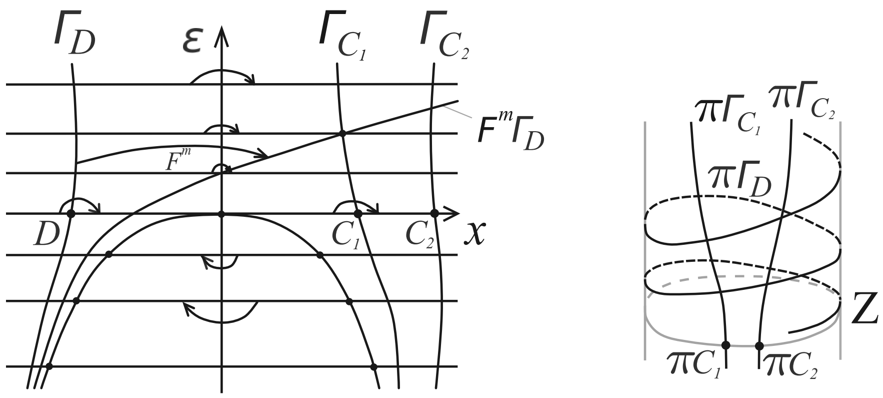

Figure 8: A fundamental domain of a marked saddle-node family

The terms are motivated by the following facts: a $(v, \tilde{v})$-isometry brings the $v$-distance to the $\tilde{v}$-distance; a $(v, \tilde{v})$-affine map is affine with respect to the $v$ and $\tilde{v}$-distances.

Now we can define the map $H_\varepsilon$. Let $Z^+ = Z \cap \{\varepsilon > 0\}, \tilde{Z}^+ = \tilde{Z} \cap \{\varepsilon > 0\}$. Let us define the map $\hat{H} : Z^+ \to \tilde{Z}^+,\ \hat{H} = (h, \hat{H}_\varepsilon)$. The map $h$ is already defined. Let the map

$$\hat{H}_\varepsilon : \pi(\Sigma^\delta \cap \{\varepsilon\} \times S^1) \to \tilde{\pi}(\tilde{\Sigma}^\delta \cap \{h_1(\varepsilon)\}) \times S^1$$

be a $(v, \tilde{v})$ isometry. This implies that on this set $\hat{H}_\varepsilon$ is a bijection, and

$$\hat{H}(\Gamma_C) = \tilde{\Gamma}_{\tilde{C}}, \hat{H}(F^l(\Gamma_C)) = \tilde{F}^l \tilde{\Gamma}_{\tilde{D}} \tag{15}$$

for $l$ large. On the complement to $\pi\Sigma^\delta$, the map

$$\hat{H}_\varepsilon : C(\pi\Sigma^\delta) \cap \{\varepsilon\} \times S^1 \to C(\pi\tilde{\Sigma}^\delta) \cap \{h_1(\varepsilon)\} \times S^1$$

is $(v, \tilde{v})$-affine.

Thus we have defined $\hat{H}$ on $Z^+$, hence $H$ on $\Pi^+ = \Pi \cap \{\varepsilon > 0\}$. By the dynamics, we extend $H$ from $\Pi^+$ to the set $\varepsilon > 0$ near 0 using the formula

$$H \circ F = \tilde{F} \circ H,\ H \circ F^{-1} = \tilde{F}^{-1} \circ H.$$

Thus $H$ is defined on the set $\varepsilon > 0$.

### 5.5.6 Construction of $H_0$

Let us now extend $H$ to the map $H_0$ on the set $\varepsilon = 0$. Let first $x > 0$. Let $\hat{H}_0 : \{0\} \times S^1 \to \{0\} \times S^1$ bring the points $\pi C_j$ to $\pi\tilde{C}_j$ and be

- the $(v, \tilde{v})$-isometry from $\pi C^\delta$ to $\pi\tilde{C}^\delta$;

- $(v,\tilde{v})$-affine from the compliment $\pi[C_1, f_0(C_1)) \setminus \pi C^\delta$ to $\pi[\tilde{C}_1, \tilde{f}_0(\tilde{C}_1)) \setminus \tilde{\pi}\tilde{C}^\delta$.

Thus $\hat{H} : \bar{\bar{Z}}^+ \to \bar{\tilde{Z}}^+$ is continuous on $C$ because the field $v$ is continuous. Thus we defined $H_0$ on $[C_1, f_0(C_1)]$.

Let us extend $H_0$ from $\pi[C_1, f_0(C_1))$ to $\{\varepsilon = 0, x > 0\}$ be dynamics.

The map $H$ on $\{\delta > 0\} \cup \{\varepsilon = 0\} \times \{x > 0\}$ is continuous at $C$ as $\hat{H}$ is.

Let us now construct $H_0$ on $\{\varepsilon\} \times \{x \le 0\}$. By (15), $H(F^l(\Gamma_{D_j})) = \tilde{F}^l(\tilde{\Gamma}_{\tilde{D}_j})$ for large $l$. As $H$ conjugates $F$ and $\tilde{F}$, we conclude:

$$H(\Gamma_{D_j}) = \tilde{\Gamma}_{\tilde{D}_j}.$$

Moreover, as $v$ and $\tilde{v}$ are $F$ and $\tilde{F}$-invariant.

$$H(\Gamma^\delta_{D_j}) = (\tilde{\Gamma}_{\tilde{D}_j})^\delta,$$

and $H$ is a $(v,\tilde{v})$-isometry on this set. Now let us construct $H_0$ on $\{\varepsilon = 0\} \times \{x < 0\}$. Let

$$H_0(D^\delta_j) = \tilde{D}^\delta_j,$$

and $H_0$ be a $(v,\tilde{v})$-isometry on $D^\delta_j$. Extend $H_0$ to $[D_1, f_0(D_1)]$ as a $(v,\tilde{v})$-affine map in the complement to $D^\delta$. Let us then extend $H_0$ to $\{\varepsilon = 0\} \times \{x < 0\}$ by dynamics. Let $H_0(0) = 0$.

Then $H$ now defined for $\varepsilon \le 0$ is continuous at $D$ and 0. It is continuous at $D$ because $v$ and $\tilde{v}$ are continuous at their domains.

It is continuous at 0 because it brings $\gamma_l = F^l\Gamma_{D_1}$ to $\tilde{\gamma}_l = \tilde{F}^l\tilde{\Gamma}_{\tilde{D}_1}$ and $\delta_l = F^{-l}\Gamma_{C_1}$ to $\tilde{\delta}_l = \tilde{F}^{-l}\tilde{\Gamma}_{\tilde{C}_1}$. But the curvilinear triangles bounded by arcs of $\gamma_l, \delta_l$ and the segment $[f_0^l D_1, f_0^{-l} C_1]$ form a shrinking sequence of neighborhoods of 0 in the set $\varepsilon \le 0$. Same for $\tilde{\gamma}_l, \tilde{\delta}_l$ and $[\tilde{f}_0^l \tilde{D}_1, \tilde{f}_0^{-l} \tilde{C}_1]$.

The most difficult part, construction of $H$ in the set $\varepsilon \ge 0$ is over.

### 5.5.7 Construction of $H$ for $\varepsilon < 0$

In the set $\varepsilon < 0$, the map $H$ is constructed in a trivial way: it preserves $\varepsilon$, brings $\Gamma_C$ to $\tilde{\Gamma}_{\tilde{C}}$ in the domain $x > \sqrt{-\varepsilon}$ ($\Gamma_D$ to $\tilde{\Gamma}_{\tilde{D}}$ in the domain $x < -\sqrt{-\varepsilon}$). In the domain $x^2 + \varepsilon < 0$ it preserves $\varepsilon$. It is constructed in a fundamental domain (a cone with a vertex 0) and extended to the domain $x^2 + \varepsilon < 0$ by dynamics.

This proves Lemma 3 modulo Proposition 1. □

Note that the maps $H_\varepsilon$ constructed above may have no limit at all as $\varepsilon \searrow 0$.

#### 5.5.8 Proof of Proposition 1

**Proof** Proposition 1 follows from the non-synchronization condition. The detailed arguments follow.

Let us prove that for $(l,i,j) \neq (l',i',j')$,

$$\text{either } \varepsilon^+_{l,i,j} < \varepsilon^-_{l',i',j'} \text{ or } \varepsilon^-_{l,i,j} > \varepsilon^+_{l',i',j'}. \tag{16}$$

Let $T$ be the time function related to the generator $v_\varepsilon$ for $\varepsilon > 0$:

$$T(x,y,\varepsilon) = \int_x^y \frac{dz}{v_\varepsilon(z)}.$$

Let

$$t(\varepsilon) = T(D_1(\varepsilon), C_1(\varepsilon)),\ \tau_j(\varepsilon) = T(D_1(\varepsilon), D_j(\varepsilon), \varepsilon),\ \rho_i(\varepsilon) = T(C_1(\varepsilon), C_i(\varepsilon), \varepsilon).$$

For $\varepsilon = 0$, $\tau_j(\varepsilon)$ and $\rho_i(\varepsilon)$ are well defined, and $t(0) = \infty$.

The non-synchronization condition requires: for $(i,j) \neq (i',j')$,

$$\tau_j(0) - \rho_j(0) \neq \tau_{j'}(0) - \tau_{i'}(0) (\bmod \mathbb{Z}). \tag{17}$$

Equations for $\varepsilon_{l,i,j}, \varepsilon^+_{l,i,j}$ and $\varepsilon^-_{l,i,j}$ are:

$$\tau_j(\varepsilon) + l = t(\varepsilon) + \rho_i(\varepsilon),$$

$$\tau_j(\varepsilon^+) + l = t(\varepsilon^+) + \rho_i(\varepsilon^+) - 2\delta, \tag{18}$$

$$\tau_j(\varepsilon^-) + l = t(\varepsilon^-) + \rho_i(\varepsilon^-) + 2\delta. \tag{19}$$

Now let us prove (16). We will use the monotonicity of the function $t : t \nearrow \infty$ as $\varepsilon \searrow 0$. WLOG, $l' \geq l$. When $l \to \infty, \varepsilon, \varepsilon^+, \varepsilon^- \to 0$. Hence, for $l$ large,

$$\tau_j(\varepsilon) = \tau_j(0) + o(1),\ \rho_i(\varepsilon) = \rho_i(0) + o(1).$$

If $l' > l + 1$, then

$$l' + \tau_{j'}(0) - \rho_{i'}(0) - 2\delta + o(1) > l + \tau_j(0) - \rho_i(0) + 2\delta.$$

Hence,

$$\varepsilon^-_{l,i,j} > \varepsilon^+_{l;,i',j'}.$$

In case $l' = l + 1$ or $l' = l$, the set

$$(l' + \tau_{j'}(0) - \rho_{i'}(0)) - (l + \tau_j(0) - \rho_i(0))$$

is finite, and contains no zero elements for $(l,i,j) \neq (l',i',j')$ by (17). Together with monotonicity of the function $t$, this implies (16) for small $\varepsilon$ and $\delta$. □

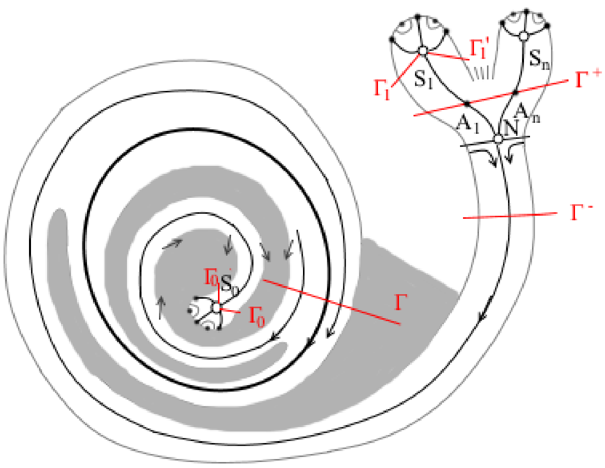


Figure 9: A large bifurcation support of an SN - SN family and its neighborhood

## 5.6 Choice of a neighborhood of the LBS$(V)$

Let $V$ be an SN - SN family, $v_0$ be the unperturbed vector field of $V$. Let $S_j(\varepsilon, \lambda)$ be the saddles of $v_{\varepsilon,\lambda}$ continuous in $\varepsilon, \lambda$ and such that $S_j(0,0) = S_j$. For $\lambda > 0$ let $S_0(\varepsilon, \lambda)$ and $N(\varepsilon, \lambda)$ be the saddle and the node to which the saddle-node $N(0, \lambda)$ is split;

$$S_0(\varepsilon, \lambda) \to N(\varepsilon, 0), N(\varepsilon, \lambda) \to N(\varepsilon, 0)$$

as $\lambda \to 0$. Sometimes we skip the dependence on $\varepsilon, \lambda$ in our notations.

Let $u_1, ..., u_n$ be the separatrices of $S_1, ..., S_n$ that cross $\Gamma^+$, $A_j = u_j \cap \Gamma^+$, see Fig. 1.

For $\lambda < 0$ let $u_0(\varepsilon, \lambda)$ be the unstable separatrix of $S_0$ that crosses $\Gamma^-$. Let $B(\varepsilon, \lambda) = u_0(\varepsilon, \lambda) \cap \Gamma^-$. For $\lambda = 0$ let $u_0(\varepsilon, 0)$ be the separatrix of the hyperbolic sectors of the saddle-nodes $N(\varepsilon, 0)$. Let $B(\varepsilon, 0) = u_0(\varepsilon, 0) \cap \Gamma^-$. Then $B(\varepsilon, \lambda) \to B(\varepsilon, 0)$ as $\varepsilon \nearrow 0$.

For $\lambda < 0$, the saddle-node $N$ vanishes. The separatrices $u_j$ penetrate to $\Gamma^-$ and cross it at the points $B_j(\varepsilon, \lambda) = u_j(\varepsilon, \lambda) \cap \Gamma^-$.

The map $Q : \Gamma^- \to \Gamma$ along the orbits of $v_{(\varepsilon,\lambda)}$ is well defined for all the small parameter values. Let $C = Q(B), C_j = Q(B_j)$. Formally, $C_j$ are defined for $\lambda > 0$ only. But we will set $C_j = C$ for $\lambda \leq 0$.

Let $D = D(\varepsilon, \lambda)$ be the last intersection point of the unstable separatrix $s(\varepsilon, \lambda)$ of $I\varepsilon, \lambda$; it is defined on all the base of $V$.

Let $\tilde{V}$ be a nearby family to $V$. Then all the objects defined for $V$ are defined for $\tilde{V}$ and denoted by the same symbols with tilde.

Let $Ł = LBS(V), \tilde{Ł} = LBS(\tilde{V})$. Let us choose a special neighborhood $U$ of Ł with the following properties, see Fig. 9. The boundary $\partial U$ of $U$ is a smooth curve with $2n$ arcs $\alpha_1, ..., \alpha_n, \sigma_1, ..., \sigma_n$ outside $\gamma$, and two arcs $\omega$ and $\sigma$ inside. The endpoints of the arcs $\sigma_j$ are the intersection points of the stable separatrices of the saddles $S_j$ with $\partial U$;

all these points smoothly depend on $(\varepsilon, \lambda)$. The arcs $\sigma_j,\ j = 1, ..., n-1$ have exactly two contact points with $v_{\varepsilon,\lambda}$. The arcs $\alpha_1, ..., a_{n-1}$ go in between the arcs $\sigma_j$: the arc $\alpha_j$ has one endpoint common with $\sigma_j$, and one endpoint common with $\sigma_{j+1}$. The arc $\alpha_n$ is more complicated: it goes back and forth along the separatrix $u_0$, and once around the semistable cycle $\gamma$. The orbits of the vector fields $v_{\varepsilon,\lambda}$ enter $U$ through all the arcs $\alpha_j$.

The arc $\sigma$ is constructed near the saddle $I$ in the same way as $\sigma_j$ near $S_j$. The arc $\omega$ goes twice along the separatrix $s$, and once around $\gamma$. The orbits of the vector fields $v_{\varepsilon,\lambda}$ exit $U$ through all the arcs $\sigma, \omega$.

Let $\tilde{U}$ be a similar neighborhood of $\tilde{L}$, and $\tilde{\sigma}_j, \tilde{\alpha}_j, \tilde{\sigma}, \tilde{\omega}$ be the similar arcs of $\partial\tilde{U}$.

## 5.7 Moderate equivalence near the large bifurcation supports

We now construct the required equivalence.

Consider a saddle-node family $F$ of the Poincare maps of the cycle $\gamma$ depending on the parameter $\varepsilon, \lambda$ for the family $V$. Let $X$ be the normalizing chart for $F$ on $\Gamma$ and $C, D, C_i(\varepsilon, \lambda)$ (defined for $\varepsilon > 0$), $C(\varepsilon, \lambda)$ (defined for $\varepsilon \leq 0$), $D(\varepsilon, \lambda)$ (defined for $\varepsilon \in (\mathbb{R}, 0))$ be the same as in Section 2.2. As $\lambda$ is fixed, we do not mention dependence on $\lambda$. In order to fit the notations of Lemma 3, let:

$$k = 1, D_1(\varepsilon) = D(\varepsilon), C_i(\varepsilon) = C_i(\varepsilon, \lambda)$$

for $\varepsilon > 0; C_i(\varepsilon) = C(\varepsilon, \lambda)$ for $\varepsilon \leq 0$. Thus the curves $\Gamma_i$ are defined; they differ for $\varepsilon > 0$ and coincide for $\varepsilon \leq 0$. Let $\tilde{F}, \tilde{C}_i, \tilde{D}, \tilde{\Gamma}$ be the analogous objects for $\tilde{V}$.

Let us now apply Lemma 3. It provides a weak topological equivalence between the families $F$ and $\tilde{F}$ with the marked points $C_i(\varepsilon), D(\varepsilon), \tilde{C}_i(\varepsilon), \tilde{D}(\varepsilon)$. The map $H$ is continuous at the points $C(0), D(0)$. The inverse map $H^{-1}$ is continuous at $\tilde{C}(0), \tilde{D}(0)$. The parameter change

$$h : (\varepsilon, \lambda) \mapsto (\tilde{\varepsilon}, \tilde{\lambda}), \tilde{\varepsilon} = h_1(\varepsilon), \tilde{\lambda} = \lambda \tag{20}$$

will be used in the construction of the moderate equivalence between $V$ and $\tilde{V}$ in $U$ and $\tilde{U}$.

Denote by $\Sigma_j, \Sigma_0$ the domains adjacent to the saddles $S_j, I$ and constructed in the following way. The domain $\Sigma_j(\varepsilon_0)$ is bounded by the arc $\sigma_j$ (respectively, $\sigma$), and the stable manifold of $S_j(S)$.

Let us now construct the moderate equivalence $\mathbf{H} = (h, \mathbf{H}_{\varepsilon,\lambda})$ between $V$ and $\tilde{V}$ in $\bar{U}$. The map of the bases $h$ is already constructed; it is provided by Lemma 3 and guaranties that the vector fields of the family $V$ with sparkling saddle connections correspond to those of the family $\tilde{V}$.

The homeomorphism $H_{\varepsilon,\lambda}|_{\Sigma_j} \to \tilde{\Sigma}_j$ continuous in the parameters and conjugating $v_{\varepsilon,\lambda}$ and $\tilde{v}_{h(\varepsilon,\lambda)}$ in those domains may be easily constructed.

Let us now construct $\mathbf{H}_{\varepsilon,\lambda}$ on the complement $W$ of $U$ to $\cup\Sigma_j$. The cross-section $\Gamma^-$ splits $U$ in two parts. Denote the part that contains $\gamma$ by $W$, and the part that contains $N$ by $Z$. Note that $W \supset \Gamma \cap \Gamma$, and $Z \supset \Gamma^+ \cap U$. Let as first construct the map $\mathbf{H}$ in $W$ (times the base). The map $\mathbf{H}_{\varepsilon,\lambda}$ on $\Gamma$ coinsides with the map $H_{\varepsilon,\lambda}$ constructed in Lemma 3.

Any backward orbit of $v_{\varepsilon,\lambda}$ that starts on $\Gamma$ at a point $p$ either intersects $\Gamma$ at the point $q = F^{-1}(p, \varepsilon, \lambda)$, or intersects $\partial U^+ \cup \Gamma^-$ at exactly one point $q$. The same property holds for $\tilde{v}_{h_1(\varepsilon),\lambda}$; any point $\tilde{p} \in \tilde{\Gamma}$ generates a point $\tilde{q} \in \tilde{\Gamma} \cup \partial\tilde{U}^+ \cup \tilde{\Gamma}^-$ on the backward orbit of $\tilde{p}$. For any $p \in \Gamma$, we set $\mathbf{H}_{\varepsilon,\lambda}(p) = H_{h_1(\varepsilon),\lambda}(p)$. Let

$$\mathbf{H}_{\varepsilon,\lambda}(q) = \tilde{q}.$$

Moreover, $\mathbf{H}_{\varepsilon,\lambda}$ brings the arc of the orbit of $v_{\varepsilon,\lambda}$ between $p$ and $q$ to the arc of the orbit of $\tilde{v}_{h_1(\varepsilon),\lambda}$ between $\tilde{p}$ and $\tilde{q}$ with the relative length preserved (if a point $r$ splits the arc from $p$ to $q$ with the ration $\frac{\lambda}{1-\lambda}$ then the point $\mathbf{H}_{\varepsilon,\lambda}(r)$ splits the arc between $\tilde{p}$ and $\tilde{q}$ with the same ratio). Such a map between two arcs is called relatively linear.

Note that the orbits passing through $C_j(\varepsilon, \lambda)$ are separatrices of the saddles $S_j(\varepsilon, \lambda)$, and that $\mathbf{H}_{(\varepsilon,\lambda)}$ brings them to corresponding separatrices, because $\mathbf{H}_{(\varepsilon,\lambda)} C_j(\varepsilon, \lambda) = \tilde{C}_j(h(\varepsilon, \lambda))$.

Let $\Gamma_0$ and $\Gamma_0'$ be two segments that connect the saddle $I$ and points on $\partial U$. Any forward orbit of $v_{\varepsilon,\lambda}$ that starts at $p \in \Gamma$ either intersects $\Gamma$ at the point $q = F(p, \varepsilon, \lambda)$ or at a point $q \in \partial U \cup \Gamma_0 \cup \Gamma_0'$. Let

$$\mathbf{H}_{\varepsilon,\lambda}(p) = H_{\varepsilon,\lambda}(p) = \tilde{p}, \mathbf{H}_{\varepsilon,\lambda}(q) = \tilde{q},$$

and the arc between $p$ and $q$ is mapped by $\mathbf{H}_{\varepsilon,\lambda}$ in the relatively linear way to the arc between $\tilde{p}$ and $\tilde{q}$.

In a similar way the map $\mathbf{H}_{\varepsilon,\lambda}$ is defined in the curvilinear triangle bounded: one by $\Gamma_0$, a stable separatrix of $S_0$, and an arc of $\partial U$; another by $\Gamma_0'$, another stable separatrix of $S_0$, and another arc of $\partial U$, see Fig. 9.

Thus the map $\mathbf{H}_{\varepsilon,\lambda}$ is defined in $W$. It brings saddles $S_0$ to saddles $\tilde{S}_0$, and arcs of separatrices of the saddles $S_j$ that intersect $\Gamma$ to those of the saddles $\tilde{S}_j$ that intersect $\tilde{\Gamma}$.

Let us now construct $\mathbf{H}$ in $B \times Z$. Consider first the domain $Y$ between $\Gamma^+$ and $\Gamma^-$. In this domain the family $V$ is smoothly orbital equivalent to (3). Let the points $B, B_j$ on $\Gamma$ ( $\tilde{B}, \tilde{B}_j$ on $\tilde{\Gamma}$) correspond to the family $V$ (respectibely $\tilde{V}$). It is easy to construct an automorphism of (3) that brings $\Gamma^-$ to $\tilde{\Gamma}^-$, $B$ to $\tilde{B}$, $B_j$ to $\tilde{B}_j$. Automatically, the points $A_j$ will be mapped to $\tilde{A}_j$.

Connect the saddles $S_j$ and $\tilde{S}_j$ by two segments with two points on $\partial U$ and $\partial\tilde{U}$. The same construction as above allows us to extend the conjugacy between the closures of the saturations of $\Gamma^+$ and $\tilde{\Gamma}^+$ by the backward orbits.

The conjugacy $\mathbf{H}$ thus constructed is continuous on the LBS $(V)$, as well as $\mathbf{H^{-1}}$ on the LBS $(\tilde{V})$. Now, Theorem 6 may be applied. It implies structural stability of $V$. Theorem 3 is proved.

# 6 Embedding theorem for parabolic germs with a parameter

In this section we prove Theorem 5. The proof of the original theorem (without the parameter) is rather lengthy. Here we recall the sketch of this proof and show that the presence of the parameter does not affect it.

## 6.1 Takens formal embedding theorem

**Theorem 7.** *[T1] Suppose that a germ $F : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ has a linear part identity. Then it is formally equivalent to an embeddable germ: there exists a formal vector field $\hat{v}$ such that $\hat{F} = g^1_{\hat{v}}$. Here $\hat{F}, \hat{v}$ are formal Taylor series of $F$ and $v$ at zero.*

The same theorem holds for smooth parameter-depending germs. If $F_\lambda : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ is a germ of a $C^\infty$-map depending on a parameter $\lambda \in B = (\mathbb{R}^k, 0)$, then in the previous theorem

$$\hat{F}_\lambda = g^1_{\hat{v}_\lambda},$$

where coefficients of the formal series $\hat{F}_\lambda, \hat{v}_\lambda$ depend smoothly on $\lambda$.

**Proof** The proof of Theorem 7 is based on the fact that the coefficients of the $k$-jet of $\hat{v}$ depend polynomially on the coefficients of the $k$-jet of $F$. If the entries of these formulas are smooth in $\lambda$, then the outcome has the same property. □

Note that a smooth or formal family of vector fields on $(\mathbb{R}, 0)$ is smoothly (or formally) equivalent to a family

$$w_\alpha = \frac{x^2 + \varepsilon}{1 + a(\alpha)x}, \ \alpha = (\varepsilon, \lambda), \ \varepsilon \in (\mathbb{R}, 0), \ \lambda \in (\mathbb{R}^{k-1}, 0). \tag{21}$$

In the formal case $a$ is a formal series in $\varepsilon, x$ with the coefficients depending on $\lambda$, denoted by $\hat{w}_\alpha$.

By the Borel-Whitney theorem, there exists a smooth vector field $v_0$ in a neighborhood of $(0, 0, \lambda)$ such that $\hat{F} = \hat{g}^1_{v_0}$, where hat means here transition to formal Taylor series at $(0, 0, \lambda)$ in the variables $(x, \varepsilon)$. The difference $F - g^1_{v_0}$ is flat on the plane $(0, 0, \lambda)$.

## 6.2 The generator for $f_{0,\lambda}$.

**Theorem 8.** *A smooth family of parabolic germs*

$$x \mapsto x + x^2 + x^3 f(x, \lambda) \tag{22}$$

*has a generator that depends smoothly on $\lambda$.*

**Proof** The proof of this theorem without parameter is well known [T2]. We repeat it here in a modified form adjusted to the presence of the parameter, The family (22) is formally equivalent at $(0, \lambda)$ to

$$g_{0,\lambda} := g^1_{w_{0,\lambda}}, \ w_{0,\lambda} = \frac{x^2}{1 + a(\lambda)x.}$$

There exists a smooth coordinate change that brings the family (1) to the form

$$f_{0,\lambda} = g^1_{w_{0,\lambda}} + R_0,$$

where $R_0$ is flat at $(0, \lambda)$.

The generator is found separately for small positive and negative $x$. Consider the case $x < 0$, the case $x > 0$ is treated in the same way. Take a map

$$t_\lambda : (\mathbb{R}^-, 0) \to (\mathbb{R}, \infty), \ x \mapsto -\frac{1}{x} + a(\lambda) \ln x.$$

The map $t_\lambda$ rectifies the vector field $w_{0,\lambda}$ and brings its time-one phase flow tranwsformation to the unit shift

$$T : t \mapsto t + 1.$$

The family $f_{0.\lambda}$ in the coordinate $t_\lambda$ has the form:

$$\tilde{f}_\lambda : t \mapsto t + 1 + R_\lambda(t), \tag{23}$$

where $R_\lambda$ is flat at infinity. This implies that for any $N$ there exists a smooth function $R_N$, bounded and with bounded derivatives such that

$$R_\lambda(t) = t^{-n} R_N(t, \lambda) \tag{24}$$

We will prove that the family $\tilde{f}_\lambda$ is smooth equivalent to the shift $T : t \mapsto t + 1$ near infinity, and the conjugacy $H_\lambda = \text{id } + h_\lambda$ is a solution of a so called Abel equation

$$h_\lambda - h_\lambda \circ f_{0,\lambda} = R_\lambda.$$

A solution to this equation has the form

$$h_\lambda = \sum_0^\infty R_\lambda \circ f_{0,\lambda}^{[k]}. \qquad (25)$$

The orbit $f_{0,\lambda}^{[k]}(t)$ resembles an arithmetic progression: $f_{0,\lambda}^{[k]}(t) > t + \frac{k}{2}$. Hence, (24) implies that the series for $f{+}_\lambda$ converges near infinity, and all its derivatives in $t, \lambda$ tend to zero as $t \to \infty$. This provides a generator $u^-(0,\lambda)$ for $x < 0$.

In the same way the generator $u^+(0,\lambda)$ for $x > 0$ is contrudiction.

The formal series of $u^-$ and $u^+$ coincide; so, $u^+$ is a $C^\infty$ extension of $u^-$ to the whole neighborhood of zero in $x$. □

From now on we consider a family $F = \{f_{(\varepsilon,\lambda)}\}$ such that $f_{(0,\lambda)}$ is embedded in a smooth flow.

### 6.3 A semiformally invariant vector field

A vector field is invariant under $F$ iff

$$F'_x v - v \circ F = 0.$$

We have already constructed the field $v$ for $\varepsilon = 0$. Denote this field by $w_0(x,\lambda)$.

Let

$$\hat{F}(x,\varepsilon,\lambda) = f_j(x,\lambda)\varepsilon^j.$$

This is a *semiformal series*: it is a formal Taylor series in $\varepsilon$ whose coefficients are smooth functions $f_j(x,\lambda)$. Let us find a formal generator of $F$ as a semiformal series:

$$\hat{v} = \sum w_j(x,\lambda)\varepsilon^j$$

such that

$$\hat{F}'_x \hat{v} - \hat{v} \circ \hat{F} \equiv 0.$$

We will find by induction in $k$ a vector field $v_k$ polynomial in $\varepsilon$ such that

$$F'_x v_k - v_k \circ F = o(\varepsilon^k). \qquad (26)$$

Base of induction: $k = 0$. Take $v_0(x,\varepsilon,\lambda) = w_0(x,\varepsilon)$. Then

$$F'_x v_0 - v_\circ F = o(1).$$

Induction step. Suppose that the vector field $v_k$ is already found; let us find

$$v_{k+1} = v_k + w_{k+1}(x,\lambda)e^{k+1}.$$

Equation (26) for $k$ replaced by $k+1$ implies:

$$\frac{df_0}{dx} w_{k+1} - w_{k+1} \circ f_0 = R_{k+1},$$

where $R_{k+1}$ is a polynomial in $w_j, j \leq k$ and their derivatives with the smooth coefficients flat at $(0, \lambda)$. In this section, the functions written in the chart $t_\lambda$ are denoted by the same symbol with tilde. In the chart $(t_\lambda, \lambda)$, the previous equation becomes a so called Abel equation (we skip the subscript $\lambda$ from $t_\lambda$ for brevity):

$$\tilde{w}_{k+1}(t, \lambda) - \tilde{w}_{k+1}(t+1, \lambda) = \tilde{R}_{k+1}(t, \lambda) \tag{27}$$

where $\tilde{R}_{k+1}$ is a polynomial in $\tilde{w}_j, j \leq k$ and their derivatives with the smooth coefficients flat at $(\infty, \lambda)$. This means that for any $N > 0, \alpha$

$$\tilde{D}^\alpha \tilde{R}_{k+1} = t^{-N} R_{N,\alpha}$$

where $R_{N,\alpha}$ is bounded together with all its derivatives. Equation (27) has a solution

$$\tilde{w}_{k+1} = \sum_{l=0}^{\infty} \tilde{R}_{k+1}(t+l, \lambda).$$

Thus series converges to a function flat at infinity.

Thus we constructed a semiformal $F$-invariant vector field

$$\hat{v} = \sum_{l=0}^{\infty} w_k \varepsilon^k$$

flat in $(x, \varepsilon)$ at the points $(0, 0, \lambda)$. Let $v$ be an extension of $v$, that is, a smooth function whose semiformal Taylor series in $\varepsilon$ equals $\hat{v}$. Then

$$F'_x v - v \circ F = R; \tag{28}$$

$R$ is flat in $\varepsilon$ on the hyperplane $L = (\varepsilon = 0)$. Hence, for any $N, \alpha$

$$D^\alpha R = \varepsilon^N R_N^\alpha. \tag{29}$$

Moreover, on $L$

$$D^\alpha R|_{\varepsilon=0} = x^N Q_N^\alpha. \tag{30}$$

## 6.4 From the semiformal invariant vector field to almost generator

Consider the time distance between the points $(x, \varepsilon, \lambda)$ and $(F(x, \varepsilon, \lambda), \varepsilon, \lambda)$ related to the vector field $v$ for $\varepsilon > 0$:

$$T(x, \varepsilon, \lambda) = \int_x^{F(x,\varepsilon,\lambda)} \frac{dy}{v(y, \varepsilon, \lambda)}.$$

We have:

$$T'_x = \frac{F'_x}{v \circ F} - \frac{1}{v} = \frac{F'_x \cdot v - v \circ F}{v \cdot v \circ F}.$$

Should the vector field $v$ be invariant, this derivative would be identically zero, and $T$ would not depend on $x$: $T(x, \varepsilon, \lambda) = \tau(\varepsilon, \lambda)$. In our case, $T'_x$ is flat on $L$. Take a small $\delta$ and let $\tau(\varepsilon, \lambda) = T(-\delta, \varepsilon, \lambda)$. Then

$$T(x, \varepsilon, \lambda) = \tau(\varepsilon, \lambda) + Q(x, \varepsilon, \lambda),$$

$Q$ is flat on $L$. Let $V = v\tau$. We will prove that

$$g_V^1 = F + G, \tag{31}$$

where $G$ is flat on $L$. The field $V$ may be called *almost generator* of $F$. We have:

$$\int_x^{F(x,\varepsilon,\lambda)} \frac{dy}{V(y, \varepsilon, \lambda)} = \frac{1}{\tau} \int_x^{F(x,\varepsilon,\lambda)} \frac{dy}{v(y, \varepsilon, \lambda)} = 1 + \frac{Q}{\tau}.$$

The function $\frac{Q}{\tau}$ is flat on $L$. We have:

$$g_V^{1+\frac{Q}{\tau}} = F = g_V^1 \circ g_V^{\frac{Q}{\tau}}.$$

The map $g_V^{\frac{Q}{\tau}} = id + g$, $\tilde{g}$ is flat on $L$. Hence,

$$F = g_V^1 - G,$$

$G$ is flat on $L$. This proves (31).

## 6.5 Conjugacy

**Lemma 4.** *Let a family $F$ (1) satisfy (31), where $G$ is flat on $L$. Then $F$ is conjugate to $g_V^1$; the conjugacy is $C^\infty$ in $(x, \varepsilon, \lambda)$ for $\varepsilon \leq 0$.*

**Proof** The vector field $V$ is smooth equivalent to $\frac{x^2+\varepsilon}{1+a(\varepsilon,\lambda)x}$. Consider a domain

$$\Omega^- : \{\varepsilon > 0,\ |x| \leq \delta\} \cup \{\varepsilon = 0,\ -\delta \leq x < 0\}$$

Let us define a chart $(t, \varepsilon, \lambda)$ in $\Omega^-$ given by the formulas:

$$t_{\varepsilon,\lambda} = \frac{1}{2\sqrt{\varepsilon}} \arctan(x^2 + \varepsilon) + \frac{1}{2} \ln(x^2 + \varepsilon)$$

for $\varepsilon \neq 0$;

$$t_{0,\lambda} = -\frac{1}{x} + a(0, \lambda) \ln x$$

for $\varepsilon = 0$. This map rectifies the vector field $V$.

Let us globalize the map $F$, multiplying $G$ by a cutting smooth function $\varphi$:

$$0 \leq \varphi \leq 1,\ \varphi \equiv 1 \text{ near } 0;\ \varphi \equiv 0 \text{ for } |x| \geq \delta.$$

Under the map $t$ the set $\Omega^-$ comes to a set that contains a set

$$\tilde{\Omega}^- : \{\varepsilon > 0,\ -\delta' \leq x < \psi(\varepsilon)\},$$

where $\psi(\varepsilon) < C\varepsilon^{-\frac{1}{2}}$. The map $F$ in the domain $\Omega^-$ becomes a map $\tilde{F}$ in $\tilde{\Omega}^-$; $\tilde{F}$ is an "almost shift":

$$\tilde{F} : (t, \varepsilon, \lambda) \mapsto (t + 1 + \tilde{G},\ \varepsilon, \lambda).$$

Note that $\tilde{G}$ is identically zero for $x > \psi(\varepsilon)$. Hence, the map $\tilde{F}$ is globally defined for $t \geq \delta'$, and all the positive iterates of $\tilde{F}$ are well defined in $\tilde{\Omega}^-$.

Consider now the conjugacy equation for $T : (t, \varepsilon, \lambda) \mapsto (t+1, \varepsilon, \lambda)$ and $\tilde{F} = T + \tilde{G}$. It is important to mention that for any $N$,

$$\tilde{G} = \varepsilon^N t^{-N} J_N \tag{32}$$

together with all its derivatives:

$$D^\alpha \tilde{G} = \varepsilon^N t^{-N} J_{N,\alpha}, \tag{33}$$

where functions $J_N, J_{N,\alpha}$ are smooth and bounded with all their derivatives.

Let us find $\tilde{H} = id + \tilde{h}$ such that

$$\tilde{H} \circ \tilde{F} = T \circ \tilde{H}.$$

This equation has a solution

$$h = \Sigma_0^\infty \tilde{G} \circ \tilde{F}^{[k]}. \tag{34}$$

The map $\tilde{F}$ is close to a shift, and its orbits are close to arithmetic progressions. Equations (32) and (33) imply that the series $h$ converges to a map flat on $L^- : \varepsilon = 0,\ x < 0$.

Lemma 4 is proved in $\Omega^-$. But what about $\varepsilon = 0, x \geq 0$?

The segments $\{\lambda, x \geq 0$ fixed, $\varepsilon \in [\varepsilon_0, 0), \varepsilon_0 > 0\}$, are mapped by $t$ to curves $t = \psi_{\lambda,x}(\varepsilon),\ t = O(1)\varepsilon^{-\frac{1}{2}}$. The points on $L^+ : \varepsilon = 0, x \geq 0$ are the ideal endpoints of these curves. The function $\tilde{h}$ still tends to 0 together with all its derivatives along these curves. Let $H = id + h$ be the map $\tilde{H}$ written in the original chart $(x, \varepsilon, \lambda)$. It is defined in $|x| \leq \delta,\ \lambda \in (\mathbb{R}^k, 0),\ \varepsilon \in (\mathbb{R}^k, 0)$ outside $L^+ : x \geq 0,\ \lambda \in (\mathbb{R}^k, 0),\ \varepsilon = 0$. But, according to the previous estimate, $h \to 0$ together with all its derivatives, as $(x, \varepsilon, \lambda)$ approaches $L^+$. Hence, $h$ may be extended by 0 on $L^+$, and $H$ will remain infinitely smooth on $L^+$. Hence, the Takens generator $v_0$ on $L^+$ is the limit of the generator $v$ already found. This proves Theorem 5 $\square$

# 7 Sketch of the proof of Conjecture 1

Here we present an idea how to modify the example from Secton 4 to get a continuum of pairwise different simple bifurcation diagrams. The presentation is quite informal. Begin with a three-parameter family $TH$ from Section 4 and consider the one-parameter family $\mathcal{E}$ described there. There are three sequences $\varepsilon_n$, $i_m$ and $\lambda_k$ that monotonically tend to 0 and have the following property: vector field $v_{\varepsilon_n}$ has a sparkling saddle connection between $E$ and $L$; vector field $v_{i_m}$ has a sparkling saddle connection between $L$ and $I$, vector field $v_{\lambda_k}$ has a sparkling saddle connection between $E$ and $I$. The first two sequences where already considered. On the coloured bifurcation diagram they are distinwished, and generate the numeric moduli of these diagrams. On the simple bifurcation diagram all the three sequences are seen as merely one sequence converging to 0.

We want to mark these sequences even on a simple bifurcation diagram. For this sace replace $TH$ by a five-parameter family, constructed from $TH$ with the following changes. Replace the hyperbolic saddle $E$ by a saddle-node still denoted by $E$ and governed by an additional parameter $\alpha_1$; the separatrix of the hyperbolic sectors of $E$ winds towards the polycycle $\gamma$. The same for the hyperbolic saddle $I$: it is replaced by a saddle-node still denoted by $I$ and governed by an additional parameter $\alpha_2$. The separatrix of the hyperbolic sectors of $I$ winds towards the loop $l$ in the negative time. Suppose that there are exactly three hyperbolic saddles $E_j, j = 1, 2, 3$; exactly one unstable separatrix of each saddle enters $E$. Suppose that there are exactly two hyperbolic saddles $I_k, k = 1, 2$; exactly one stable separatrix of each saddle emerges from $I$. Consider a one-parameter subfamily $\mathcal{E}$ of the unfolding of the vector field thus modified. Vector fields of this family have the polycycle "heart" unbroken, and saddle-nodes $E$ and $I$ preserved. Only the loop is broken. Again there are three sequences $e_n, i_m, \lambda_k$ corresponding to sparkling saddle connections $EL, LI, EI$ respectively. But now these three sequences are marked even inside the simple bifurcation diagram of the family: when the saddle-nodes vanish, a bunch of three, two and six connections occurs near $EL, LI, EI$ respectively.

This is quite a heuristic argument; to make it formal one need to prove that the family $\mathcal{E}$ is topologically distinguished even inside the simple bifurcation diagram of the unfolding, and many other things.

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