Finiteness Theorems for Limit Cycles

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To my family:
Lena
Serezha
Lizochka
Aleksandr

Foreword

This book is devoted to a proof of the following finiteness theorem:

A polynomial vector field on the real plane has a finite number of limit cycles.

Some related results are proved along with it.

At the time of the discovery of limit cycles more than one hundred years ago Poincaré posed the question of whether the number of these cycles is finite for polynomial vector fields. He proved that the answer is yes for fields not having polycycles (separatrix polygons).

In a series of papers between 1889 and 1923, Dulac, a student of Poincaré, advanced greatly the local theory of differential equations (his achievements were finally understood only in the 1970s and early 1980s), and he presented a proof of the finiteness theorem in the memoir "Sur les cycles limites" (1923). In 1981 an error was found in this proof. The 1923 memoir practically concluded the mathematical creativity of Dulac. In the next thirty-two years (he died in 1955) he published only one survey (1934). Did he discover the error in his paper? Did he attempt to correct it during all his last years? These questions will surely remain forever unanswered.

To prove the finiteness theorem it suffices to see that limit cycles cannot accumulate on a polycycle of an analytic vector field (the nonaccumulation theorem). For this it is necessary to investigate the monodromy transformation (also called the Poincaré return mapping or the first return mapping) corresponding to this cycle. The investigation in this book uses the following five sources.

1. The theory of Dulac. This theory enables us to investigate the power asymptotics of the monodromy transformation. However, there exists a polycycle of an analytic vector field whose monodromy transformation has a non-identity flat correction which thus decreases more rapidly than any power (the correction of a mapping is the difference between it and the identity). Therefore, power asymptotics are clearly insufficient for describing monodromy transformations.

2. Going out into the complex domain. The first systematic investigation of the global theory of analytic differential equations on the complex projective plane was undertaken by Petrovskiĭ and Landis in 1955. By extending the solutions of an analytic differential equation into a neighborhood of a polycycle in the complex plane, the author was able to prove the nonaccumulation theorem for a polycycle whose vertices are nondegenerate saddles (1984). This step was taken under the influence of the work of Petrovskiĭ and Landis.

3. Resolution of singularities. This procedure, which reduces in its simplest variant to a finite series of polar blowing-ups (transitions from Cartesian coordinates to polar coordinates), enables us to essentially simplify the behavior of the solutions in a neighborhood of singular points of a vector field. The theorem on resolution of singularities asserts that in finitely many polar blowing-ups a compound singular

point of an analytic vector field can be replaced by finitely many elementary singular points. The latter is the name for singular points at which the linearization of the field has at least one nonzero eigenvalue. The greatest complexity in the structure of the monodromy transformation is introduced by degenerate elementary singular points with one eigenvalue equal to zero and the other not equal to zero. They are investigated by methods of the geometric theory of normal forms.

4. The geometric theory of normal forms. Formal changes of variables enable us to reduce the germs of vector fields at singular points and the germs of diffeomorphisms at fixed points to comparatively simple so-called "resonant" normal forms (synonym: Poincaré–Dulac normal forms). As a rule, the normalizing series diverge when there are resonances, including the vanishing of an eigenvalue of the linearization (Bryuno, 1971; the author, 1981).

In this case the normal form is given not analytically as a series with a "relatively small number" of nonzero coefficients, but geometrically as a so-called "normalizing atlas." Namely, a punctured neighborhood of a singular point in a complex space is covered by finitely many domains of sector type that contain this singular point on the boundary. In each of these neighborhoods the vector field is analytically equivalent to its resonant normal form; a change of coordinates conjugating the original field with its normal form is said to be normalizing. A collection of normalizing substitutions is called a normalizing atlas. All the information about the geometric properties of the germ is contained in the transition functions from one normalizing substitution to another. The nontriviality (difference from the identity transformation) of these transition functions constitutes the so-called "nonlinear Stokes phenomenon." (A collection of papers by Elizarov, Shcherbakov, Voronin, Yakovenko, and the author will be devoted to this phenomenon.) It was first investigated for one-dimensional mappings by Ecalle, Malgrange, and Voronin in 1981. Normalizing atlases for germs of one-dimensional mappings are so-called functional cochains and play a fundamental role in the description of monodromy transformations of polycycles.

5. Superexact asymptotic series. These series are for use in describing asymptotic behavior with power terms and exponentially small terms simultaneously taken into account, and perhaps also iterated-exponentially small terms.

The structure of the book is as follows. In the Introduction we present all results about the Dulac problem obtained up to the writing of this book, with full proofs. An exception is formed by results in the local theory and theorems on resolution of singularities; their proofs belong naturally in textbooks, but such texts have unfortunately not yet been written. Superexact asymptotic series are discussed at the end of the Introduction and historical comments are given.

In the first chapter we give a complete description of monodromy transformations of polycycles of analytic vector fields and prove the nonaccumulation theorem. The main part of the chapter is the definition of regular functional cochains, which are used to describe monodromy transformations. This description is based on the group properties of regular functional cochains. Their verification recalls the proving of identities. However, since the definitions are very cumbersome, many details must be checked, and this takes a lot of space: Chapters II, IV, and, in part, V.

One of the most important properties of regular functional cochains is that they are uniquely determined by their superexact asymptotic series (STAR)[1] This is an assertion of the same type as the Phragmén–Lindelöf theorem for holomorphic functions of a single variable; it is used without proof in Chapter I and is proved in Chapter III and part of Chapter V.

Finally, the partial sums of STAR do not oscillate. This is established in §4.10. The proof of the nonaccumulation theorem is thus based on the following chain of implications.

A monodromy transformation has countably many fixed points $\Rightarrow$ the STAR for its correction is zero (since the partial sums of the nonzero series do not oscillate) $\Rightarrow$ the correction is zero by the Phragmén–Lindelöf theorem.

The introductory chapter overcomes all the difficulties connected with differential equations and reduces the finiteness theorem to questions of one-dimensional complex analysis.

The ideas for the proof presented were published by the author in the journal *Uspekhi Mathematicheskikh Nauk* **45** (1990), no. 2. This paper is the first part of the proposed work, of which the second part—the present book—is formally independent. In the first part the nonaccumulation theorem is proved for the case when the monodromy transformation has power and exponential asymptotics but not iterated-exponential asymptotics. The scheme of the paper is close to that of the book: the four sections of the paper are parallel to the first four chapters of the book and contain the same ideas, but there are essentially fewer technical difficulties in them. The reading of the paper should facilitate markedly the reading of the book. However, the text of the book is independent of the first part, and it is intended for autonomous reading.

Some words about the organization of the text. The book is divided into chapters and sections; almost all sections are divided into subsections. The numbering of the lemmas begins anew in each chapter, and the formulas in a chapter are labelled by asterisks; the labelling for formulas and propositions begins anew in each section. References to formulas and propositions in other sections are rare. In referring to a subsection of the same section we indicate only the letter before the heading of the subsection, and in referring to a subsection of another section of the same chapter we indicate the number of the chapter and the section; in referring to another chapter we indicate the chapter, section, and subsection.

I received a great deal of diverse help in writing this book from R. I. Bogdanov, A. L. Vol'berg, E. A. Gorin, V. P. Gurarii, P. M. Elizarov, A. S. Il'yashenko, E. M. Landis, A. Yu. and V. M. Nemirovich-Danchenko, and A. G. Khovanskiĭ. To all of them I express warm gratitude.

The several years spent writing this book were a heavy burden on my wife and children. Without their understanding, patience, and love the work would certainly never have been completed. I dedicate the book to my family.

[1]This is the Russian abbreviation of Super Exact Asymptotic Series (Сверх Точные Асимптотические Ряды). It is chosen because it seems to sound better in English than the English abbreviation.

Introduction

§ 0.1. Formulation of results: finiteness theorems and the identity theorem

In this section we formulate the main results: three finiteness theorems, the nonaccumulation theorem, and the identity theorem. Then we derive the first four theorems from the fifth.

A. Main theorems. This book is devoted to a proof of the following results.

THEOREM I. *A polynomial vector field on the real plane has only finitely many limit cycles.*

THEOREM II. *An analytic vector field on a closed two-dimensional surface has only finitely many limit cycles.*

THEOREM III. *A singular point of an analytic vector field on the real plane has a neighborhood free of limit cycles.*

These three theorems are called finiteness theorems.

As known from the times of Poincaré and Dulac ([**?, ?**]; a detailed reduction is carried out in §0.1B), the first two theorems are consequences of the following theorem.

THEOREM IV (nonaccumulation theorem). *An elementary polycycle of an analytic vector field on a two-dimensional surface has a neighborhood free of limit cycles.*

Recall that a polycycle of a vector field is a separatrix polygon; more precisely, it is a union of finitely many singular points and nontrivial phase curves of this field, with the set of singular points nonempty; solutions corresponding to nontrivial phase curves tend to singular points as $t \to +\infty$ and $t \to -\infty$; a polycycle is connected and cannot be contracted in itself to some proper subset of itself (Figure 1, next page).

A polycycle is said to be elementary if all its singular points are elementary, that is, their linearizations have at least one nonzero eigenvalue.

The monodromy transformation of a polycycle is defined in the same way as for an ordinary cycle, except that a half-open interval is used in place of an open interval (Figure 2). It is convenient to regard monodromy transformations as germs of mappings $(\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$.

FIGURE 1 f1

FIGURE 2 f2

THEOREM V (identity theorem). *Suppose that the monodromy transformation of polycycle of an analytic vector field on a two-dimensional surface has countably many fixed points. Then it is the identity.*

B. Reductions: A geometric lemma. The logical connections between Theorems I to V are shown in Figure 3. We strengthen the nonaccumulation Theorem IV by replacing the word "elementary" by the word "any." At the end of the subsection this new theorem will be derived from the identity theorem. In turn, it immediately yields Theorem III: a singular point is a one-point polycycle. Theorem III is singled out because its assertion has been widely used in the mathematical literature and folklore without reference to Dulac's memoir [**?**], the only source laying claim to a proof, though not containing one.

Theorem II also follows from the strengthened nonaccumulation theorem. Indeed, assume that it is false, and there exists an analytic vector field with countably many limit cycles on a closed two-dimensional surface.

GEOMETRIC LEMMA. *Suppose that an analytic vector field on a closed two-dimensional surface has a sequence of closed phase curves. Then there exists a subsequence of this sequence accumulating on a closed phase curve or a polycycle* (*recall that a polycycle can degenerate into a point*).

REMARK. This lemma is true for smooth fields with singular points of finite multiplicity, and is false for arbitrary smooth fields: the limit set for such fields can be a "separatrix polygon with infinitely many sides" (Figure 4).

Everywhere below, "smoothness" means "infinite smoothness," and "diffeomorphism" means a C^∞-diffeomorphism.

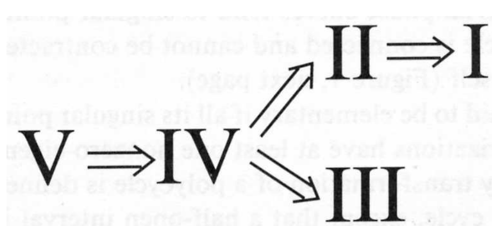


FIGURE 3 f3

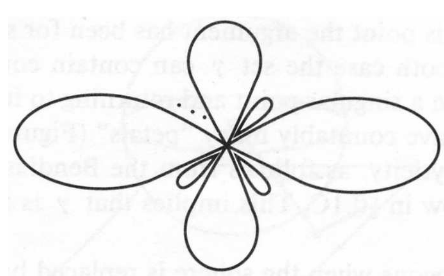


FIGURE 4 f4

Proof. We prove the geometric lemma for the case when the surface in it is a sphere. Consider a disk on the sphere that does not intersect the countable set of curves in the given sequence. The stereographic projection with center at the center of the disk carries the original vector field into an analytic vector field on the plane that has a countable set of closed phase curves in some disk. It can be assumed without loss of generality that the number of singular points of the field in each disk is finite: otherwise the analytic functions giving the components of the field would have a common noninvertible factor, and by dividing both components by a suitable common analytic noninvertible factor we could make the resulting field have only finitely many singular points in each compact set. The closed phase curves of the original field remain phase curves of the new field.

Only finitely many curves in the sequence under consideration can be located pairwise outside each other: inside each of these curves is a singular point of the field, and the number of singular points is finite. Consequently, our sequence decomposes into finitely many subsets called nests: each curve of a nest bounds a domain with a countable set of curves of the same nest outside it or inside it. Take a sequence of curves of a nest; it is possible to take one point on each of them in such a way that the sequence of points converges. Then the chosen sequence of curves accumulates on a connected set γ. By the theorem on continuous dependence of the solutions on the initial conditions, this set consists of singular points of the equation and of phase curves. The considerations used in the Poincaré–Bendixson theorem enable us to prove that these curves go from some singular points to others if γ is not a cycle. Up to this point the argument has been for smooth vector fields. However, in the smooth case the set γ can contain countably many phase curves going out from a singular point and returning to it: a singular point of a smooth field can have countably many "petals" (Figure 4). This pathology is prevented by analyticity, as follows from the Bendixson–Dumortier theorem formulated below in §0.1C. This implies that γ is a closed phase curve or a polycycle.

The proof is analogous when the sphere is replaced by an arbitrary closed surface, but additional elementary topological considerations are needed, and we do not dwell on them. □

All the subsequent arguments are also given for the case when S is a sphere.

This concludes the derivation of Theorem II from the strengthened nonaccumulation theorem.

We derive this last theorem from the identity theorem. To do that is suffices to prove that a monodromy transformation is defined for the polycycle γ in the geometric lemma (the limit cycle corresponds to a fixed point of the monodromy transformation). For this, in turn, it is necessary to make more precise the definition given in §0.1A on an intuitive level.

Definition 1. A semitransversal to a polycycle of a vector field on a surface S is defined to be a curve $\varphi\colon [0,1) \to S$ satisfying the following conditions: the point $\varphi(0)$, called the vertex of the semitransversal, lies on the cycle; the curve φ is transversal to the field at all points except perhaps the vertex.

Using the word loosely, we also call the image of φ a semitransversal.

Definition 2. A polycycle γ of a vector field is said to be monodromic if for an arbitrary neighborhood $\mathcal{U}$ of the cycle there exist two semitransversals Γ and Γ' with vertex on γ, one belonging to the other, that have the following properties:

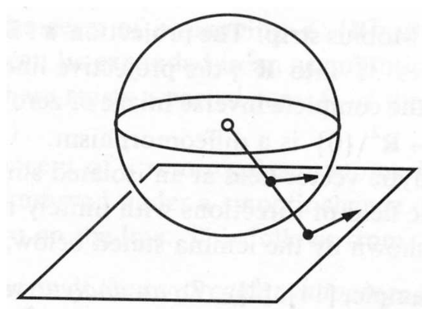


FIGURE 5 f5

the positive semitrajectory beginning at an arbitrary point q on Γ intersects Γ' at a positive time, with the first such point of intersection denoted by $\Delta(q)$; the arc of the semitrajectory with initial point q and endpoint $\Delta(q)$ lies in the neighborhood $\mathcal{U}$.

The germ of the mapping $\varphi^{-1} \circ \Delta \circ \varphi \colon (\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$ is called the monodromy transformation of the polycycle γ and denoted by Δ_γ.

We now take a transversal to the polycycle γ constructed in the geometric lemma; let γ_n be a sequence of closed phase curves accumulating on γ. One of the semitransversals of this transversal intersects countably many curves in this sequence. The curves γ_n and γ_{n+1} bound a domain homeomorphic to an annulus; let Γ_n be the intersection of this "annulus" with Γ. It can be assumed without loss of generality that there are no singular points inside this annulus, since there are only finitely many such points. Consequently, by the theorem on extension of phase curves, the monodromy transformation $\Gamma_n \to \Gamma_n$ is defined. This implies that the polycycle γ is monodromic. The strengthened nonaccumulation theorem thereby follows from the identity theorem.

Finally, Theorem I is a simple consequence of Theorem II. The reduction is carried out with the help of a well-known construction of Poincaré (Figure 5). Consider a sphere tangent to the plane at its South Pole, and a polynomial vector field on the plane. We project the sphere from the center onto this plane. Everywhere off the equator of the sphere there arises an analytic vector field that is "lifted from the plane" and tends to infinity on approaching the equator. Multiplying the constructed field by a suitable power of the analytic function "distance to the equator," we get a new field with finitely many singular points on the equator, and hence on the entire sphere. By Theorem II, it has finitely many limit cycles. Thus, the original field on the plane also has finitely many limit cycles. This proves Theorem I.

We remark that the identity theorem is obvious in the case when γ is a closed phase curve. In this case the monodromy transformation Δ_γ is the germ of an analytic mapping $(\mathbb{R}, 0) \to (\mathbb{R}, 0)$. If the limit cycles accumulate on γ, then the mapping Δ_γ has a countable set of isolated fixed points that accumulate at an interior point of the domain of definition, which contradicts the uniqueness theorem for analytic functions.

The proof of the identity theorem in the general case goes according to the same scheme. The only difficulty is that isolated fixed points of the monodromy transformation accumulate not at an interior point, but at a boundary point of the domain of definition, and this is not forbidden for a biholomorphic mapping.

In the next subsection the strengthened nonaccumulation theorem is derived from Theorem IV.

C. The reduction: Resolution of singularities. We recall the definition of resolution of singularities (otherwise known as the σ-process or blowing-up), following Arnol'd [**?**]. Consider the natural mapping of the punctured real plane $\mathbb{R}^2 \setminus \{0\}$ onto the projective line $\mathbb{R}P^1$: with each point of the punctured plane we associate the line joining this point to zero. The graph of this mapping is denoted by M; its closure $\overline{M}$ in the direct product $\mathbb{R}^2 \times \mathbb{R}P^1$ is diffeomorphic to the Möbius strip. The projection $\pi\colon \mathbb{R}^2 \times \mathbb{R}P^1 \to \mathbb{R}^2$ along the second factor carries $\overline{M}$ into $\mathbb{R}^2$; the projective line $L = \mathbb{R}P^1$ (called a glued-in line below) is the complete inverse image of zero under this mapping; the projection $\pi\colon M \to \mathbb{R}^2 \setminus \{0\}$ is a diffeomorphism.

The germ of an analytic vector field at an isolated singular point becomes the germ of an analytic field of directions with finitely many singular points on a glued-in line, as shown by the lemma stated below.

Lemma (see, for example, [**?**, **?**]). *To an analytic vector field v given in a neighborhood of the isolated singular point 0 in $\mathbb{R}^2$ there corresponds an analytic field of directions α defined in some neighborhood of a glued-in line L on the surface $\bar{M}$ everywhere except for finitely many points located on L and called singular points. Under the projection $\pi\colon M \to \mathbb{R}^2 \setminus \{0\}$ the field α passes into the field of directions generated by the field v. In a neighborhood of each singular point the field α is generated by the analytic vector field $\tilde{v}$.*

The last assertion allows the σ-process to be continued by induction.

A singular point of the field of directions is elementary if the germ of the field at this point is generated by the germ of the vector field at an elementary singular point.

The Bendixson–Dumortier theorem ([**?**, **?**, **?**]). *By means of finitely many σ-processes a real-analytic vector field given in a neighborhood of a real-isolated singular point on the plane $\mathbb{R}^2$ can be carried into an analytic field of directions given in a neighborhood of a union of glued-in projective lines and having only finitely many singular points, each of them elementary and different from a focus or a center.*

The composition of σ-processes described in the Bendixson–Dumortier theorem is called a nice blowing-up. A nice blowing-up enables us to turn an arbitrary polycycle of an analytic vector field on the plane into an elementary polycycle with the same monodromy transformation. This gives a reduction of the strengthened nonaccumulation theorem to Theorem IV and concludes the proof of the chain of implications represented in Figure 3.

§ 0.2. The theorem and error of Dulac

In this section we present a proof of the main true result in Dulac's memoir [**?**] and point out the error in his proof of the finiteness theorem. The scheme of his argument lies at the basis of the proof of the identity theorem given below (see subsection D).

A. Semiregular mappings and the theorem of Dulac.

Definition 1. A Dulac series is a formal series of the form

$$\sigma = cx^{\nu_0} + \sum_1^\infty P_j(\ln x)x^{\nu_j},$$

where $c > 0$, $0 < \nu_0 < \cdots < \nu_j < \cdots$, $\nu_j \to \infty$, and the P_j are polynomials.

Definition 2. The germ of a mapping $f\colon (\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$ is said to be semiregular if it can be expanded in an asymptotic Dulac series. In other words, for any N there exists a partial sum Σ of the above series such that $f(x) - \Sigma(x) = o(x^N)$.

Remark. The concept of a semiregular mapping is invariant: semiregularity of a germ is preserved under a smooth change of coordinates in a full neighborhood of zero on the line. This follows from

Lemma 1. *The germs of the semiregular mappings form a group.*

The lemma follows immediately from the definition. The main true result in [**?**] is

Dulac's Theorem. *A semitransversal to a monodromic polycycle of an analytic vector field can be chosen in such a way that the corresponding monodromy transformation is a flat, or vertical, or semiregular germ.*

The proof of this theorem is presented in subsections E to H.

B. The lemma of Dulac and a counterexample to it. Dulac derived the finiteness Theorem I from the preceding theorem and a lemma.

Lemma. *The germ of a semiregular mapping* $f\colon (\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$ *is either the identity or has the isolated fixed point zero.*

This lemma is proved in §23 of [**?**] with the help of the following argument. The fixed point of the germ of f is found from the equation $f(x) = x$. If the principle term in the Dulac series for f is not the identity, then this equation has the isolated solution 0. If the principal term in the series is the identity but f itself is not the identity, then the equation $f(x) = x$ is equivalent to the equation

e:0.2-1 (2.1) $$x^{\nu_1}P_1(\ln x) + o(x^{\nu_1}) = 0,$$

where P_1 is a nonzero polynomial. This equation has the isolated root 0. Namely, dividing the equation by x^{ν_1}, we get an equation not having a solution in a sufficiently small neighborhood of zero. Indeed, the first term on the right-hand side of the new equation has a nonzero (perhaps infinite) limit as $x \to 0$, while the second term tends to zero. This concludes the proof of the lemma.

The lemma is false: a counterexample is supplied by the semiregular mapping $f\colon x \to x + e^{-1/x}\sin\frac{1}{x}$, which has countably many fixed points accumulated at zero. The asymptotic Dulac series for f consists in the single term x. The error in the proof above amounts to the fact that the Dulac series for a semiregular mapping can be "trivial"—it may not contain terms other than x. Then the left-hand side of equation (2.1) is equal to $o(x^{\nu_1})$, and we cannot investigate its zeros.

Actually, we proved here

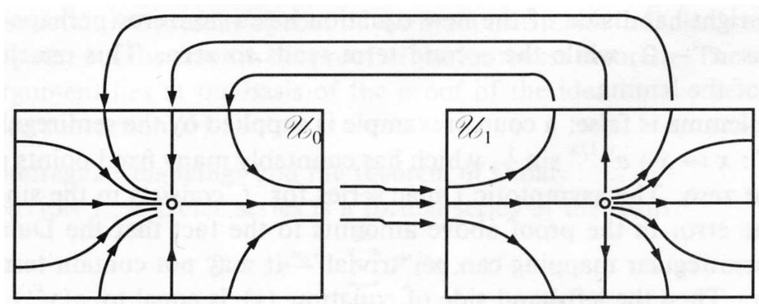


FIGURE 6 f6

THE CORRECTED LEMMA OF DULAC. *The germ of a semiregular mapping* $f\colon (\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$ *has either a trivial (equal to x) Dulac series or an isolated fixed point at zero.*

The difficulty of the problem of finiteness is due to the fact that triviality of the Dulac series for a monodromy transformation does not imply that this transformation is the identity, as shown by the example in the next subsection.

C. Monodromy transformations with nonzero flat correction. It may seem likely that nonzero flat functions cannot arise in the theory of analytic differential equations. The example below destroys this illusion. The construction is carried out with the help of gluing, which is a powerful tool in the nonlocal theory of bifurcations and differential equations. As a result we obtain the following

PROPOSITION 1. *There exists an analytic vector field on a two-dimensional analytic surface having a polycycle with two vertices—a separatrix lune—whose monodromy transformation has a nonzero flat correction* [**?**].

PROOF. The analytic surface mentioned in the proposition is obtained by gluing together two planar domains with vector fields on them; we proceed to describe the latter.

In the rectangle $\mathcal{U}\colon |x| \le 1,\ |y| \le e^{-1}$ on the plane (e is the base of natural logarithms) consider a vector field giving a standard saddle node:

$$v(x, y) = x^2\, \partial/\partial x - y\, \partial/\partial y$$

(Figure 6). In the same rectangle consider the field w obtained from v by symmetry with respect to the vertical axis and by time reversal:

$$w(x, y) = x^2\, \partial/\partial x + y\, \partial/\partial y.$$

Take two copies of the rectangle $\mathcal{U}$: $\mathcal{U}_0 = \mathcal{U}\times\{0\}$ and $\mathcal{U}_1 = \mathcal{U}\times\{1\}$. We glue together points on two pairs of boundary segments of each of the rectangles (Figure 6; the arrows outside the rectangles indicate the gluing maps). Namely, we identify the points of the segments

$$\Gamma_0^+ = [-\tfrac{1}{2}, \tfrac{1}{2}] \times \{e^{-1}\} \times \{0\},$$
$$\Gamma_1^- = [-\tfrac{1}{2}, \tfrac{1}{2}] \times \{e^{-1}\} \times \{1\}$$

with the abscissae x and $-x$, and the points of the segments

$$\Gamma_0^- = \{1\} \times [-\tfrac{1}{4}, \tfrac{1}{4}] \times \{0\},$$
$$\Gamma_1^+ = \{-1\} \times [-\tfrac{5}{16}, \tfrac{3}{16}] \times \{1\}$$

with the ordinates y and $f(y) = y - y^2$, respectively. In the notation for the transversals $\Gamma_l^+, \Gamma_l^- \subset \mathcal{U}_l$ the plus sign indicates entry of the trajectories into the domain $\mathcal{U}_l$ across the transversal, while the minus sign indicates exit.

As a result of the gluings we get from the rectangles $\mathcal{U}_0$ and $\mathcal{U}_1$ a two-dimensional manifold homeomorphic to an annulus. By means of a well-known construction [**?**] we can introduce on it an analytic structure coinciding with the original structure on $\mathcal{U}_0$ and $\mathcal{U}_1$ and such that the vector field coinciding with v on $\mathcal{U}_0$ and with w on $\mathcal{U}_1$ is analytic on the whole surface. This vector field will have the separatrix lune γ obtained from segments of the coordinate axes in the gluings.

It is easy to compute the monodromy transformation for this polycycle γ. First we compute the correspondence mapping $\Delta\colon \Gamma_0^+ \to \Gamma_0^-$ along the trajectories of the field v where it is defined (Figure 6). The function $ye^{-1/x}$ is a first integral of v. Consequently, for $x > 0$,

$$e^{-1} \cdot e^{-1/x} = \Delta(x) \cdot e^{-1}, \quad \Delta(x) = e^{-1/x}.$$

Similarly, the germ of the correspondence mapping at the point $y = 0$ of the semitransversal $\Gamma_1^+ \cap \{y > 0\}$ onto Γ_1^- is equal to

$$-\Delta^{-1}(y) = 1/\ln y.$$

By construction of the gluing mappings, the germ Δ_γ has the form

$$\Delta_\gamma = \Delta^{-1} \circ f \circ \Delta, \quad f(y) = y - y^2$$

and acts according to the formula

$$x \xmapsto{\Delta} e^{-1/x} \xmapsto{f} e^{-1/x}(1 - e^{-1/x}) \xmapsto{\ln} \left(-\frac{1}{x} + \ln(1 - e^{1/x})\right)$$
$$\xmapsto{-1/y} x[1 - x\ln(1 - e^{-1/x})]^{-1}.$$

This mapping has a nonzero flat correction at zero, which is what was required. $\square$

Thus, Dulac series do not suffice for describing the asymptotic behavior of monodromy transformations.

D. The scheme for proving the identity theorem. For the proof we construct a set of germs mapping $(\mathbb{R}^+, \infty) \to (\mathbb{R}^+, \infty)$ that contains the monodromy transformations of polycycles of analytic vector fields written in a suitable chart, but is broader. The germs in this new set have two properties: they can be expanded and they can be extended.

That they can be expanded means that to each germ corresponds an asymptotic series containing information not only about power asymptotics but also about exponential asymptotics. Such series are called STAR—superexact asymptotic series; see §0.5 for more details. The triviality of such a series, that is, the condition that it equals the identity, means that the correction of this series is very rapidly decreasing as $x \to \infty$. In turn, the terms of the series do not oscillate, and hence the existence of a countable number of fixed points for the germ implies that the corresponding series is trivial.

That they can be extended means that a germ can be extended into the complex domain to be a map-cochain—a piecewise continuous mapping holomorphic off the lines of discontinuity. The Phragmén–Lindelöf theorem holds for the map-cochains arising upon extension of a monodromy transformation: if the correction of a germ decreases too rapidly, then it is identically equal to zero. The triviality of the STAR ensures precisely a "too rapid" decrease of the correction.

The following implication is obtained (Δ_γ is the monodromy transformation of the polycycle γ, $\hat{\Delta}_\gamma$ is the STAR for Δ_γ, and Fix_∞ is the set of germs with countably many fixed points):

$$\Delta_\gamma \in \mathrm{Fix}_\infty \implies \hat{\Delta}_\gamma = x \implies \Delta_\gamma - x \equiv O.$$

If all the singular points on an elementary polycycle are hyperbolic saddles, then it is said to be hyperbolic, otherwise it is said to be nonhyperbolic. In the hyperbolic case the program presented was carried out in [**?**] with the use of ordinary and not superexact asymptotic series; see §0.3 below. In the general case the geometric theory of normal forms of resonant fields and mappings is used to describe the monodromy transformation (§0.4).

We return to the presentation of the proof of Dulac's theorem.

E. The classification theorem. Dulac's theorem describes the power asymptotics of a monodromy transformation. To get these asymptotic expressions it suffices to use the theory of smooth, and not analytic, normal forms.

Definition 3. Two vector fields are smoothly (analytically) orbitally equivalent in a neighborhood of the singular point 0 if there exists a diffeomorphism (an analytic diffeomorphism) carrying one neighborhood of zero into another that leaves 0 fixed and carries phase curves of one field into phase curves of the other (perhaps reversing the direction of motion along the phase curves).

Remark. In the definition of orbital equivalence it is usually required that the directions of motion along phase curves be preserved; to simplify the table below we do not require this.

Theorem. *An analytic vector field in some neighborhood of an isolated elementary singular point on the real plane is smoothly orbitally equivalent to one of the vector fields in the table.*

Here the numbers k, m, and n are positive integers, m and n are coprime, a is a real number, $\underline{x} \in \mathbb{R}^2$, $\underline{x} = (x, y)$, $r^2 = x^2 + y^2$, I is the operator of rotation through the angle $\pi/2$, the fraction m/n is irreducible, and $\varepsilon \in \{0, 1, -1\}$.

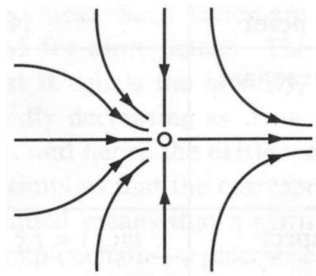


FIGURE 7 f7

Type of singular point	Normal form
1. A saddle with nonresonant linear part $v(\underline{x}) = \Lambda x + \cdots$	$w(\underline{x}) = \Lambda \underline{x}$
2. A center with respect to the linear terms	$w(\underline{x}) = I\underline{x} + \varepsilon(r^{2k} + ar^{4k})x$
3. A resonant node	$w(x,y) = (kx + \varepsilon y^k)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$
4. A resonant saddle with the eigenvalue ratio $-\lambda = \lambda_2/\lambda_1 = -m/n$	$w(x,y) = x[1 + \varepsilon(u^k + au^{2k})]\frac{\partial}{\partial x} - \lambda y\frac{\partial}{\partial y}$ $u = x^m y^n$ the resonant monomial
5. A degenerate elementary singular point	$w(x,y) = x^{k+1}(\pm 1 + ax^k)^{-1}\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$

A closely related assertion was formulated as a conjecture by Bryuno [?]. A large part of the list given above is contained in the article [?] of Bogdanov. The classification theorem in its present form was formulated in [?]; fragments of the proof are contained in [?], [?], and [?]; a complete proof is given in [?].

COROLLARY (topological classification of elementary singular points). *An elementary singular point of an analytic vector field is one of five topological types: a saddle, a node, a focus, a center, and a saddle node.*

PROOF. Points of types 1, 3, or 4 in the table are among the saddles, nodes, or foci and are identified according to the linear part. For $\varepsilon = 1$ the points of type 2 are unstable foci; for $\varepsilon = -1$ they are stable; for $\varepsilon = 0$ they are centers. Degenerate elementary singular points reduce to normal forms with separating variables; depending on the sign $+$ or $-$ in front of the 1 in parentheses and the parity of k, these can be saddles, nodes, or saddle nodes (Figure 7). □

REMARK. This corollary was known as far back as Bendixson [?], Another proof of it is a derivation from the reduction principle by Shoshitaishvili [?], [?].

As noted by Bogdanov, the normal forms given by the classification theorem can be integrated in elementary functions. The proof of Dulac's theorem is based on this remark.

F. The scheme for proving Dulac's theorem, and the correspondence mappings. For a one-point elementary monodromic polycycle Dulac's theorem is trivial: the corresponding singular point is a focus or a center. Its monodromy

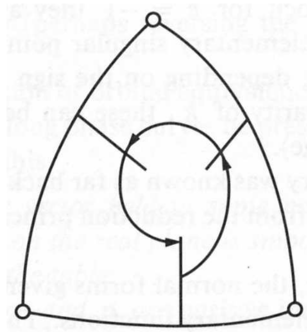


FIGURE 8 f8

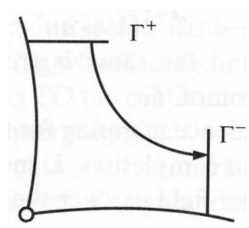


FIGURE 9 f9

transformation extends analytically to a full neighborhood of zero on the line, and hence can be expanded in a convergent (and not just asymptotic) Taylor series (a special case of a Dulac series). Everywhere below we consider an elementary polycycle with more than one point.

Note that if an elementary polycycle with more than one point is monodromic, then all the singular points on it have the topological type of a saddle or saddle node.

This follows immediately from the preceding corollary.

The monodromy transformation of an elementary polycycle can be decomposed in a composition of correspondence mappings for hyperbolic sectors of elementary singular points (Figures 8 and 9). Note that the vector fields in a neighborhood of a singular point with a hyperbolic sector may be normalized not only locally (in a small neighborhood of a point) but also "semiglobality" (in a neighborhood of the union of stable and unstable separatrixes). The conjugacy is extended by dynamics. A hyperbolic sector is represented in Figure 9; the correspondence mapping carries a semitransversal across which phase curves enter the sector into a semitransversal across which they leave the sector; the image and inverse image belong to a single phase curve. The inverse mapping is also called a correspondence mapping.

The first part of the proof of Dulac's theorem consists in a computation of the correspondence mappings for hyperbolic sectors of saddles and saddle nodes (rows 1, 4, and 5 of the table in subsection E). The second part is an investigation of compositions of these mappings with smooth changes of coordinates and with each other.

LEMMA 2. *A correspondence mapping for a hyperbolic sector of a nondegenerate saddle of a smooth vector field is semiregular.*

REMARK. It is natural to prove the lemma for smooth vector fields, and to use it for analytic vector fields.

PROOF. This lemma was proved for analytic vector fields in the first part of the memoir [**?**]. The proof below uses the classification theorem in subsection E.

By Lemma 1 in subsection A, the germ of a mapping smoothly equivalent to a semiregular mapping is also semiregular; therefore, it suffices to prove Lemma 2 for fields written in normal form (rows 1 and 4 of the table in E). In more detail, the conjugacy between the original vector field and its normal form induces coordinate changes on the cross-sections $h^+ : (\Gamma^+, 0) \to (\Gamma^+, 0)$ and $h^- : (\Gamma^-, 0) \to (\Gamma^-, 0)$. Let Δ and Δ_{norm} be Dulac maps in the original and normalizing charts respectively. Then

$$\Delta = h^- \circ \Delta_{\text{norm}} \circ h^+.$$

The maps $h^\pm$ are C^∞ in a full neighborhood of zero, Δ_{norm} is semiregular as proved below. Hence, Δ is semiregular as well.

Let us now pass to the proof of semiregularity of Δ_{norm}.

Suppose that the saddle under consideration is nonresonant: the eigenvalue ratio is irrational. Then the corresponding normal form has the form

$$w(x, y) = x\frac{\partial}{\partial x} - \lambda y \frac{\partial}{\partial y}, \quad \lambda > 0$$

(the normal form in the first row of the table, multiplied by λ_1^{-1}, where $\Delta = \operatorname{diag}(\lambda_1, \lambda_2)$). In this and the next subsections Γ^- is the semitransversal $\{x = 1, y \in [0, 1]\}$ with chart y, and Γ^+ is the semitransversal $\{y = 1, x \in [0, 1]\}$ with chart x. The correspondence mapping $\Gamma^+ \to \Gamma^-$ can be computed in an elementary way and has the form

$$x \mapsto \gamma = \Delta(x) = x^\lambda;$$

this mapping is semiregular. Lemma 2 is proved in the nonresonant case.

Suppose now that the saddle is resonant, $\lambda = m/n$. The corresponding normal form has a form equivalent to formula 4 of the table in subsection E:

$$w(x, y) = x\frac{\partial}{\partial x} - y(\lambda + \tilde{\varepsilon}(u^k + \tilde{a}u^{2k}))\frac{\partial}{\partial y},$$

where $u = x^m y^n$ is the resonant monomial, $\tilde{\varepsilon} \in \{0, 1, -1\}$, and $\tilde{a} \in \mathbb{R}$. The case $\varepsilon = 0$ is treated as above. Let $\varepsilon \neq 0$. The proof of the lemma is based on the fact that the equation $\underline{\dot{x}} = w(\underline{x})$ can be integrated. The integration is carried out as follows. The factor system is written with respect to the resonant monomial:

$$\dot{u} = f(u).$$

Here $\dot{u} = L_w u$ and $f = -\tilde{\varepsilon} n u^{k+1}(1 + \tilde{a}u^k)$. The factor system can be integrated in quadratures, and the same is true for the equation $\dot{x} = x$. Then y is found as a function of t.

To compute the correspondence mapping for the field w it is not necessary to carry these computations to completion. Denote by g_v^t the transformation of the phase flow of the vector field v over the time t (where it is defined). Let

$$\xi = (x, 1) \in \Gamma^+, \quad \eta = (1, \Delta(x)) \in \Gamma^-.$$

The phase curve of the field w passes from the point ξ to the point η in the time $t = -\ln x$. Therefore

$$u(\eta) = g_f^{-\ln x} u(\xi)$$

where f is the right-hand side of the factor system with respect to u. But

$$u(\xi) = x^m, \quad u(\eta) = (\Delta(x))^n.$$

Finally,

$$\Delta(y) = [g_f^{-\ln x}(x^m)]^{1/n}.$$

The last formula gives a semiregular mapping. We prove this first for $m = n = 1$. The local phase flow of the field $f(\partial/\partial u)$ at the point $(0,0)$ in (t,u)-space is given by the germ of an analytic function of two variables; denote it by F. We extend the germ F into the complex domain and prove that for sufficiently small δ the Taylor series for F converges on the curve $L\colon t = -\ln u$, $(t,u) \in \mathbb{R}$, $u \in (0,\delta)$. For this we consider the equation $\dot{u} = f(u)$ with the complex phase space $\{u\}$ and with complex time. The solution φ of this equation with the initial condition $\varphi(0) = u$ is holomorphic in a disk with radius of order $|u|^{-k}$. For $a = 0$ ($f = -\varepsilon n u^{k+1}$) this follows from the explicit formula $\varphi(t) = u(1+\varepsilon nktu^k)^{-1/k}$ for the solution; for $a \neq 0$ it can be proved by simple estimates. Hence, the Taylor series for F converges in the domain $|t| \le A|u|^{-k}$, where A is some positive constant. This domain contains the curve L for sufficiently small δ. Consequently, the mapping $u \mapsto F(-\ln u, u)$ is semiregular. This shows that the mapping $y \mapsto \Delta(y)$ is semiregular for $m = n = 1$.

The semiregularity of the mapping $y \mapsto \Delta(y)$ for arbitrary m and n follows from Lemma 1.

Lemma 2 is proved. □

G. The correspondence mapping for a hyperbolic sector of a saddle node or degenerate saddle. A hyperbolic sector of a degenerate elementary singular point includes in its boundary part of the stable (or unstable) and center manifolds. We recall that in this case the stable manifold is a smooth invariant curve of the field that passes through the singular point and is tangent there to an eigenvector of the linear part with nonzero eigenvalue. A center manifold is an analogous curve tangent at the singular point to the kernel of the linear part.

Definition 4. For a hyperbolic sector of a degenerate elementary singular point a correspondence mapping whose image is a semitransversal to a center manifold is called the mapping TO the center manifold for brevity; its inverse is the mapping FROM the center manifold.

Example. For a suitable choice of semitransversal the correspondence mapping of the hyperbolic sector of the standard saddle node $x^2(\partial/\partial x) - y(\partial/\partial y)$ has the form

$f_0(x) = e^{-1/x}$, TO the center manifold,

$f_0^{-1}(x) = -1/\ln x$, FROM the center manifold;

see C.

Definition 5. The germ of the mapping $f\colon (\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$ is said to be a flat semiregular germ if the composition $h = f_0^{-1} \circ f$ is semiregular.

Remark. Below in subsection H it is proved that a germ smoothly equivalent to a flat semiregular germ is itself a flat semiregular germ (corollary to Lemma 4 in H).

Lemma 3. *The germ of a mapping TO a center manifold for an analytic vector field is a flat semiregular germ.*

Proof. By the preceding remark, it suffices to prove the lemma for the corresponding smooth orbital normal form w; see row 5 in the table in E. In this case the correspondence mapping is said to be standard and denoted by Δ_{st}. It can be assumed without loss of generality that the quadrant $x \geq 0$, $y \geq 0$ contains a hyperbolic sector of the field w. Therefore, the sign $+$ should be chosen in the indicated formula for w:

$$w(x, y) = x^{k+1}(1 + ax^k)^{-1}\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

The corresponding differential equation has separating variables. It can be integrated, and the correspondence mapping can be computed explicitly. Namely, let the semitransversals Γ^+ and Γ^- be the same as in subsection F. Let $\Delta_{\mathrm{st}} : (\Gamma^+, 0) \to (\Gamma^-, 0)$ be the germ of the mapping TO the center manifold for the field w. Then the phase curve of the field falls from the point $(x, 1)$ to the point $(1, \Delta_{\mathrm{st}}(x))$ in the time

$$t = -\ln \Delta_{\mathrm{st}}(x) = \int_x^1 \frac{1 + a\xi^k}{\xi^{k+1}}\, d\xi.$$

An elementary computation yields:

$$\int_x^1 \frac{1 + a\xi^k}{\xi^{k+1}}\, d\xi = \frac{1}{h_{k,a}} - \frac{1}{k},$$

where $h_{k,a}(x) = kx^k/(1 - akx^k \ln x)$ is a semiregular mapping. Consequently,

$$\Delta_{\mathrm{st}} = C \exp(-1/h_{k,a}), \quad C = \exp 1/k,$$

and Δ_{st} is a flat semiregular mapping. □

H. Conclusion of the proof of Dulac's theorem. By results in subsections E to G, it remains to prove that a composition of semiregular germs, flat semiregular germs, and inverses of them (perhaps after a cyclic permutation of the germs that corresponds to a proper choice of semitransversal) is a flat, vertical, or semiregular germ. Recall that the germs of semiregular mappings form a group; see Lemma 1 in A.

Lemma 4. *Suppose that f_1 and f_2 are two flat semiregular germs, and h is a semiregular germ $(\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$. Then the composition $f = f_2^{-1} \circ h \circ f_1$ is semiregular, and the asymptotic series for f depends on the principal term in the asymptotic expansion for h and does not depend on the remaining terms of this expansion.*

It can be said that the composition f "forgets" all the terms of the asymptotic series for h except for the principal term.

Proof of the lemma. Suppose that $f_1 = f_0 \circ h_1$ and $f_2 = f_0 \circ h_2$, where f_0 is the standard flat mapping $x \to \exp(-1/x)$, and the germs h_1 and h_2 are semiregular. Then

$$f = h_2^{-1} \circ f_0^{-1} \circ h \circ f_0 \circ h_1.$$

By virtue of Lemma 1, it suffices to prove that the composition $f_0^{-1} \circ h \circ f_0$ is semiregular. Let

$$h(x) = cx^{\nu_0}(1 + \tilde{h}(x)),$$
$$\tilde{h}(x) = O(x^{\varepsilon}), \quad \varepsilon > 0,\ c > 0.$$

Then

$$\begin{aligned} f_0^{-1} \circ h \circ f_0(x) &= \left[\frac{\nu_0}{x} - \ln c - \ln(1 + \tilde{h} \circ f_0)\right]^{-1} \\ &= \frac{x}{\nu_0 - x \ln c - x \ln(1 + \tilde{h} \circ f_0)} \\ &= \frac{x}{\nu_0 - x \ln c} + \cdots , \end{aligned}$$

where the dots stand for an exponentially decreasing (as $x \to 0$) component.

Consequently, the mapping $f_0^{-1} \circ h \circ f_0$ is semiregular. Lemma 4 is proved. □

Corollary. *A germ smoothly equivalent to a flat semiregular germ is itself a flat semiregular germ.*

Proof. Let $f_0 \circ h$ be a flat semiregular germ, and f a germ smoothly equivalent to it. Here we have in mind so-called RL-equivalence: the substitutions in the image and the inverse image can be different. In other words, there exist germs of diffeomorphisms h_1, and h_2 such that

$$f = h_1 \circ f_0 \circ h \circ h_2$$

It must be proved that the composition $f_0^{-1} \circ f$ is semiregular. This follows immediately from Lemma 4 above. □

We proceed to the proof of Dulac's theorem. Let γ be an elementary polycycle. The monodromy transformation can be decomposed into a composition of the correspondence mappings Δ_γ described in subsections F and G:

e:0.2-2

$$\Delta_\gamma = \Delta_N \circ \cdots \circ \Delta_1. \tag{2.2}$$

Definition 6. An elementary polycycle is said to be balanced if in (2.2) the number of the mappings FROM the center manifold is equal to the number of mappings TO the center manifold. Otherwise the cycle is said to be unbalanced.

Useful for describing the composition (2.2) is the function χ called the characteristic of this composition and defined on $[-N, 0]$ as follows. The function χ is continuous and linear on the closed interval between two adjacent integers, and $\chi(0) = 0$. If the mapping Δ_j in (2.2) corresponds to a nondegenerate singular point, then let $\chi(-j) = \chi(-j+1)$. If Δ_j is the mapping FROM the center manifold, then $\chi(-j) = \chi(-j+1) + 1$. If Δ_j is the mapping TO the center manifold, then $\chi(-j) = \chi(-j+1) - 1$. Obviously, a polycycle γ is balanced if and only if $\chi(0) = \chi(-N) = 0$. The characteristic of a balanced polycycle is determined up to an additive constant and a "cyclic shift of the argument": $j \to j + k \pmod N$, both of which depend on the choice of the semitransversal.

Definition 7. A semitransversal to a balanced polycycle is said to be properly chosen if the cycle characteristic defined with the help of the decomposition (2.2) for the corresponding monodromy transformation is nonpositive.

Remark. For a balanced polycycle a proper choice of a semitransversal is always possible. It corresponds to a proper cyclic permutation of factors in (2.2).

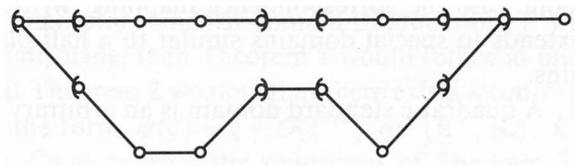


FIGURE 10 f10

LEMMA 5. *For a suitable choice of semitransversal the monodromy transformation for a balanced polycycle is semiregular, while for an unbalanced polycycle it is flat or vertical.*

PROOF. In the composition (2.2) read from right to the left, we put a left parenthesis after each mapping FROM the center manifold and a right parenthesis before each mapping TO the center manifold (Figure 10). If the cycle is balanced and its characteristic is nonpositive, then the parentheses turn out to be placed correctly, in particular, the number of left parentheses is equal to the number of right parentheses; the first parenthesis is a left one, and the last is a right one. In this case all the flat and vertical mappings fall in parentheses. Inside all the parentheses the products are semiregular in view of Lemma 4 (this is demonstrated intuitively in Figure 10; the obvious general argument is omitted). Lemma 5 now follows from Lemma 1 for balanced cycles. The proof is analogous for unbalanced cycles. □

Dulac's theorem follows immediately from Lemma 5.

The classification theorem, and with it Dulac's theorem, admits a generalization to the smooth case: it is necessary only to require that the vector field satisfy at all singular points a Lojasiewicz condition—upon approach of the singular point the modulus of the vector of the field decreases no more rapidly than some power of the distance to the singular point. The finiteness theorem is false, of course, for such fields.

To prove the finiteness theorem it turns out to be necessary to go out into the complex domain. In the next section the nonaccumulation theorem (the Theorem IV in §0.1A) is proved for a polycycle with hyperbolic singular points. Here essential use is made of the corrected lemma of Dulac (subsection B)—the strongest of the results in [**?**].

§ 0.3. Finiteness theorems for polycycles with hyperbolic vertices

In this section we introduce the class of almost regular germs, which contains the correspondence mappings of hyperbolic saddles in the analytic case, and we prove Theorems I, II, and IV for vector fields with nondegenerate singular points. Beginning with this section, all the vector fields under consideration are analytic; explicit mention of analyticity is often omitted. The presentation follows [**?**] and [**?**].

A. Almost regular mappings.

THEOREM IV bis. *The limit cycles of an analytic vector field with nondegenerate singular points cannot accumulate on a polycycle of this field.*

Theorems I and II for fields with nondegenerate singular points (including singular points at infinity in Theorem I) can be derived from this as in §0.1.

A chart x that is nonnegative on a semitransversal, equal to zero at the vertex, and can be extended analytically as $x = \operatorname{Re} z$ to a full neighborhood of the vertex on the transversal containing the semitransversal, is said to be a natural chart. The chart $\xi = -\ln x$ is called a logarithmic chart. It may be extended as $\xi = \operatorname{Re} \zeta$, $\zeta = -\ln z$.

It is convenient to write the correspondence mapping for a hyperbolic sector of an elementary (not necessarily hyperbolic) singular point in a logarithmic chart. In a natural chart this is the germ of the mapping $\Delta\colon (\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$, and in a logarithmic chart it is the germ of $\tilde{\Delta}\colon (\mathbb{R}^+, \infty) \to (\mathbb{R}^+, \infty)$; passage to the logarithmic chart is denoted by a tilde.

In the hyperbolic case the correspondence mapping, written in the logarithmic chart, extends to special domains similar to a half-plane and called quadratic domains.

def:qstand

Definition 1. A quadratic standard domain is an arbitrary domain of the form

$$\Omega_C = \varphi_C(\mathbb{C}^+ \setminus K), \quad \varphi_C = \zeta + C(\zeta + 1)^{1/2},\ C > 0,\ K = \{|\zeta| \le R\}.$$

Definition 2. A Dulac exponential series is a formal series of the form

$$\Sigma = \nu_0\zeta + c + \sum P_j(\zeta) \exp \nu_j \zeta,$$

where $\nu_0 > 0$, $0 > \nu_j \searrow -\infty$, and the P_j are polynomials; the arrow $\searrow$ means monotonically decreasing convergence.

Definition 3. An almost regular mapping is a holomorphic mapping of some quadratic standard domain Ω in $\mathbb{C}$ that is real on $\mathbb{R}^+$ and can be expanded in this domain as an asymptotic real Dulac exponential series. Expandability means that for any $\nu > 0$ there exists a partial sum approximating the mapping to within $o(\exp(-\nu\xi))$ in Ω.

Theorem 1. *An almost regular mapping is uniquely determined by an asymptotic series of it. In particular, an almost regular mapping with asymptotic series ζ is the identity.*

Remark. This theorem explains the term "almost regular." At the beginning of the century "regularity" was often used as a synonym for "analyticity." Apparently, Dulac called the mappings he introduced "semiregular" because of their similarity to regular mappings: Dulac series are similar to Taylor series. However, semiregular mappings, in contrast to almost regular mappings, are not determined by their asymptotic series. On the other hand, Dulac exponential series for almost regular mappings diverge in general. "Almost regular mappings" are thus more regular than "semiregular mappings," but are still not exactly regular.

Proof. The dilference between two semiregular mappings with a common Dulac series is a holomorphic function f defined and bounded in some standard domain and decreasing on $(\mathbb{R}^+, \infty)$ more rapidly than any exponential ($\mathbb{R}^+$ is the positive semi-axis). By a theorem of Phragmén–Lindelöf type, this function is identically equal to zero. We prove this: the theorem is obtained immediately from it.

Let $\mathbb{C}^+$ be the right half-plane $\operatorname{Re} \zeta > 0$.

The following result is known to specialists and will be proved in §3.1C.

THEOREM 2. *If a function g is holomorphic and bounded in the right half-plane and decreases on* $(\mathbb{R}^+, \infty)$ *more rapidly than any exponential* $\exp(-\nu\xi)$, $\nu > 0$, *then* $g \equiv 0$.

If instead of a quadratic standard domain the function f were holomorphic in the right half-plane, then Theorem 1 would follow at once from Theorem 2. To exploit Theorem 2 we note that there exists a conformal mapping $\psi\colon \mathbb{C}^+ \to \Omega$ with the form $\psi(\xi) = \xi + O(\xi^{1/2})$ on $(\mathbb{R}^+, \infty)$. Consequently, the function $g = f \circ \psi$ satisfies the conditions of Theorem 2. From this, $g \equiv f \equiv 0$. $\square$

B. Going out into the complex plane, and the proof of Theorem IV bis.

THEOREM 3. *The correspondence mapping of a hyperbolic saddle, written in a logarithmic chart, extends to an almost regular mapping in some quadratic domain.*

This theorem will be proved in subsections C and D. We derive Theorem IV bis from it.

DEFINITION 4. Two almost regular mappings are equivalent if they coincide in some quadratic standard domain. An equivalence class of such mappings is called an almost regular germ.

It follows from the definition of almost regular germs that these germs form a group with the operation of "composition." Therefore, it follows from Theorem 3 that the monodromy transformation Δ_γ of a polycycle with hyperbolic vertices, written in the logarithmic chart, extends to an almost regular germ. We now prove the chain of implications in §2D. Let $\Delta_\gamma \in \mathrm{Fix}_\infty$. Then by the corrected lemma of Dulac in §2B, the corresponding Dulac series is equal to id: $\hat{\Delta}_\gamma = \mathrm{id}$. This implies that the mapping $\tilde{\Delta}_\gamma$—the monodromy transformation Δ_γ written in the logarithmic chart—expands in an asymptotic Dulac exponential series equal to id. It follows from Theorem 1 and the almost regularity of $\tilde{\Delta}_\gamma$ that $\tilde{\Delta}_\gamma = \mathrm{id}$, which proves Theorem IV bis.

C. Hyperbolicity and almost regularity. Here we prove Theorem 3 in subsection B. By the definition of an almost regular germ, it must be proved that the mapping under investigation, written in the logarithmic chart (it is denoted by $\tilde{\Delta}$):
(a) extends biholomorphically to some quadratic standard domain Ω (regularity);
(b) can be expanded in an asymptotic Dulac series in this domain (expandability).

The proof is broken up into four steps.

STEP 1. GEOMETRY AND ANALYTIC EXTENSION. Let us begin with an example. Consider the correspondence mapping of the linear saddle given by the field $v = z(\partial/\partial z) - \lambda w(\partial/\partial w)$ in $\mathbb{C}^2$. Let Γ^+ and Γ^- be the intervals $[0,1] \times \{1\}$ and $\{1\} \times [0,1]$. The correspondence mapping $\Delta\colon \Gamma^+ \to \Gamma^-$ has the form $z \mapsto z^\lambda$ in suitable natural coordinates. This mapping extends to the Riemann surface of the logarithm over the punctured disk (delete the center $z = 0$) on the line $w = 1$. On the real plane the image and inverse image of the mapping Δ are joined by phase curves of the field v. Which lines on the complex phase curves of v join the image and inverse image of the extended correspondence mapping?

The construction answering this question is easily analyzed in the linear case and is used in the general case. In the linear case it allows us to extend the correspondence mapping to the whole right half-plane in the logarithmic chart. The construction is proceeded as follows.

Let (Figure 11)

$$B = \{|z| \le 1\} \times \{|w| \le 1\},$$
$$\mathcal{D}_0 = \{0 < |z| \le 1\} \times \{0\}, \quad \mathcal{D}_1 = \{0 < |z| \le 1\} \times \{1\}.$$

For each $\zeta \in \mathbb{C}^+$, $\zeta = \xi + i\eta$, denote by μ^ζ (ζ is an index, not a power) the curve with initial point 0 and endpoint ζ consisting of the two intervals $[0, \xi]$ and $[\xi, \zeta]$, parametrized by the arclength s, and let $S = s(\zeta) = |\xi| + |\eta|$. The point corresponding to the parameter s is denoted by $\mu^\zeta(s)$; this defines a mapping $\mu^\zeta \colon [0, S] \to \mathbb{C}$, $s \mapsto \mu^\zeta(s)$.

We define the following curves (Figure 11):

$$\gamma = \gamma^{\zeta,1} \colon [0, S] \to \mathcal{D}_1, \quad s \mapsto (\exp(-\mu^\zeta(s)), 1),$$
$$\gamma_0 = \gamma^\zeta \colon [0, S] \to \mathcal{D}_0, \quad s \mapsto (\exp(\mu^\zeta(s) - \zeta), 0).$$

Let $\gamma_T^\zeta = \gamma^\zeta|_{[0,T]}$. Denote by φ_p the complex phase curve of v passing through the point p. On the Riemann surface φ_p let $\hat{\gamma}^\zeta$ be an arc with initial point p that covers the curve γ^ζ under the projection $\pi_z \colon (z, w) \mapsto z$.

We prove that the endpoint of the arc $\hat{\gamma}^\zeta$ is the result of analytic extension of the correspondence mapping Δ (see the beginning of Step 1) along the curve $\gamma^{\zeta,1}$ with initial value $\Delta(1) = 1$.

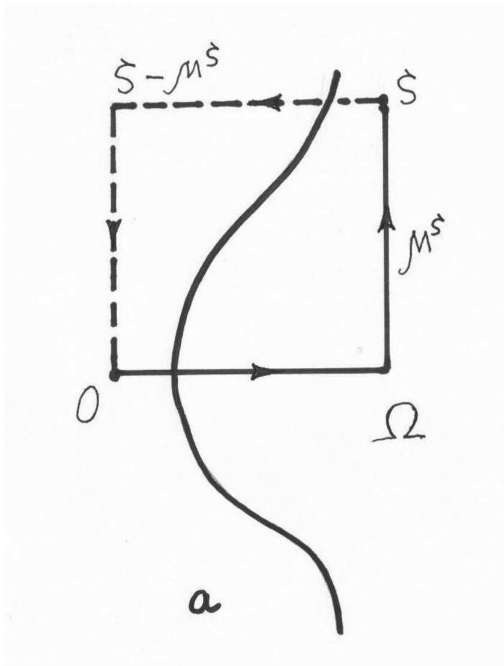
We first prove that the covering $\hat{\gamma}^\zeta$ is defined and belongs to B. Indeed, the curve γ^ζ consists of two parts: a segment of a radius of the disk $\mathcal{D}_0$, and an arc of the corresponding circle. For $p = \gamma^{\zeta,1}(S) = (\exp(-\zeta), 1)$ the Riemann surface φ_p belongs to the real hypersurface $|w||z|^\lambda = \exp(-\lambda\xi)$, where $\xi = \operatorname{Re}\zeta$. Along the arc of the curve $\hat{\gamma}^\zeta$ lying over the radius, $|z|$ is increasing and $|w|$ is decreasing. The arc of $\hat{\gamma}^\zeta$ lying over the arc of the circle $|z| = 1$ belongs to the torus $|z| = 1$, $|w| = \exp(-\lambda\xi)$. Therefore, the whole curve belongs to B.

Further, the endpoint of $\hat{\gamma}^\zeta$ depends continuously on ζ by construction. It depends on ζ analytically according to the theorem on analytic dependence of a solution on the initial conditions. Consequently, it is the result of the analytic extension under investigation.

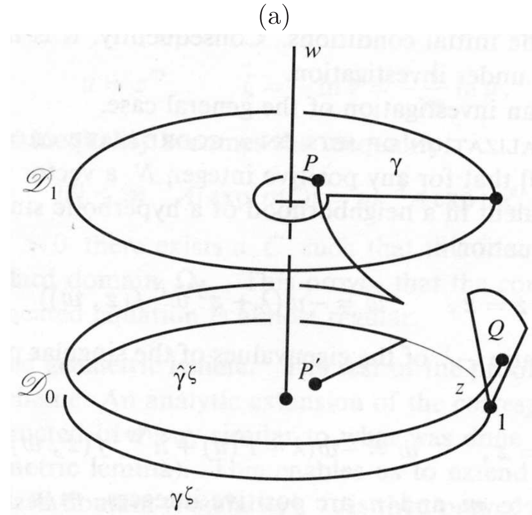
We proceed to an investigation of the general case.

Step 2. Normalization of jets on a coordinate cross. To study the asymptotics of the correspondence map near zero in the natural chard, we have to follow real curves on the leaves of the foliation that connect the points $z \in \Gamma^+$, $|z| << 1$, and $\Delta(z) \in \Gamma^-$, $|\Delta(z)| << 1$. These curves are located in narrow neighborhoods of the coordinate cross. To compare these curves with the analogous curves for the normalized equation, we need to make the difference between the original and normalized equations small on the coordinate cross in the unit bidisc centered at $(0, 0)$. This is the normalization named in the title of step 2. It is proved in the memoir [**?**] that for any positive integer N a vector field is orbitally analytically equivalent in a neighborhood of a hyperbolic singular point to a field giving the equation

$$\dot{z} = z, \quad \dot{w} = -w(\lambda + z^N w^N f(z, w))$$



(a)



(b)

FIGURE 11

for an irrational ratio $-\lambda$ of the eigenvalues of the singular point, and giving the equation

$$\dot{z} = z, \quad \dot{w} = -w(\lambda + P(u) + u^{N+1} f(z, w)) \tag{3.1}$$

for $\lambda = m/n$. Here m and n are positive integers, m/n is an irreducible fraction, $u = z^m w^n$ is the resonant monomial, P is a real polynomial without free term and of degree at most N, and f is a function holomorphic at zero. In [?] a real neighborhood is considered, but the arguments work for a complex one. the details for the complex case are presented in [?]. It can be assumed without loss of generality that in the bidisk B the function $|f|$ is less than an arbitrarily preassigned constant, and the correspondence mapping $\Gamma^+ \to \Gamma^-$ is defined, where Γ^+ and Γ^- are the same as in Step 1; this can be achieved by a change of scale.

The case of rational λ will be treated below; the proof of Theorem 3 for irrational λ is similar, only simpler.

STEP 3. THE CORRESPONDENCE MAPPING OF THE TRUNCATED EQUATION. This equation is obtained by discarding the last term in parentheses in (3.1):

e:0.3-2

$$\dot{z} = z, \quad \dot{w} = -w(\lambda + P(u)). \tag{3.2}$$

We prove that the correspondence mapping of this equation, written in the logarithmic chart, is almost regular. This was actually already done in §2F. It was proved there that

$$\Delta(z) = [g_{\tilde{f}}^{\ln z}(z^n)]^{1/m}, \qquad \text{where } \dot{u} = \tilde{f}(u)$$

is the factor system for the truncated equation, $\tilde{f} = nwP(u)$.

We prove first that the mapping $u \mapsto g_{\tilde{f}}^{(\ln u)/n} u$, written in the logarithmic chart, is almost regular. Consider it first in the chart u. As in §2F, let $F(t,u)$ be the local phase flow for $\tilde{f}(\partial/\partial u)$ at the point $(0,0)$ of (t,u)-space. The Taylor series for F converges in the domain $|t| \le A|u|^{-1}$, as proved in §2F. Under the substitution $t = (\ln u)/n$ this series becomes a Dulac series in the variable u. It converges in the domain where

$$|\ln u| < A|u|^{-1}.$$

Here and below we consider the branch of the logarithm that is real on the positive semi-axis.

On the disk $\mathcal{D}_1$ containing the semitransversal Γ^-, the natural chart z, the function u, and the logarithmic chart ζ are connected by the relations:

$$u = z^m, \quad \zeta = -\ln z = -\frac{1}{m}\ln u.$$

The previous inequality becomes the inequality

$$|\zeta| \le m^{-1}A|\exp(m\zeta)| = m^{-1}A \exp m\xi.$$

For every $A > 0$ there exists a C such that this inequality holds in the quadratic standard domain Ω_C. This proves that the correspondence mapping of the truncated equation is almost regular.

D. The second geometric lemma. The rest of the proof goes according to the following scheme. An analytic extension of the correspondence mapping of (3.1) is constructed in a way similar to what was done for the linear case in Step 1 (geometric lemma). This enables us to extend the mapping to a quadratic standard domain (regularity). It is then proved that the difference between the correspondence mappings of the original and the truncated equations is small if N is large. This proves that the mapping under investigation can be expanded in a Dulac series (expandability).

STEP 4. GEOMETRIC LEMMA. Suppose that the curves $\mu^\zeta, \gamma^\zeta, \gamma_T^\zeta$ and $\gamma^{\zeta,1}$ are the same as in Step 1. Let φ_p be the phase curve of the field (3.1) containing the point p, and let $\hat{\gamma}^{\zeta}$ $(\hat{\gamma}^{\zeta}_{T})$ be a covering on φ_p over the curve γ^{ζ} (γ^{ζ}_{T}) with initial point $p = (\exp(-\zeta), 1)$.

Geometric lemma. *For any equation* (3.1) *the scale can be chosen in such a way that in the bidisk B the following holds. There is a $C > 0$ such that*:

1. *the arc $\hat{\gamma}^{\zeta}$ is defined for every $\zeta \in \Omega_C$, where Ω_C is a quadratic standard domain, see Definition 1* ;
2. *the endpoint of this arc is the result of analytic extension of the correspondence mapping $\Delta\colon \Gamma^{+} \to \Gamma^{-}$ of* (3.1) *along the curve γ^{ζ}_{1} with initial value $\Delta(1) = 1$*;
3. *the first integral $\tilde{u} = -(\lambda \ln z + \ln w)$ of the linearized equation* (3.1) *varies in modulus at most by* 1 *from the initial value $\tilde{u}_0 = \tilde{u}(p)$, $p = (\exp(-\zeta), 1)$, upon extension along the curve $\hat{\gamma}^{\zeta}$*;
4. *if $\hat{\gamma}^{\zeta}_{0}$ is the curve analogous to $\hat{\gamma}^{\zeta}$ on the phase curve of the truncated equation with initial point p, then the values of the function $\tilde{u}$ differ at most by* $\exp(-(N+1)\lambda\xi)$ *at the endpoints of the curves $\hat{\gamma}^{\zeta}_{0}$ and $\hat{\gamma}^{\zeta}$.*

Remark. Note that $\tilde{\Delta}(\zeta)$ is the function Δ written in the logarithmic chart: if $z = \exp(-\zeta)$, then $\tilde{\Delta}(\zeta) = -\ln \Delta(z)$. On the other hand, $\tilde{u}(1, \Delta(z)) = -\ln \Delta(z)$. The branches of the logarithm in both expressions are the same. Hence, $\tilde{u}(1, \Delta(z)) = \tilde{\Delta}(\zeta)$. Note that almost regularity of $\Delta(z)$ means that $\tilde{\Delta}(\zeta)$ is defined in some quadratic standard domain, and may be expanded there in an exponential Dulac series. So we will prove these properties for the germ $\tilde{u}(1, \Delta(\exp(-\zeta)))$.

Proof. Note that for arbitrary ζ and sufficiently small T the curve $\hat{\gamma}^{\zeta}_{T}$ is defined. We prove that it is defined for arbitrary $T \in [0, S]$ if $|z|$ is sufficiently small.

We prove first that over the part of γ^{ζ} belonging to a radius of the disk $\mathcal{D}_0$ the extensions of the solutions of equations (3.1) and (3.2) with initial point p are defined for sufficiently small values of $\exp(-\zeta)$. Consider the domain $\mathcal{U} = \{|u| \leq \alpha\}$; smallness requirements are imposed on α below. Along the trajectories of equations (3.1) and (3.2) regarded as equations with realtime, $\arg z$ does not change, but $|w|$ decreases in $\mathcal{U} \cap B$: for equation (3.1)

$$\frac{1}{2}(w\bar{w})^{\cdot} = \operatorname{Re}(\bar{w}\dot{w}) = -w\bar{w}(\lambda + \operatorname{Re}(P(u) + u^{N+1}f)).$$

In the domain $\mathcal{U} \cap B$, where u and f are sufficiently small, $|w|^{\cdot} < 0$ for $w \neq 0$. The derivative of $|w|$ in $\mathcal{U} \cap B$ along the field of the truncated equation can be estimated similarly. We must still prove that the curves under investigation do not leave $\mathcal{U} \cap B$.

We carry out the proof for the system (3.1); it is analogous for (3.2), only simpler. In the variables $z, \tilde{u}$ the system (3.1) has the form

e:0.3-3

$$\begin{aligned} \dot{z} = z, \quad \dot{\tilde{u}} &= V(\tilde{u}) + R, \\ V(\tilde{u}) &= -P(\exp(-n\tilde{u})), \\ R &= \exp(-(N+1)\tilde{u})f(z, w). \end{aligned} \tag{3.3}$$

To conclude the passage to the new variables it would be necessary to express the function $f(z, w)$ in terms of z and $\tilde{u}$ in the expression for R, but this expression will not be used, since $|f|$ will be estimated from above by a constant. The solution of the system (3.3) with initial point $(z_0, \tilde{u}_0) = (\exp(-\zeta), \lambda\zeta)$ is considered over the curve $\mu^{\zeta}_{T} - \zeta$. When the time t runs through this curve, the point $(z(t), 0)$ runs through the curve γ_T^ζ, while the point $(z, \tilde{u})(t)$ runs through the curve $\hat{\gamma}_T^\zeta$. The latter curve is defined for small T. We prove that it is defined for $T = S$ (S is the length of the curve μ^ζ); this will mean that the curve $\hat{\gamma}^\zeta$ is defined.

In the domain $\mathcal{U} \cap B$

$$|V(\tilde{u}) + R| \leq \beta \exp(-n \operatorname{Re} \tilde{u})$$

for some $\beta > 0$. If $|\exp(-\zeta)|$ is small, then $\operatorname{Re} \tilde{u}_0$ is large; for an arbitrary sufficiently large value of $\operatorname{Re} \tilde{u}_0$ and for arbitrary $\delta \in \mathbb{C}$ with $|\delta| < 1$,

e:0.3-4 (3.4)
$$\beta |2\lambda^{-1} \tilde{u}_0| \exp(-n \operatorname{Re}(\tilde{u}_0 + \delta)) < 1.$$

The number $|2\lambda^{-1}\tilde{u}_0|$ exceeds the time S over which the curve γ^ζ runs. Consequently, if $\operatorname{Re} \zeta$ is sufficiently large, then the curve $\hat{\gamma}_\xi^\zeta$ lying over the radius $\arg z = \text{const}$ is defined and belongs to the intersection $\mathcal{U} \cap B$: for $T < \xi$ the curve $\hat{\gamma}_T^\zeta$ does not go out to the boundary of this intersection. Similarly, it follows from the inequality (3.4) that the curve does not go out to the boundary of $\mathcal{U}$ for $T \in [\xi, \xi + |\eta|]$, and the curve $\hat{\gamma}^\zeta$ is defined by the theorem on extension of phase curves.

It can be proved similarly that the curve $\hat{\gamma}_0^\zeta$ is defined. Suppose now that $\tilde{\varphi}$ is a solution of (3.3) with the initial condition $\tilde{\varphi}(-\zeta) = (\exp(-\zeta), \lambda\zeta)$. Then $\tilde{u}(\tilde{\varphi}(0)) = \tilde{\Delta}(\zeta)$ (see the remark after the formulation of the lemma). Let be the solution of the truncated equation $\dot{z} = z$, $\dot{\tilde{u}} = V(\tilde{u})$ with the same initial condition; then $\tilde{u}(\tilde{\varphi}(0)) = \tilde{\Delta}_0(\zeta)$. It was proved above that the solution $\tilde{u} \circ \tilde{\varphi}|_{\mu^\zeta - \zeta}$ runs through values lying in the disk K with center $\tilde{u}_0$ and radius 1. Let $L = \max_K(1, |V'(\zeta)|)$. Then, by Gronwall's lemma,

$$|\tilde{u} \circ \tilde{\varphi}(0) - \tilde{u} \circ \tilde{\varphi}_0(0)| \leq \max |R| \exp LS = o(\exp(-N\lambda\xi)) = o(\exp(-\nu\xi))$$

for any previously assigned $\nu > 0$ if N is sufficiently large. This proves the geometric lemma. □

Theorem 3 now follows from the fact that the germ $\tilde{\Delta}_0$ is almost regular.

This finishes the proof of the finiteness theorem for fields with hyperbolic singular points.

§ 0.4. Correspondence mappings for degenerate elementary singular points. Normalizing cochains

The correspondence mappings in the heading can be described with the help of the geometric theory of normal forms. According to this theory, the germ of a vector field or mapping in a punctured neighborhood of a fixed point gives an atlas of normalizing charts with nontrivial transition functions. The normalizing charts conjugate the germ with its formal normal form; the transition functions contain all the information about the geometric properties of the germ.

A. Formulations. The correspondence mappings in the heading decompose into a product of three factors, of which two must be defined; we proceed to do this. The germ of a holomorphic vector field at an isolated elementary singular point is formally orbitally equivalent to the germ

e:0.4-1 (4.1)
$$\dot{z} = z^{k+1}(1 + az^k)^{-1}, \quad \dot{w} = -w.$$

Here $k + 1$ is the multiplicity of the singular point, and a is a constant that is real if the original germ is real. For a formal normal form the manifold $z = 0$ is contractive, and the manifold $w = 0$ is the center manifold. The correspondence mapping of a semitransversal to the first manifold onto a semitransversal to the second (briefly, the mapping TO the center manifold) is denoted by Δ_{st} for the normalized system, and it has the form (see §0G)

$$\Delta_{\mathrm{st}} = C\exp(-1/h_{k,a}(z)), \text{ where } h_{k,a}(z) = kz^k/(1 - akz^k \ln z), \ C = \exp\frac{1}{k}.$$

The factors of this form introduce exponentially small terms into the asymptotic expression for the monodromy transformations.

The complexified germ of a real holomorphic vector field at an isolated degenerate elementary singular point always has a one-dimensional holomorphic invariant manifold that is contractive after a suitable time change, and it does not as a rule have a holomorphic center manifold. Corresponding to a contractive manifold is a monodromy transformation that has the following form after a suitable scale change with a positive factor:

e:0.4-2 (4.2) $$f\colon z \mapsto z - 2\pi i z^{k+1} + \cdots .$$

This transformation is formally equivalent to a time shift $-2\pi i$ along the trajectories of the vector field $v(z) = z^{k+1}/(1 + az^k)$. Here k and a are the same as in the formal orbital normal form of the germ. The corresponding normalizing formal series diverge as a rule, but they are asymptotic series for the normalizing cochains; we proceed to the definition of the latter.

A nice k-partition of the punctured disk is defined to be a partition of this disk into $2k$ equal sectors, one of which has a boundary ray on the real axis.

THEOREM 1 (on sectorial normalization [**?**], [**?**]). *For an arbitrary parabolic germ* (4.2) *(not necessary a monodromy transformation) there exists a tuple of holomorphic functions, called a cochain normalizing the germ, having the following properties.*

1. *The functions in the tuple are in bijective correspondence with the sectors of a nice k-partition of some disk with center zero and radius R; each function is defined in a corresponding sector.*

2. *Each function in the tuple extends biholomorphically to a sector S_j with the same bisector and a larger angle $\alpha \in (\pi/k, 2\pi/k)$; the radius of the sector depends on α.*

3. *All the functions in the tuple have a common asymptotic Taylor series at zero with linear part the identity.*

4. *In the intersections of the corresponding sectors the functions in the tuple differ by $o(\exp(-c/z^k))$ for some $c > 0$.*

5. *Each of the functions in the tuple conjugates the germ* (4.2) *in the sector S_j with the time shift by $-2\pi i$ along the trajectories of the field $v(z)$.*

There is a unique normalizing cochain whose correction decreases more rapidly than the correction of the germ (4.2)*, that is, a cochain* $\mathrm{id} + o(z^{k+1})$.

DEFINITION 1. The set of all normalizing cochains described in the preceding theorem and its supplement below is denoted by $\mathcal{NC}$ (= normalizing cochains); the set of mappings in the tuple corresponding to the sector adjacent from above (from below) to $(\mathbb{R}^+, 0)$ is denoted by $\mathcal{NC}^u$ ($u =$ upper) (respectively, $\mathcal{NC}^l$ ($l =$ lower)).

The following result is also known, but it will be proved below in C because it is contained "between the lines" in [**?**] and [**?**].

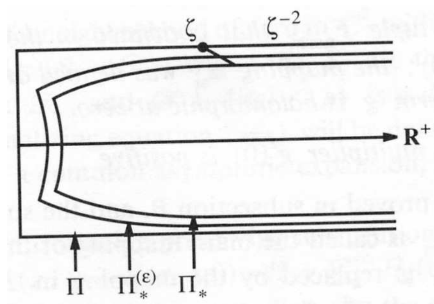


FIGURE 12 f12

To state it let us introduce the following notations:

eqn:pistar

$$\Pi = \{\xi \geq a, |\eta| \leq \pi/2\},$$
$$\Pi_*^{(\varepsilon)} = \Phi_{1-\varepsilon}\Pi, \tag{4.3}$$
$$\Phi_{1-\varepsilon} = \zeta + (1-\varepsilon)\zeta^{-2}, \quad a = a(\varepsilon), \quad \varepsilon \in [0, 1),$$

see Figure 2.

SUPPLEMENT. *A function in a tuple forming a normalizing cochain extends holomorphically to a domain broader than the sector S_j. For the sector S^u (S^l) adjacent from above (from below) to the positive semi-axis in a nice k-partition, this domain has the form $k^{-1}\Pi_*^{(\varepsilon)}$ in the logarithmic chart $\zeta = -\ln z$ ($\xi = -\ln x$, $\xi = \operatorname{Re}\zeta$, $x = \operatorname{Re} z$).*

An asymptotic decomposition to a Taylor series for $\mathcal{NC}^u$ may be extended to the same domain. The very same statement holds for S^l and $\mathcal{NC}^l$.

An analogous result is valid for the remaining functions in the tuple. The mapping corresponding to S^u in the normalizing cochain is denoted by F^u_{norm}.

REMARK. Let $\Pi_*^{(0)} = \Pi_*$. The domains $\Pi_*^{(\varepsilon)}$ are ordered by inclusion: $\Pi_*^{(\varepsilon)} \subset \Pi_*^{(\varepsilon')}$ for $\varepsilon < \varepsilon'$. The domain $\Pi_*^{(\varepsilon)}$ is called the generalized ε-neighborhood of the curvilinear half-strip Π_* (Figure 2).

Everything is now ready for a description of the correspondence mappings in the section heading.

thm:bound

THEOREM 2 ([?], [?]). *The correspondence mapping $\Delta\colon (\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$ TO a center manifold of a degenerate elementary singular point of a real-analytic vector field is the restriction to $(\mathbb{R}^+, 0)$ of any of the two compositions*

$$\Delta = g^u \circ \Delta_{\text{st}} \circ F^u_{\text{norm}} = g^l \circ \Delta_{st} \circ F^l_{\text{norm}},$$

where F_{norm} is the normalizing map-cochain (see the theorem on sectorial normalization) for the corresponding monodromy transformation; F^u_{norm} (F^l_{norm}) is that mapping in the tuple F_{norm} that is defined in the sector S^u (S^l) adjacent from above (respectively from below) to $(\mathbb{R}^+, 0)$; the mapping Δ_{st} was defined at the beginning of the subsection, and the germs g^u, g^l are holomorphic at their fixed point zero.

SUPPLEMENT. *The multipliers $(g^u(0))'$ and $(g^l(0))'$ are positive.*

Theorem 2 will be proved in subsection B, and the supplement to it in D.

The mapping F^u_{norm} is called the main mapping of the tuple F_{norm}.

REMARK. If F^u_{norm} is replaced by the mapping F^l_{norm} corresponding to the sector adjacent to $\mathbb{R}^+$ from below, and the germ g is replaced by another holomorphic germ, then the product Δ does not change on $\mathbb{R}^+$. After choosing the main mapping in Theorem 2, we thereby dwelt upon one of two mutually symmetric variants, each one being by itself asymmetric.

Normalizing map-cochains make it necessary to use functional cochains for investigating monodromy transformations of polycycles.

add smth

The set of all germs described in the preceding theorem, will be denoted by **TO**, the set of germs inverse to them by **FROM**, and the set of all almost regular germs by $\mathcal{R}$. The identity theorem follows from the next result.

THEOREM A. *A composition of germs in the classes* **TO**, $\mathcal{R}$, *and* **FROM** *either has no fixed points near zero, or is the identity.*

Chapters I–V are devoted to a proof of this theorem.

B. Proof of the theorem on the correspondence mapping. In this subsection we prove Theorem 2 in A.

B.1. *Preliminary results.* For germs of vector fields at a degenerate elementary singular point there is a theorem on sectorial normalization analogous to Theorem 1 in A. To formulate it the germ must be reduced to a "preliminary normal form."

DULAC'S THEOREM ([?]). *The germ of an analytic vector field at an isolated elementary singular point of multiplicity* $k+1$ *is orbitally analytically equivalent to the germ giving the equation*

e:0.4-3

$$\dot z = z^{k+1}, \quad \dot w = -w + F(z,w), \qquad F(0,0)=0, \quad dF(0,0)=0. \tag{4.4}$$

REMARK. An isolated singular point of an analytic vector field on the complex plane is always of finite multiplicity. If the original equation is real, then the normalizing substitution and equation (4.4) are also real.

We proceed to formulate the theorem on sectorial normalization for equation (4.4). A nice k-covering of the punctured disk $\mathcal{D}_0\colon 0<|z|<\varepsilon$ is a tuple of sectors S_j as described in the formulation of Theorem 1 in subsection A. Namely, we consider a nice partition of the punctured disk and replace each sector of this partition by a sector with the same bisector, the same radius, and a larger opening $\alpha \in (\pi/k, 2\pi/k)$. The resulting tuple of sectors S_j forms a *nice* k*-covering* of the punctured disk.

A nice k-covering of a neighborhood of zero in $\mathbb{C}^2$ with the w-axis deleted is defined to be a tuple $\{S_j \times \mathcal{D}\}$, where the S_j are the sectors of a nice k-covering of the pictured disk $\mathcal{D}_0$, and $\mathcal{D} = \{|w| \le \rho\}$ is a disk on the w-axis. The substitutions normalizing equation (4.4) will be defined in the "sectors" $S_j \times \mathcal{D}$ and will have a common asymptotic expansion, to the definition of which we proceed.

DEFINITION 2. A semiformal z-preserving substitution is a substitution H of the form $(z,w) \mapsto (z, w+\hat H(z,w))$, $\hat H = \sum_1^\infty H_n(w) z^n$; the functions H_n are holomorphic in one and the same disk $\mathcal{D}$; the series $\hat H$ of powers of z is formal (a

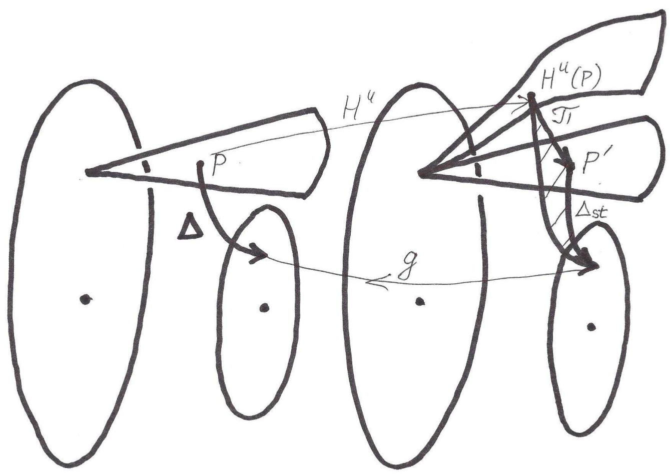


FIGURE 13 f20002

z-preserving substitution is denoted in the same way as the correction of its second component).

We can now state a proposition on a semiformal normalization of a saddlenode.

PROPOSITION ([?]). *For an arbitrary equation* (4.4) *there exists a unique substitution of the form* $h \circ \hat{H}$, *where* $\hat{H}$ *is a semiformal* z*-preserving substitution, and* h *is a holomorphic substitution of the form* $(z, w) \mapsto (h(z), w)$, $h(z) - z = o(z^{k+1})$, *carrying equation* (4.4) *into the equation*

e:0.4-4 (4.5) $$\dot{z} = z^{k+1}, \quad \dot{w} = -w(1 + az^k)$$

(*which can be reduced to equation* (4.1) *by a change of time*).

Let us now state a theorem about sectorial holomorphic normalization of a saddlenode.

THEOREM 3 (on sectorial normalization [?]). *For an arbitrary equation* (4.4) *there exists in each sector* $S_j \times \mathcal{D}$ *of a nice* k*-covering of a neighborhood of zero in* $\mathbb{C}^2$ *with the* w*-axis deleted a unique biholomorphic mapping* $h \circ H_j$ *carrying equation* (4.4) *into equation* (4.5) *and such that the series* $\hat{H}$ *is asymptotic for* H_j *in* $S_j \cap \mathcal{D}$ *as* $z \to 0$, $h(z) - z = o(z^{k+1})$.

The mappings $\tilde{H}_j = h \circ H_j$ are said to be normalizing equation (4.4) in the sectors $S_j \times \mathcal{D}$. Let S_*^u (S_*^l) be the sector of a nice k-covering containing the sector S^u (S^l) adjacent to $(\mathbb{R}^+, 0)$ from above (below) in a nice k-partition.

B.2. *End of proof of Theorem 2.* As before, consider a germ V of a saddle-node vector field V at 0 of the form (4.4). Let V_{st} be the normal form of V. Let $\mathbf{H}^u$ be

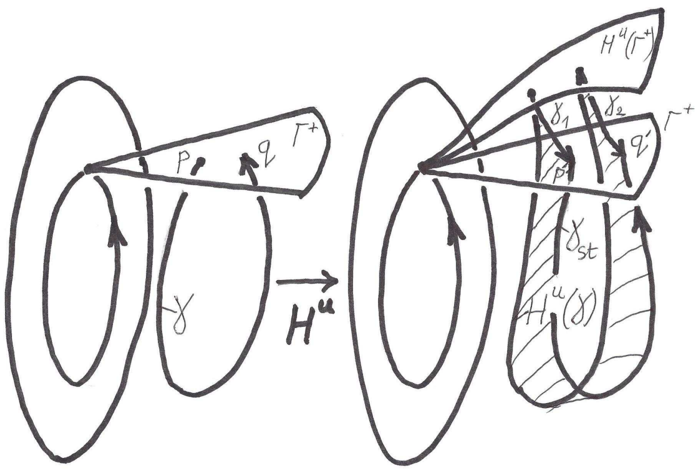


FIGURE 14 f20003

a holomorphic normalizing transformation defined in a sector $S_*^u \times D$, see Theorem 3 above:

$$\mathbf{H}^u(z,w) = (h(z), H^u(z,w)).$$

The map $\mathbf{H}^u$ is an orbital analytic conjugacy between V and V_{st}. Let Γ^+ and Γ^- be two disks in the lines $w = w^+$ and $z = z^-$ respectively, z^- and w^+ so small that the disk Γ^- and the sector $S_* = S_*^u \times \{w^+\}$ belongs to the domain of $\mathbf{H}^u$. Consider the correspondence map $\Delta : S_* \to \Gamma^-$. For any $p \in S_*$, let $q = \Delta(p)$.

Let π be a projection along the solutions of a neighborhood of $(0, w^+)$ to Γ^+ along the orbits of the normalized equation. Note that $\mathbf{H}^u(\Gamma^-) \subset \{h(z^-) \times \mathbb{C}\}$, and $g^{-1} = \mathbf{H}^u|_{\Gamma^-}$ is biholomorphic. On the other hand, the restriction $\mathbf{H}^u|_{S_*}$ can not be holomorphically extended onto Γ^+ in general. Below we prove the following relation: if Δ is a map **TO** defined in S_*, then

eqn:dec (4.6) $$\Delta|_{S_*} = g \circ \Delta_{\text{st}} \circ (\pi \circ \mathbf{H}^u|_{S_*}).$$

As above, denote by f the monodromy transformation for V that corresponds to a positively oriented loop around zero on the w-axis, the *holomorphic* invariant manifold of V.

prop:norm PROPOSITION 1. *The map $\pi \circ \mathbf{H}^u|_{S^*}$ coincides with F^u_{norm}, a component of the normalizing cochain for the monodromy map f defined in the sector S^*.*

Together, formula (4.6) and Proposition 1 imply Theorem 2.

Formula (4.6) is sort of tautology, see Figure 13. Let $p \in S^u \times \{w^+\}$, then $H^u(p) \in S_*^u$. Let $p' = \pi \circ H^u(p)$. Let $q = \Delta(p) \in \Gamma^-$, $q' = \Delta_{\text{st}}(p')$. Figure (13) shows that

$$\Delta : p \overset{F}{\mapsto} p' \overset{\Delta_{\text{st}}}{\mapsto} q' \overset{g}{\to} q.$$

This implies (4.6).

PROOF. of Proposition 1 The proof is based on the fact that the monodromy transformation f_{st} of V_{st} is imbeddable, and F conjugates f and f_{st} in S_*.

Let us proof the first statement: the map f_{st} is a phase flow transformation of a holomorphic vector field. Consider the transversal Γ^+ with the chart z. The inverse image z and the image $f_{\text{st}}(z)$ of the transformation f_{st} are by definition the z-coordinates of the initial point and the endpoint of the arc γ with initial point (z, w^+) covering on the solution of (4.1) the loop $\{w = w^+ e^{i\varphi} \mid \varphi \in [0, 2\pi]\}$ which belongs to the w-axis. The system (4.1) has separating variables; the desired arc γ has the form

$$\gamma = \{g^t(z, w^+) \mid t \in [0, -2\pi i]\}.$$

Here $\{g^t\}$ is the local phase flow of (4.1); the arc γ is defined if $|z|$ is sufficiently small. Consequently,

$$f_{\text{st}} = g^{-2\pi i}_{v(z)}, \quad v(z) = z^{k+1}/(1 + az^k),$$

and the monodromy transformation f_{st} is imbeddable.

Let $p = (z, w^+) \in \overline{S^u} \times \{w^+\}$, and γ be the curve on the leaf of V through p that covers S^1; the endpoint of γ is $q = f(p)$, where f is the monodromy transformation for V and S^1. let $p' = \pi \circ \mathbf{H}^u(p),\ q' = \pi \circ \mathbf{H}^u(q)$. Then

eqn:mst

$$q' = f_{\text{st}}(p'), \tag{4.7}$$

see Figure 14.

Indeed, let γ_1 and γ_2 be two arcs on a leaf of V_{st} that connect $\mathbf{H}^u(p)$ and p', $\mathbf{H}^u(q)$ and q' respectively, and cover two segments on the w axis. Then the arcs $\gamma_1^{-1}\mathbf{H}^u(\gamma)\gamma_2$ and γ_{st} are homotopic on the leaf of V_{st}, because their projections to the punctured w axes are homotopic on this axis. This proves (4.7), hence, Proposition 1. □

REMARK. The substitutions F^u and F^l, as well as g and $\tilde{g}$, are not real on $(\mathbb{R}^+, \infty)$, not even for real equations (4.1), (4.4). It is proved in subsection D that the multiplier $g'(0)$ is nevertheless positive.

C. Proof of the supplement to the theorem on sectorial normalization.

C.1. *First steps in the proof of the supplement.*

PROOF. We consider the extension of the normalization mapping from the sectors S^u and S^l; the remaining ones are investigated similarly, but this investigation is unnecessary for us. Denote for brevity $\mathcal{NC}^u = F^u,\ \mathcal{NC}^l = F^l$.

We repeat here the proof of the sectorial normalization theorem due to Malgrange [**?**] with some improvements necessary for the proof of the Supplement. We deal with F^u only. For F^l the proof is literally the same.

Let f be a germ (4.2), and g be its formal normal form:

$$g = g^1_{w(z)},\ w(z) = -\frac{2\pi i z^{k+1}}{1 + az^k},$$

where g^1_w is a time 1 shift along the phase curves of the vector field w. The functional equation for the map F^u has the form

eqn:fe

$$F^u \circ f = g \circ F^u. \tag{4.8}$$

Let us pass to the coordinate t that rectifies the vector field w:

$$t = \int \frac{dz}{w(z)}.$$

We have:

$$t = \frac{1}{2\pi i k} z^{-k} - \frac{a}{2\pi i} \ln z.$$

Denote by $\hat{f}, \hat{g}, \hat{F^u}$ the germs f, g, F^u written in the chart t. It is important to notice that

$$\hat{g}(t) = t + 1.$$

Thus the functional equation above becomes the so called Abel equation:

$$\hat{F^u} \circ \hat{f} = \hat{F^u} + 1.$$

The sector S^u in the chart t for $a = 0$ becomes a "left halfplane sector"

$$\hat{S}_0^u = \{t | \arg t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], |t| > R\}$$

for some R. For arbitrary $a \in \mathbb{R}$, S^u becomes a curvilinear sector $\hat{S}^u$, which is $\hat{S}_0^u$ slightly distorted. We are interested in a broader domain described in the Supplement. Let us describe this domain in the chart t.

The domain under consideration is

$$\hat{S} = \hat{S}^u \cup \hat{\Pi}^\varepsilon,\ \hat{\Pi}^\varepsilon = t \circ (\exp) \circ (-k^{-1}) \Pi_*^{(\varepsilon)}$$

prop:dom

Proposition 2. *The germ at ∞ of the $\hat{S}$ above belongs to a germ at infinity of a domain*

$$S_0 = \{t||t| > R\} \setminus \{t = t_1 + it_2,\ t_1 > 0,\ t_2^2 \le t_1\}.$$

lem:expan

Lemma 1. *For any $M > 0$ there exists N with the following property. Let $\hat{f} = t + 1 + O(t^{-N})$ be a holomorphic function in a punctured neighborhood of infinity with the ray $\mathbb{R}^+$ deleted. Then the Abel equation with this $\hat{f}$ in the left hand side has a solution $\hat{F}$ of the form*

$$\hat{F} = id + (t^{-M})$$

in the domain S_0.

Together, Lemma 1 and Proposition 2 imply the Supplement, as shown in the next subsection.

C.2. *Deduction of the Supplement from Lemma 1 and Proposition 2.* Consider the functional equation (4.8) in the original coordinates. It has a unique solution in S^u of the form $F^u = id + o(z^{k+1})$. This solution may be expanded in a formal Taylor series denoted by $\hat{F}$. Let Σ_f, Σ_g be the Taylor series of f and g (formal and convergent at the same time). The following equality in formal series holds:

$$\hat{F} \circ \Sigma_f = \Sigma_g \circ \hat{F},$$

or

$$\hat{F} \circ \Sigma_f \circ \hat{F}^{-1} = \Sigma_g.$$

Let Σ be a partial sum of $\hat{F}$ up to a term of degree K; this degree will be chosen later. Then

eqn:abel1

$$f_K := \Sigma \circ f \circ \Sigma^{-1} = g + o(z^K). \tag{4.9}$$

Consider a functional equation

eqn:fe1 (4.10)
$$F_K \circ f_K = g \circ F_K. \tag{4.10}$$

We will prove that for arbitrary L, K may be so chosen that (4.10) has a solution F_K of the form:

eqn:slnfe (4.11)
$$F_K = z + o(z^L) \tag{4.11}$$

in S. Then substituting in (4.10) the expression for f_K from (4.9), we get:

$$F^u = F_K \circ \Sigma = \Sigma + o(z^L)$$

in S. This is exactly the statement of the Supplement, which is now proved, modulo the choice of K.

Let us prove the possibility to chose K so that (4.11) holds. As $f_K = g + o(z^K)$ in a full neighborhood of zero, then in the chart t, $\hat{f}_K = t + 1 + o(t^{-N})$ in $\hat{S}$ as $t \to \infty$. The degree N tends to infinity together with K.

In the chart t the functional equation (4.10) takes the form:

$$\hat{F}_K \circ \hat{f}_K = \hat{F}_K + 1, \ \hat{f}_K = t + 1 + o(t^{-N})$$

in S.

Now, by Lemma 1, this equation has a solution $\hat{F}_K = t + o(t^{-M})$ in S. Going back to the original chart z, we get: $F_K = z + o(z^L)$. Not only N, but M and L also, tend to infinity as $K \to \infty$. So the desired choice of K is possible. This proves the Supplement, modulo Lemma 1 and Proposition 2. $\square$

C.3. *Proof of Lemma 1.*

Proof. For this subsection only, replace $\hat{f}$ from Lemma 1 by f. Consider the Abel equation

eqn:abel (4.12)
$$F \circ f = F + 1, \ f = t + 1 + o(t^{-N}) \tag{4.12}$$

in S_0. Note that the orbits of the shift $t \mapsto t + 1$ are not well defined for any

$$t \in S_r^0 = S^{0 \cap \{|t|>r\}},$$

however large r be. On the contrary, the orbits of the inverse shift $t \mapsto t - 1$ are well defined for any $t \in S_r^0$ for r large enough, see Figure 15. We will prove below that the same holds for the orbits of f^{-1}. Let us find F in the form $F = t + h$, and denote $f = t + 1 + \varphi$, $\varphi = o(t^{-N})$.

The Abel equation takes the form

$$\varphi + h \circ f = h.$$

The series

$$h = -\sum_1^\infty \varphi \circ f^{-k}$$

solves equation (4.12) for $F = t + h$ whenever it is well defined and converges.

Consider $t \in S_{0r}$ for r to be chosen later. For r large enough, a cone

$$\mathcal{K}_t = \{\tau \in \mathbb{C} | \arg(\tau - t) - \pi| < |t|^{-3}\}$$

belongs to S_0. Moreover, the orbit $\{f^{-k}(t) | k \geq 1\}$ belongs to $\mathcal{K}_t$, see Figure 15. We have: $|\tau| \geq |t|^{1/2}$ in $\mathcal{K}_t$.

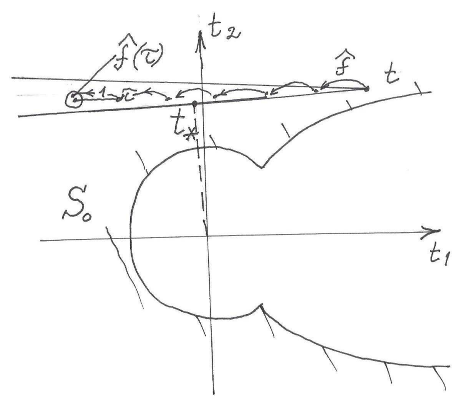


FIGURE 15 f13

Let t_* be a point on this orbit on which $|\tau|$ restricted to this orbit takes a minimal value, $t_* = f^{-k_*}(t)$. By definition of the domain S_0, and the cone $\mathcal{K}_t$ $|t_*| \geq \frac{1}{2}|t|^{1/2}$. Moreover,

$$\sum_1^\infty |\varphi \circ f^{-k}(t)| \leq C \sum_1^\infty |f^{-k}(t)|^{-N} \leq C \sum_{-k_*}^\infty |f^{-k}(t_*)|^{-N} \leq C_1 \sum_{-k_*}^\infty \left| t_* + \frac{k}{2} \right|^{-N} \leq$$

$$C_1 \sum_{-\infty}^\infty \left| t_* + \frac{k}{2} \right|^{-N} \leq C_2 \int_{-\infty}^\infty |t_* + s|^{-N} ds \leq C_2 \int_{-\infty}^\infty (|t_*|^2 + s^2)^{-\frac{N}{2}} ds =$$

$$C_2 |t_*|^{-N+1} \int_{-\infty}^\infty (1+u^2)^{-\frac{N}{2}} du = C_3 |t_*|^{-N+1} \leq C_4 |t|^{-\frac{N}{2}+\frac{1}{2}}.$$

This implies Lemma 1. □

C.4. *Proof of Proposition 2.*

PROOF. Recall that

$$\hat{S} = \hat{S}_u \cup t \circ \exp \circ (-k^{-1})(\Pi_*^{(\varepsilon)}),$$
$$S_0 = \{t \,|\, |t| > R\} \setminus \{t = t_1 + it_2,\ t_1 > 0,\ t_2^2 \leq t_1\}$$

We have to prove that

$$(\hat{S}, \infty) \subset (S_0, \infty).$$

For this it is sufficient to prove that the upper boundary curve of $(\hat{S}, \infty)$ located in $\mathbb{C}^+$ lies above that of S_0. Let the first curve be the graph of a function η : $\{t | t_1 + it_2,\ t_2 = \eta(t_1)\}$ We have to prove that

$$\eta(t_1) \succ \sqrt{t_1}.$$

For this let us calculate $\eta(t_1)$, or better some function $\eta_0 \prec \eta$; we will then prove that $\eta_0 \succ \sqrt{t_1}$. We have: for some $c > 0, c_1 \in \mathbb{R}$, (in fact, $c = \frac{1}{2\pi k}$).

$$t \circ \exp \circ (-k^{-1}\zeta) = ic \exp \zeta + ic_1 \zeta.$$

On the other hand, for any $\varepsilon \in (0,1)$,

$$(\Pi_*^{(\varepsilon)}, \infty) \subset (P, \infty), \text{ where } P = \{\xi + i\eta \,|\, |\eta| < \frac{\pi}{2} - \xi^{-4}\}$$

The lower boundary L of P has the form $\{\xi - \frac{i\pi}{2} + i\xi^{-4} | \xi > 0\}$. Hence, for some real c, c_1 whose explicit values are of no importance,

$$\begin{aligned}\Gamma := t \circ \exp \circ (-k^{-1})(L) &= ic \exp(\xi - \frac{i\pi}{2} + i\xi^{-4}) + ic_1(\xi - \frac{i\pi}{2} + i\xi^{-4}) \\ &= c \exp(\xi + i\xi^{-4}) + ic_1(\xi + i\xi^{-4}).\end{aligned}$$

The curve Γ is a graph of a function (denote it η_1) that satisfies an inequality for some $C > 0$

$$\eta_1(t_1) \succ \frac{Ct_1}{(\ln t_1)^4} := \eta_0.$$

Obviously, $\eta_0 \succ \sqrt{t_1}$. □

This completes the proof of the proposition, and together with it, the proof of the supplement to the theorem on the sectorial normalization.

D. The realness of the derivative $g'(0)$ in the expression for the correspondence mapping of a degenerate elementary singular point (a supplement to Theorem 2 in A). Since equation (4.4) is real, the complex conjugation involution $I\colon (z, w) \mapsto (\bar{z}, \bar{w})$ preserves it. The semiformal normalizing substitution in the proposition in subsection B passes into itself under this involution, by uniqueness. This implies that the normalizing substitution in Theorem 3, which is also uniquely determined, passes into itself under the involution of conjugation $I\colon \mathbf{H}^u \circ I = I \circ \mathbf{H}^l$. Then the substitutions F^u and F^l, which normalize the monodromy transformation, have the same property: $F^u(\bar{z}) = \overline{F^l(z)}$. In what follows we will say that the corresponding normalizing cochain is weakly real. Consequently, the Taylor series common for F^u and F^l is real. Accordingly, in the formula of Theorem 2 for the correspondence mapping,

$$\Delta = g \circ \Delta_{\mathrm{st}} \circ F^u_{\mathrm{norm}},$$

the first factor F^u_{norm} on the right-hand side can be expanded in a real asymptotic Taylor series. Recall that

$$\Delta_{\mathrm{st}} = f_0 \circ h_{k,a}, \quad f_0 = \exp(-1/z);$$

$$h_{k,a} = \frac{kz^k}{1 - akz^k \ln z}$$

can be expanded in a real Dulac series (subsection 0.2G). Thus, the Dulac series for the composition $\Delta_1 = h_{k,a} \circ F^u_{\mathrm{norm}}$ is real. Suppose now that

$$g'(0) = \nu, \quad \tilde{\nu} = \mathrm{Ad}(f_0)\nu\colon \zeta \mapsto \frac{\zeta}{1 - \zeta \ln \nu}.$$

Let $g = g_1 \circ \nu$; then $g_1'(0) = 1$. The Dulac series for the composition

$$f_0^{-1} \circ \Delta = (\mathrm{Ad}(f_0) g_1 \circ \tilde{\nu}) \circ \Delta_1$$

is real because Δ is real. The Dulac series for $\mathrm{Ad}(f_0)g_1$ is equal to x (see Lemma 4 in §2). Consequently, the Dulac series for $\tilde{\nu}$ is real, being a composition quotient of two real Dulac series. This implies that $\nu > 0$ and proves a supplement to Theorem 2.

The compositions of the correspondence mappings are better described in the logarithmic chart. We pass now to this description.

§ 0.5. Transition to the logarithmic chart. Extension of the class of normalizing cochains

A. Transition to the logarithmic chart. A semitransversal to an elementary polycycle can always be chosen to belong to an analytic transversal: to an open interval transversal to the field. A chart on the semitransversal equal to zero at the vertex and analytically extendible to the transversal is said to be natural; its logarithm with the minus sign is called the logarithmic chart. A natural chart is denoted by x and the corresponding logarithmic chart by ξ; the transition function is $\xi = -\ln x$. In a natural chart the monodromy transformation of the polycycle is the germ of a mapping $(\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$, and in the logarithmic chart it is the germ of a mapping $(\mathbb{R}^+, \infty) \to (\mathbb{R}^+, \infty)$. The notation $z = x + iy$, $\zeta = \xi + i\eta$ is used upon extension to the complex domain. Transition to the logarithmic chart is denoted by tilde: if f is a function that represents some map in the natural chart, then $\tilde{f}$ represents the same map in a logarithmic chart.

The following table contains examples of mappings used repeatedly in what follows.

	Mapping in a natural chart	The same mapping in the logarithmic chart
1	Power: $z \mapsto Cz^\nu$	Affine: $\zeta \mapsto \nu\zeta - \ln C$
2	Standard flat: $z \mapsto \exp(-1/z)$	Exponential: $\zeta \mapsto \exp \zeta$
3	A mapping defined in a sector with vertex 0 and expandable in a convergent or asymptotic Taylor series $\hat{f} = z(1 + \sum_1^\infty a_j z^j)$	A mapping defined in a horizontal half-strip and expandable in a convergent or asymptotic Dulac (exponential) series $\tilde{f} = \zeta + \sum_1^\infty b_j \exp(-j\zeta)$
4	$h_{k,a} \colon z \mapsto kz^k(1 - az^{-k}\ln z)^{-1}$	$\tilde{h}_{k,a} \colon \zeta \mapsto k\zeta - \ln k - \ln(1 - a\zeta\exp(-k\zeta))$
5	An almost regular mapping with asymptotic Dulac series at zero $z \mapsto Cz^\nu + \sum P_j(z)z^{\nu_j}$, where $C > 0$, $\nu > 0$, $0 < \nu_j \nearrow \infty$, and and the P_j are real polynomials	An almost regular mapping with asymptotic Dulac exponential series at infinity $\zeta \mapsto \nu\zeta - \ln C + \sum Q_j(\zeta) \cdot \exp(-\mu_j\zeta)$ where $C > 0$, $\nu > 0$, $0 < \mu_j \nearrow \infty$, and the Q_j are real polynomials

The set of almost regular germs with affine principal part the identity is denoted by $\mathcal{R}^0$.

The most important example is a normalizing cochain (see the example in §1.1) written in the logarithmic chart. Upon transition to the logarithmic chart the cochain F_{norm} becomes a map-cochain denoted by $\tilde{F}_{\text{norm}}$ and defined in the half-plane $\mathbb{C}_a^+ \colon \xi \geq a$; a depends on the cochain.

The k-partition of a punctured disk by sectors becomes the partition of $\mathbb{C}_a^+$ into half-strips by the rays $\eta = \pi m/k$, $m \in \mathbb{Z}$.

The mappings making up F_{norm} extend analytically to the ε-neighborhoods of the corresponding half-strips in the partition for arbitrary $\varepsilon \in (0, \pi/2)$ (a depends also on ε).

They have an exponentially decreasing correction (difference with the identity).

The modulus of the correction of the coboundary has the upper estimate $\exp(-C \exp k\xi)$ for some $C > 0$ depending on the cochain.

The mappings making up F_{norm} can be expanded in a common asymptotic Dulac exponential series; see row 3 of the table.

This list of properties, with variations, will appear many times in the future.

The cochain $\tilde{F}_{\text{norm}}$ is periodic: it is preserved under a shift by $2\pi i$.

The set of all such cochains corresponding to different values of C and the same value of k is denoted by $\mathcal{NC}_k$.

B. Separating the affine factors. We now describe the mappings of the class **TO** in the logarithmic chart. Their appearance sharply complicates the investigation of the monodromy transformations of polycycles; this investigation is relatively simple without them, see §0.3 and [**?**]. In a natural chart a mapping Δ of class **TO** is described by the Theorem 2 in §0.4: after a suitable scale change in the inverse image it has the form $\Delta = g \circ \Delta_{\text{st}} \circ F_{\text{norm}}$ where $\Delta_{\text{st}} = C \circ f_0 \circ h_{k,a}$, $C \in \mathbb{R}$, $f_0(z) = \exp(-1/z)$, $h_{k,a} = kz^k/(1 - az^k \ln z)$, and $g\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is the germ of a holomorphic mapping with linear part the identity. Taking into account the examples in the table given, we get that the mapping Δ, written in the logarithmic chart, has the form

$$\tilde{\Delta} = \tilde{g} \circ (\text{id} - \ln C) \circ \exp \circ \tilde{h}_{k,a} \circ \tilde{F}_{\text{norm}}.$$

The corrections of the mappings $\tilde{g}$ and $\tilde{F}_{\text{norm}}$ decrease exponentially; the mappings $\text{id} - \ln C$ and $\tilde{h}_{k,a}$ have "affine principal part" that does not in general coincide with the identity mapping. This coincidence holds only when $C = 1$ and $k = 1$, respectively.

In what follows it is convenient to group together all the affine factors in the composition, and for the remaining factors (not considering the exponential, of course) to make the affine principal part the identity. We mention that

$$(\text{id} - \ln C) \circ \exp = \exp \circ h_0,$$
$$h_0 = \text{id} + \ln(1 - (\exp(-\zeta) \ln C)) \in \mathcal{R}^0;$$

the mapping h_0 is almost regular and has decreasing correction. Next, denote by a_k the affine mapping $\zeta \mapsto k\zeta - \ln k$. Then

$$\tilde{h}_{k,a} = h_1 \circ a_k,$$
$$h_1 = \zeta - \ln\left(1 + \frac{a}{k^2}(\zeta + \ln k)\exp(-\zeta)\right).$$

The exact expression for h_1 is only needed for seeing that $h_1 \in \mathcal{R}^0$. Finally, the composition

$$F = a_k \circ \tilde{F}_{\text{norm}} \circ a_k^{-1}$$

is a map-cochain corresponding to the partition into half-strips of width π by the rays $\eta = \pi l$, $l \in \mathbb{Z}$. Indeed, if the function f_l in $\tilde{F}_{\text{norm}}$ is holomorphic in the half-strip $\eta \in [\pi l/k, \pi(l+1)/k]$, then the function $f_l \circ ((\zeta + \ln k)/k)$ in the tuple $\tilde{F}_{\text{norm}} \circ a_k^{-1}$ is holomorphic in the half-strip $\eta \in [\pi l, \pi(l+1)]$. As before, the correction of the map-cochain F decreases exponentially with rate of order $\exp(-\xi)$. This construction motivates the following definition. Let:

$$\mathcal{NC} = \bigcup_k a^k \circ \mathcal{NC}_k \circ a_k^{-1},$$

a_k is the same affine mapping as above.

Denote by $\mathcal{H}$ the set of germs of mappings $(\mathbb{C}^+, \infty) \to (\mathbb{C}^+, \infty)$ obtained from germs of holomorphic mappings $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ with linear part the identity by passing to the logarithmic chart (see line 3 of the table):

$$\mathcal{H} = \{-\ln \circ\, g \circ \exp(-\zeta) \mid g \in O_0,\ g(0) = 0,\ g'(0) = 1\}.$$

Finally, denote by $\mathcal{Aff}$ the set of germs of affine mappings $(\mathbb{C}^+, \infty) \to (\mathbb{C}^+, \infty)$ with real coefficients and positive multiplier, and by $\mathcal{M}_{\mathbb{R}}$ ($\mathbb{R}$ for real mappings) the set of germs of mappings, below called real, whose restrictions to $(\mathbb{R}^+, \infty)$ act as $(\mathbb{R}^+, \infty) \to \mathbb{R}$. Then we get finally that

$$\mathbf{TO} \subset (\mathcal{H} \circ \exp \circ\, \mathcal{R}^0 \circ \mathcal{NC} \circ \mathcal{Aff}) \cap \mathcal{M}_{\mathbb{R}} \overset{\text{def}}{=} \underline{\mathbf{TO}}.$$

C. Description of the cochains of class $\mathcal{NC}$. As indicated above, all the cochains in this class correspond to one and the same partition of the half-plane $\mathbb{C}_a^+$ (a depends on the cochain $F \in \mathcal{NC}$) by the rays $\eta = \pi l$, $\xi \geq a$. The partition by these rays of an arbitrary domain in the right half-plane is denoted by Ξ_{st} and is called the *standard partition*. The domains of this partition are the half-strips $\Pi_j : \eta \in \pi(j-1), \pi j]$, $\xi \geq a$. The half-strips Π_0 and Π_1 adjacent to $(\mathbb{R}^+, \infty)$ are called the main ones. For what follows we need to extend the mapping in a normalizing cochain corresponding to the main half-strip of the standard partition, to a curvilinear half-strip close to the right half-strip $|\eta| \leq \pi/2$. The possibility of such an extension follows from the supplement to Theorem 1 in §0.4A

This implies that the cochains in the set $\mathcal{NC}$ possess the following properties.

1. Each cochain $F \in \mathcal{NC}$ corresponds to the standard partition of some right half-plane $\mathbb{C}_a^+$, where a depends on F.

2. All the mappings in the cochain extend to the ε-neighborhoods of the corresponding half-strips for some $\varepsilon > 0$. The mappings in the cochain corresponding to the main half-strips Π_0 and Π_1 of the partition extend holomorphically to a germ at infiniy of a half-strip $\Pi_*^{(\varepsilon)}$ for any $\varepsilon > 0$, see (4.3), and the correction of the extended mapping can be estimated from above by a decreasing exponential.

3. The corrections of all mappings of the cochain extended as in the previous item may be estimated in modulus from above by the decreasing exponential $\exp(-\mu\xi)$ for some $\mu > 0$ common for all the mappings in F.

4. The correction of the coboundary δF in the ε-neighborhoods of all the rays of the partition can be estimated from above by an iterated exponential:

$$|\delta F - \mathrm{id}| < \exp(-C \exp \xi)$$

for some $C > 0$.

5. The cochain F may be expanded in an asymptotic Dulac series in its domain, including the extended components mentioned in item 2.

The properties listed above for normalizing cochains of class $\mathcal{NC}_k$ become these properties under conjugation by the affine mapping $a_k \colon \zeta \mapsto k\zeta - \ln k$. The class $\mathcal{NC}$ may be called *an extended class of the normalizing cochains.*

This is all that we need to know about the mappings of class $\underline{\textbf{TO}}$ in what follows. Denote by $\underline{\textbf{FROM}}$ the set of germs of mappings inverse to the germs in the class $\underline{\textbf{TO}}$.

THEOREM. *An arbitrary finite composition of restrictions to* $(\mathbb{R}^+, \infty)$ *of germs in the classes* $\underline{\textbf{TO}}$, $\underline{\textbf{FROM}}$, *and* $\mathcal{R}$ *either is the identity or does not have fixed points on* $(\mathbb{R}^+, \infty)$.

This theorem is proved below. Its proof is purely a matter of complex analysis. We proceed to an investigation of the compositions described in it.

§ 0.6. Structural theorem and class of a monodromy transformation

A. Preliminary structural theorem. In the previous section we obtained the following relult.

THEOREM. *A monodromy transformation of an elementary polycycle may be decomposed in a composition of almost regular germs and germs of classes* $\underline{\textbf{TO}}$, $\underline{\textbf{FROM}}$ *described in the previous section.*

This result summarizes the facts that we need from the local theory of differential equations.

In this section we will structurize the compositions mentioned in the theorem above in order to prepare them to the future study started in Chapter 1.

B. The composition characteristic and class of a monodromy transformation. Definitions of §0.3 may be easily modified for compositions where the classes **TO** and **FROM** are substituted by $\underline{\textbf{TO}}$ and $\underline{\textbf{FROM}}$. This provides a definition of the characteristic χ_Δ of a composition Δ, balanced and unbalanced composition, and an analog of Lemma 5 from §0.2.

DEFINITION 1. A class of a composition Δ is the oscillation of its characteristic function:

$$(\text{class of } \Delta) = -\min \chi_\Delta,$$

provided that the semitransversal is properly chosen, that is, $\chi_\Delta \leq 0$.

The class of a composition is a major parameter of induction used throughout the whole book. It is denoted by n. For $n = 0$, the polycycle is hyperbolic, and the Finiteness Theorem for such a polycycle is proved in §0.3. For $n > 0$, the proof goes by induction in n. The induction step from 0 to 1 is proceeded separately in Part 1. On one hand, it is much simpler than the induction step from $n-1$ to n proceeded in Part 2. On the other hand, it follows the same lines, as the general induction step. Thus Part 1 serves as an elementary prototype for Part 2. Yet Part 2 is formally independent on Part 1, and may be read right after Chapter 0.

C. Elementary properties of compositions of class n. Before stating the properties mentioned in the heading, we recall some notation and introduce some. If $\mathcal{A}_1, \dots, \mathcal{A}_m$ are subsets of some group, then $\mathrm{Gr}(\mathcal{A}_1, \dots, \mathcal{A}_m)$ and $\mathrm{Gr}_+(\mathcal{A}_1, \dots, \mathcal{A}_m)$ denote the group and semigroup, respectively, generated by them. The set of all products $a_m \circ \cdots \circ a_1$, $a_j \in \mathcal{A}_j$, is denoted by $\mathcal{A}_m \circ \cdots \circ \mathcal{A}_1$. Let:

$$\mathrm{Ad}(f)g = f^{-1} \circ g \circ f, \quad Ag = \mathrm{Ad}(\exp)g,$$
$$A^{-1}g = \mathrm{Ad}(\ln)g.$$

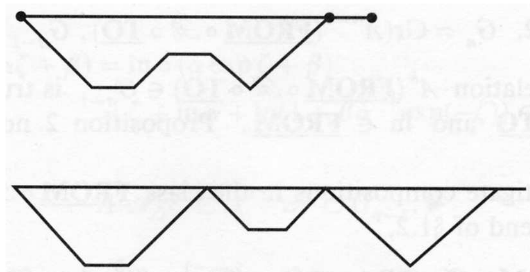


FIGURE 16 f14

If $\mathcal{A}$ and $\mathcal{B}$ are subsets of some group, then

$$\mathrm{Ad}(\mathcal{A})\mathcal{B} = \mathrm{Gr}(\mathrm{Ad}(a)b \mid a \in \mathcal{A},\, b \in \mathcal{B}).$$

If $\mathcal{B}$ is a normal subgroup of the group $G = \mathrm{Gr}(\mathcal{A}, \mathcal{B})$, then any element $g \in G$ can be represented in the form $g = ba$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$. In this case $G = \mathcal{B} \circ \mathcal{A}$. Let the classes $\mathcal{R}$, $\mathcal{R}^0$, and $\mathcal{A}\!f\!f$ be the same as in §1.2 CCC. Then $\mathcal{R} = \mathcal{R}^0 \circ \mathcal{A}\!f\!f$.

We now investigate the simplest properties of compositions of class n. Denote the group of all such compositions by G_n.

PROPOSITION 1. $G_n = \mathrm{Gr}(A^k(\underline{\mathbf{FROM}} \circ \mathcal{R} \circ \underline{\mathbf{TO}}), \mathcal{R} \mid 1 \le k \le n-1)$.

PROOF. Let us modify the characteristic of a mapping $g \in G_n$ by adding the compositions $\exp^{[k]} \circ \ln^{[k]}$ as shown in Figure 16. Such a modification is possible because $\exp \in \underline{\mathbf{TO}}$, $\ln \in \underline{\mathbf{FROM}}$, and the composition is balanced. We get the composition on the right-hand side of the equality in Proposition 1. □

prop:gn PROPOSITION 2.

$$G_n = \mathrm{Gr}(A^{n-1}(\underline{\mathbf{FROM}} \circ \mathcal{R} \circ \underline{\mathbf{TO}}), G_{n-1}).$$

PROOF. The relation $A^k(\underline{\mathbf{FROM}} \circ \mathcal{R} \circ \underline{\mathbf{TO}}) \in G_{n-1}$ is true for $k \le n-2$ because $\exp \in \underline{\mathbf{TO}}$ and $\ln \in \underline{\mathbf{FROM}}$. Proposition 2 now follows from Proposition 1. □

prop:aff PROPOSITION 3.

$$A(\mathcal{A}\!f\!f) \subset \mathcal{R}; \quad A^n(\mathcal{A}\!f\!f) \subset G_{n-1}.$$

PROOF.

$$A(\alpha\zeta) = \zeta + \ln\alpha \in \mathcal{R},$$

$$A(\zeta + \beta) = \zeta + \ln(1 + \beta\exp(-\zeta)) \in \mathcal{R}^0.$$

Moreover,

$$A^n \mathcal{A}\!f\!f \subset A^{n-1}\mathcal{R} \subset G_{n-1}.$$

□

D. Final structural theorem. The composition $\underline{\mathbf{FROM}} \circ \mathcal{R} \circ \underline{\mathbf{TO}}$ has the form

$$\underline{\mathbf{FROM}} \circ \mathcal{R} \circ \underline{\mathbf{TO}} = \mathcal{A}\!f\!f \circ \mathcal{NC}^{-1} \circ \mathcal{R}^0 \circ A(\mathcal{H} \circ \mathcal{R} \circ \mathcal{H}) \circ R^0 \circ \mathcal{NC} \circ \mathcal{A}\!f\!f.$$

We want now to separate the normalizing cochain in this composition from the factor in the set $A(\mathcal{H} \circ \mathcal{R} \circ \mathcal{H})$. For this we need the following definitions.

DEFINITION 2. $\mathcal{A}^0 = \mathrm{Gr}(f \in \mathcal{R}^0 \circ \mathcal{NC} \mid \text{there exists a } \tilde{g} \in \mathcal{H}\colon A\tilde{g} \circ f \text{ is real})$.

Note that, by definition,

$$\underline{\mathbf{TO}} \subset \mathcal{H} \circ \exp \circ \mathcal{A}^0 \circ \mathcal{A}\!f\!f.$$

Indeed, if $\Delta \in \underline{\mathbf{TO}}$, then the germ of Δ is real on $(\mathbb{R}^+, \infty)$, and

$$\Delta = g \circ \exp \circ h \circ F \circ a, \quad g \in \mathcal{H},\ h \in \mathcal{R}^0,\ F \in \mathcal{NC},\ a \in \mathcal{A}\!f\!f.$$

But the composition $\ln \circ \Delta \circ a^{-1}$ is real, and equal to $(Ag) \circ f$, $f = h \circ F$. Consequently, $f \in \mathcal{A}^0$.

The class $\mathcal{H}$ is not in the class $\mathcal{R}$, because the germs of class $\mathcal{R}$ are always real on $(\mathbb{R}^+, \infty)$, while those of class $\mathcal{H}$ are not always. This consideration, together with the preceding formula, motivates the definition

DEFINITION 3.

$$H^0 = \mathrm{Gr}(\mathcal{H}, \mathcal{R}^0).$$

THEOREM 1 (Final structural theorem).

eqn:strth

$$G_n \subset Gr(A^n H^0,\ A^{n-1}\mathcal{A}^0, G_{n-1}) \cap \mathcal{M}_{\mathbb{R}}. \tag{6.1}$$

PROOF. By Proposition 2 in C, it is sufficient to prove that

eqn:stru

$$A^{n-1}(\underline{\mathbf{FROM}} \circ \mathcal{R} \circ \underline{\mathbf{TO}}) \subset Gr(A^n H^0,\ A^{n-1}\mathcal{A}^0, G_{n-1}) \cap \mathcal{M}_{\mathbb{R}} \tag{6.2}$$

We have:

$$\underline{\mathbf{TO}} \subset \mathcal{H} \circ \exp \circ \mathcal{A}^0 \circ \mathcal{A}\!f\!f.$$

Hence,

$$\underline{\mathbf{FROM}} \circ \mathcal{R} \circ \underline{\mathbf{TO}} \subset \mathcal{A}\!f\!f \circ \mathcal{A}^0 \circ A(\mathcal{H} \circ \mathcal{R} \circ \mathcal{H}) \circ \mathcal{A}^0 \circ \mathcal{A}\!f\!f.$$

But $\mathcal{R} = \mathcal{R}^0 \circ \mathcal{A}\!f\!f$ and $\mathcal{H} = Ad(\mathcal{A}\!f\!f)\mathcal{H}$. Hence,

$$\mathcal{H} \circ \mathcal{R} \circ \mathcal{H} = H^0 \mathcal{A}\!f\!f.$$

Therefore, by Proposition 3 in Subsection C,

$$\underline{\mathbf{FROM}} \circ \mathcal{R} \circ \underline{\mathbf{TO}} \subset \mathcal{A}\!f\!f \circ \mathcal{A}^0 \circ A(H^0) \circ A(\mathcal{A}\!f\!f) \circ \mathcal{A}^0 \circ \mathcal{A}\!f\!f \subset Gr(A(H^0), \mathcal{A}^0, \mathcal{R}).$$

This implies (6.2). □

Final structural theorem is the main result of Chapter 0. Both Parts I and II start with this theorem: Part I deals with the particular case $n = 1$, Part II with the general case.

We end this chapter with a few comments.

E. Rate of decreasing of corrections of monodromy maps of class n. The group in the right hand side of (6.1) contains a germ

$$g_0 = A^n(\text{ id } + \exp(-\zeta)).$$

In fact, the group G_n itself contains this germ. We will not need this fact; it may be proved in the same way as the Proposition in §0.2C.

PROPOSITION 4. *On* $(\mathbb{R}^+, \infty)$

eqn:corr

$$g_0 - \ id \ = (1 + o(1))(\exp(-\xi)\exp(-\exp\xi)...\exp(-\exp^{[n]}\xi). \tag{6.3}$$

The proof goes by induction in n.

Base of induction: $n = 1$ (this is more instructive than the case $n = 0$.

$$A(\xi + \exp(-\xi)) = \ln(\exp\xi + \exp(-\exp\xi)) =$$
$$\xi + \ln(1 + \exp(-\xi)\exp(-\exp\xi)) = \xi + (1 + o(1))\exp(-\xi)\exp(-\exp\xi)).$$

Step of induction: from $n - 1$ to n:

$$A^n(\xi + \exp(-\xi)\exp(-\exp\xi))(-\xi)) =$$
$$A(\xi + (1 + o(1))(\exp(-\xi)\exp(-\exp\xi)...\exp(-\exp^{[n-1]}\xi) =$$
$$\xi + (1 + o(1))(\exp(-\xi)\exp(-\exp\xi)...\exp(-\exp^{[n]}\xi).$$

This proposition shows that the correction of a germ $g \in G_n$ may decrease approximately as $\exp(-\exp^{[n]}\xi)$. Yet it cannot decrease as $\exp(-\exp^{[n+1]}\xi)$. Indeed, we will show that the modulus of a non-zero correction of a germ $g \in G^n$ is bounded from below by a multiple exponent of the form $\exp(-\exp^{[n]}\varepsilon\xi)$ for some $\varepsilon > 0$.

§ 0.7. Historical comments

For almost sixty years the finiteness problem was regarded as solved. Dulac's 1923 memoir [**?**] devoted to it was translated into Russian and published as a separate book in 1980. The first doubts as to the completeness of Dulac's proof were apparently expressed by Dumortier: in a report at the Bourbaki seminar [**?**] Moussu referred to a private communication from Dumortier in 1977. In the summer of 1981 Moussu sent to specialists letters in which he asked whether they regarded Dulac's assertion about finiteness of limit cycles as proved. Two month earlier the author of these lines had found a mistake in the memoir (see [**?**], [**?**]) and mentioned this in a reply to Moussu's letter. An up-to-date presentation of the main true result in Dulac's memoir and an analysis of his mistake are sketched briefly in [**?**] and [**?**] and given in detail in [**?**].

We mention that the greatest difficulties overcome in the memoir are related to the local theory of differential equations not for analytic vector fields, as might be assumed from the context, but for infinitely smooth vector fields, and these difficulties were connected with the description of correspondence mappings for hyperbolic sectors of elementary singular points. The investigation of compositions of these mappings that leads to the appearance of asymptotic Dulac series is then carried out in an elementary manner. The first part of Dulac's memoir concerns monodromy transformations of polycycles with nondegenerate elementary singular points, and the second part those of polycycles with arbitrary elementary singular points (degenerate ones are added). In the third part the application of resolution of singularities to the investigation of nonelementary polycycles is discussed; in the fourth part polycycles consisting of one singular point are considered. The complexity of the last two parts is due to the fact that they are based on a theorem on resolution of singularities that was proved only forty-five years later [**?**].

After Dumortier's detailed study of resolution of singularities of vector fields in 1977, the arguments in the last two parts of Dulac's memoir became commonplace; they are given a few lines in the survey [**?**] (see §0.1C). The difficulties connected with the first part of the memoir [**?**] are overcome by going out into the complex plane, see §?? above. Thus, all the papers [**?**], [**?**], [**?**], and [**?**] written in the last five years on the finiteness problem have overcome in more or less explicit form the difficulties that were not overcome in the second part of the memoir.

Correspondence mappings for degenerate elementary singular points were thoroughly studied in [**?**] and [**?**] from the point of view of their extension into the complex domain. The only difficulty consists in the investigation of compositions of these and almost regular mappings. To handle this difficulty it was necessary to develop a calculus of "functional cochains" and of "superexact asymptotic series." All subsequent work is devoted to the investigation of the indicated compositions. The main ideas in the present book are presented in [**?**], where Theorem A in subsection 4A is proved for compositions of germs in the classes $\mathcal{R}$, **TO**, and **FROM** in which germs in **TO** and **FROM** alternate.

The geometric theory of normal forms of resonant vector fields and mappings began to develop in parallel in Moscow and France with work of Ecalle [**?**] and Voronin [**?**] (see also [**?**], [**?**], [**?**]). The first steps in this theory were independent of the finiteness problem and taken before it was realized to be open.

Here I take the liberty of presenting a reminiscence that can be called a parable on the connection between form and substance. In December 1981 I made a report at a session of the Moscow Mathematical Society devoted to two questions that seemed to me to be independent of each other: the Dulac problem and the Ecalle–Voronin theory. Having to motivate the combination of the two parts in a single report, I improvised the following phrase: "Dulac's theorem shows what the smooth theory of normal forms gives for the investigation of the finiteness theorem. This theory cannot give a definitive proof. To obtain such a proof it is necessary to investigate the analytic classification of elementary singular points." In uttering the phrase, which originated there at the blackboard, I understood that this was not a pedagogical device, but a program of investigation. The first formulation of Theorem 2 in §0.4 was given in Leningrad at the International Topological Conference in 1982 [**?**], and a proof was published in [**?**].

The proofs below of the finiteness theorems make essential use of the geometric theory of normal forms (§0.4). The finiteness theorem was announced in [**?**].

Using the theory of resurgent functions he created in connection with local problems of analysis, Ecalle developed the approach of the four authors of [**?**] and obtained in parallel independent proofs of all the finiteness theorems stated in §0.1. At the time of writing this both proofs (those of Ecalle and of the author) exist as manuscripts.

def

CHAPTER 1

Decomposition of a Monodromy Transformation into Terms with Noncomparable Rates of Decrease

In the first part of this chapter we give an axiomatic description of the basic concepts: germs of regular functional cochains, map-cochains (RROK)[1] and superexact asymptotic series (STAR). The finiteness theorems are derived from these axioms. In the second part of the chapter we construct a model for these axioms. In the rest of the text, Chapters $2-5$, we justify the axioms for the model constructed.

We begin with the general concept of functional cochains , for which the normalizing class of cochains is a particular case.

S 1.1. Functional cochains and map-cochains

Let $\Omega \subset \mathbb{C}$ be an arbitrary domain, and Ξ a locally finite partition of it into analytic polyhedra: each domain of the partition is the closure of an open set given by finitely many inequalities of the form $\omega \leq 0$, where ω is a real-analytic function on a subdomain of $\mathbb{R}^2$. A tuple $F = \{f_j\}$ of functions is called a functional cochain corresponding to the partition Ξ if the functions in the tuple are in bijective correspondence with the domains of the partition, and each function extends holomorphically to some neighborhood of its domain of the partition. The partition corresponding to a functional cochain F is denoted by Ξ^F.

The coboundary δF of a functional cochain F is defined to be the tuple of holomorphic functions defined as follows on the boundary lines of the partition: corresponding to an ordered pair of domains of the partition Ξ^F that have the line $\mathcal{L}$ as common border is the germ on of the holomorphic function $f_1 - f_2$, where f_1 and f_2 are the functions in F corresponding to the first and second domains of the pair. The tuple of these germs is called the coboundary of the cochain.

Map-cochains are constructed similarly: in the preceding definition it is necessary only to require that the functions f_j give biholomorphic mappings of the corresponding domains of the partition onto their images, and the difference $f_1 - f_2$ in the definition of the coboundary is replaced by the composition $f_1^{-1} \circ f_2$. It is required of the tuple f that for a pair of domains of the corresponding partition with common border along the line $\mathcal{L}$ the composition $f_1^{-1} \circ f_2$ of the corresponding mappings in the tuple be defined in some neighborhood of $\mathcal{L}$.

The preceding definition gives the difference coboundary of a functional cochain, and the latter one gives the composition coboundary.

Example. Normalizing cochains written in the logarithmic chart form the main example. The role of domain Ω is played by a right halfplane $\mathbb{C}_a^+ : \operatorname{Re} \zeta > a > 0$ for some a. The composition coboundaries of these cochains provide so called

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[1]Once more a Russian abbreviation: Ростки регулярных отображений коцепей.

Ecalle–Voronin moduli of analytic classification of parabolic germs. The corrections of the coboundaris decrease like double exponentials; see S05 for details.

Functional cochains can be added, subtracted, and multiplied. Compositions are considered for map-cochains. The sum of two functional cochains F and G is the functional cochain denoted by $F+G$ and corresponding to the product of the partitions Ξ^F and Ξ^G . This means that to the intersection $\mathcal{D}_1 \cap \mathcal{D}_2$ of two domains of the respective partitions Ξ^F and Ξ^G there corresponds the function f_1+g_1 equal to the sum of the functions in F and G corresponding to $\mathcal{D}_1$ and $\mathcal{D}_2$. Differences and products of functional cochains are defined analogously, as are compositions of map-cochains. Sums, differences, and products of functional cochains on one and the same domain Ω are always defined. The composition $F \circ G$ is not always defined; a sufficient condition for its existence is that there exists a positive number ε such that the mappings in F extend to the ε-neighborhoods of the corresponding domains of the partition, and the correction $G - \mathrm{id}$ of G is less than ε in modulus. In what follows, all the partitions will have a boundary line $(\mathbb{R}^+, \infty)$. A component of a cochain F that corresponds to a domain of partition adjacent to $(\mathbb{R}^+, \infty)$ from above (from below) is denoted by F^u (respectively, F^l); u stand for *upper*, l for *lower*.

The domains of all the map-cochains defined below are open and contain $(\mathbb{R}^+, \infty)$. Hence, the compositions of map-cochains are defined at least in some neighborhoods of $(\mathbb{R}^+, \infty)$.

S 1.2. Extending the group of the monodromy transformations of class n

sec:exmon

In this section we will extend the group G_n named in the heading in such a way that elements of the extended group G^n might be represented according to the Additive Decomposition Theorem below. This representation will be used to prove that the corrections of the germs of the group G^n do not oscillate.

The group G^0 is the group $\mathcal{R}$ of almost regular germs. It will be used as a base of induction in n (the class of the composition).

The group G^1 is also constructed in Part 1. This construction is merely the induction step from 0 to 1. Here we proceed the step from $n-1$ to n, and the construction of Part 1 is a particular case of this procedure.

Fix n and assume, by induction, that all the groups $G^m,\ m<n$ are constructed. We will then construct the group G^n, and prove various properties of this group. At the same time, we will prove the same properties for the group G^0 (base of induction). We will make an induction hypothesis that all these properties hold for the groups $G^m,\ m<n$.

Recall that $\mathcal{H}$ is a group of all parabolic germs of conformal mappings at zero written in the logarithmic chart. Recall also that

$$H^0 = Gr(\mathcal{R}^0, \mathcal{H}),$$

$$\mathcal{A}^0 = \mathrm{Gr}(f \in \mathcal{R}^0 \circ \mathcal{NC} \mid \text{there exists a } \tilde{g} \in \mathcal{H}\colon A\tilde{g} \circ f \text{ is real}).$$

According to the Final structural theorem,

$$G_n \subset Gr(A^n H^0,\ A^{n-1}\mathcal{A}^0, G_{n-1}).$$

By induction assumption, the group $G^{n-1} \supset G_{n-1}$ is constructed. Consider a normal subgroup generated by $A^{n-1}\mathcal{A}^0$ in $Gr(A^{n-1}\mathcal{A}^0, G_{n-1})$:

eqn:jn1
$$J^{n-1} = \mathrm{Ad}(G^{n-1})A^{n-1}\mathcal{A}^0. \tag{2.0}$$

We will construct a set of functional cochains of class n that consists of two subsets $\mathcal{FC}_0^n$ and $\mathcal{FC}_1^{n-1}$, called cochains of class n and type zero (class n and type 1 respectively). We will start with some properties of these cochains taken as axioms. Later on we will built a model for these axioms. That is, we will define the cochains of these classes explicitly, and then justify that the model satisfies the axioms stated above.

Note that cochains of class 1 are already defined in Part 1: $\mathcal{FC}_0^1$ and $\mathcal{FC}_1^0$ are simple and sectorial cochains respectively.

Cochains that decrease exponentially in their domain are called *fastly decreasing*; this property is marked by adding a plus sing as a right subscript.

Everywhere below, a superscript in square brackets denotes the corresponding composition power of the germ of a diffeomorphism:

$$f^{[k]} = f \circ f \circ \cdots \circ f \quad (k \text{ times}).$$

An exception is the notation for the germ of an inverse mapping: we write f^{-1} instead of $f^{[-1]}$.

The first axiom is:

eqn:ax0
$$A^n H^0 \subset \mathrm{id}\ + \mathcal{FC}_{0^+}^n \circ \exp^{[n]} \tag{2.0}$$

eqn:ax1
$$J^{n-1} \subset Gr(\mathrm{id}\ + \mathcal{FC}_{1^+}^{n-1} \circ \exp^{[n-1]} \circ g | g \in G^{n-2}). \tag{2.0}$$

The second statement of this axiom will be repeated later as a part of another axiom called The Fourth Shift Lemma.

Denote by H^n the following group:

$$H^n = Gr(id + \mathcal{FC}_{0^+}^n \circ \exp^{[n]} \circ g | g \in G^{n-1}).$$

Let

$$G^n = Gr(H^n, J^{n-1}, G^{n-1}) \cap M_{\mathbb{R}}.$$

Obviously,

$$G^n \supset G_n.$$

This completes the construction of the group G^n that will be studied all over the rest of the book.

Before stating further axioms, we need some preparations. They are made in the next section.

S 1.3. Multiplicatively Archimedean classes and proper groups

In this section we describe a grading of functions according to rate of decrease that arises naturally in the study of compositions of correspondence mappings. We also introduce proper groups.

A. Classes of Archimedean equivalence.

Definition 1. A subset of the set of all germs of functions $(\mathbb{R}^+, \infty) \to \mathbb{R}$ is ordered according to growth if the difference of any two germs in this set is a germ of constant sign:

$$f \succ g \iff f - g \succ 0.$$

The sign $\succ$ is used as the "greater than" sign for germs: $f - g \succ 0$ if and only if there exists a representative of the germ $f - g$ that is positive on the whole domain of definition.

Definition 2. Two germs of functions f and g carrying $(\mathbb{R}^+, \infty)$ into $\mathbb{R}$ are said to be multiplicatively Archimedean-equivalent if the ratio of the logarithms of the moduli of these germs is bounded and bounded away from zero. In the language of formulas, $f \sim g$ if and only if there exist c and C such that

$$0 < c < \big|\ln|f|/\ln|g|\big| < C.$$

This is clearly an equivalence relation. A class of multiplicatively Archimedean-equivalent germs is called an Archimedean equivalence class.

Examples. The germs ξ, 2, $\exp\xi$, $\exp(-\exp\mu\xi)$, $\exp(-\exp^{[2]}(\xi + C))$, and $\exp(-\exp^{[k]}\xi)$ are pairwise multiplicatively Archimedean-nonequivalent for different values of $\mu > 0$, C, and k. The germs $\exp\mu\xi$ and $\exp\nu\xi$, as well as $\exp(-\exp(\xi+\alpha))$ and $\exp(-\exp(\xi+\beta))$, are multiplicatively Archimedean-equivalent for arbitrary real α and β and arbitrary real nonzero μ and ν.

B. Proper groups.

Definition 3. A group of germs of diffeomorphisms $(\mathbb{R}^+, \infty) \to (\mathbb{R}^+, \infty)$ is said to be ordered if it is ordered in the sense of Definition 1.

def:proper

Definition 4. A group of germs of diffeomorphisms $(\mathbb{R}^+, \infty) \to (\mathbb{R}^+, \infty)$ is said to be k-proper if:

1°. the germs of the group differ from linear germs by a bounded correction;

2°. the group contains the germ $A^k(\mu\xi)$ for an arbitrary $\mu > 0$;

3°. the group is ordered.

Examples. 1. The group $G^0 = \mathcal{R}$ is 0-proper. (G^0 is, in fact, 2-proper, but this will not be used). Indeed, requirement 1° follows from the decomposability of the germ into a Dulac series. Requirement 2° simply means that real linear germs are almost regular. Requirement 3° was established in S0.3 with the help of the Phragmen–Lindelof theorem.

2. The main example: for any m, the group G^m is m-proper. For $m = 0$ this is just proved. For $m < n$ this is the induction assumption used throughout the book. For $m = n$ this is proved in S1.5.

In S1.6 we prove (modulo some auxiliary statements that are proved later) that the corrections of monodromy transformations belong to Archimedean classes that can be obtained with the help of the following construction.

Let G be an arbitrary k-proper group. For any $g \in G$ we denote by $\mathcal{A}_g^k$ the Archimedean class of the germ $\exp(-\exp^{[k]} \circ g)$. Let: $\mathcal{A}_G^k = \{\mathcal{A}_g^k \mid g \in G\}$

Examples. 1. The set $\mathcal{A}^0_{\mathcal{A}\!f\!f}$ consists of the unique Archimedean class with representative $\exp\xi$.

2. The set $\mathcal{A}^1_{\mathcal{R}}$ consists of the Archimedean classes of germs

$$f_\mu = \exp(-\exp \mu\xi), \quad \mu > 0,$$

and only of them.

Indeed, let $g \in \mathcal{R}$ be an arbitrary almost regular germ. Then there exist positive constants μ and C such that

$$\mu(\xi - C) \prec g \prec \mu(\xi + C).$$

Consequently, setting $a = \exp(-\mu C)$ and $b = \exp(\mu C)$, we get that

$$(f_\mu)^a = f_\mu \circ (\xi - C) \prec \exp(-\exp g) \prec f_\mu \circ (\xi + C) = (f_\mu)^b.$$

This means that the germs f_μ and $\exp(-\exp g)$ are multiplicatively Archimedean-equivalent.

The properties of the Archimedean classes constructed are used repeatedly in what follows.

C. Generalized multipliers and special subsets of proper groups. The main role in comparison of Archimedean classes in the set $\mathcal{A}^k_G$ (the group G is k-proper) is played by the compositions $A^{-k}g$, $g \in G$; see Proposition 2 below. We begin with a study of these compositions.

DEFINITION 5. For any germ $g : (\mathbb{R}^+, \infty) \to (\mathbb{R}^+, \infty)$, the generalized multiplier of order k, $\lambda_k(g)$ is defined as

$$\lambda_k(g) = \lim_{(\mathbb{R}^+,\infty)} A^{-k}g/\xi,$$

provided that the limit exists.

PROPOSITION 1. *Suppose that G is a k-proper group. Then for any germ $g \in G$ there exists a generalized multiplier of order k, zero, positive, or infinite.*

PROOF. Let f and g be arbitrary germs in the group G. Since f and g are germs of diffeomorphisms, the following relations are equivalent:

$$f \prec g \quad \text{and} \quad A^{-k}f \prec A^{-k}g.$$

Consequently, for any $\lambda > 0$ one of the following three relations holds:

$$A^k(\lambda\xi) \prec g, \quad A^k(\lambda\xi) = g, \quad A^k(\lambda\xi) \succ g,$$

or, what is equivalent,

$$\lambda\xi \prec A^{-k}g, \quad \lambda\xi = A^{-k}g, \quad \lambda\xi \succ A^{-k}g. \qquad \square$$

Therefore,

$$\sup\{\lambda \geq 0 \mid \lambda\xi \prec A^{-k}g\} = \lim_{(\mathbb{R}^+,\infty)} A^{-k}g/\xi = \lambda_k(g).$$

REMARK 1. When the group G is given, and the number k is fixed, for which the group G is k-proper, then the following mapping is defined:

$$\lambda_k \colon G \to 0 \cup \mathbb{R}^+ \cup \infty, \quad g \mapsto \lambda_k(g).$$

Let

$$\lambda_k^{-1}(0) = G^-_{\text{slow}}, \quad \lambda_k^{-1}(\mathbb{R}^+) = G_{\text{rap}}, \; \lambda_k^{-1}(\infty) = G^+_{\text{slow}},$$
$$G_{\text{slow}} = G \setminus G_{\text{rap}} = G^-_{\text{slow}} \cup G^+_{\text{slow}}.$$

In other words, G^-_{slow}, G_{rap}, and G^+_{slow} are the subsets of G consisting of those germs g such that the composition $A^{-k}g$ increases more slowly than any linear germ, like a linear germ, and more rapidly than any linear germ, respectively.

REMARK 2. Obviously, G_{rap} is a group, while G^-_{slow} and G^+_{slow} are semigroups. The designations rap (rapid) and slow indicate the rapidity and slowness of decrease of the corrections. Moreover, $G^{\pm}_{\mathrm{slow}} \circ G_{\mathrm{rap}} = G^{\pm}_{\mathrm{slow}}$.

EXAMPLES. 1. For the 0-proper group G^0,

$$\lambda_0(g) = \lim_{(\mathbb{R}^+,\infty)} \frac{g(\xi)}{\xi}$$

exists and is finite.

2. In what follows, we always consider the generalized multipliers of the k-th order for the elements of the groups G^k. This gives rise to the group G^k_{rap} and to semigroups $G^{k\pm}_{\mathrm{slow}}$.

D. Archimedean classes corresponding to a proper group. The following three propositions enable us to compare germs in Archimedean classes belonging to the sets $\mathcal{A}^k_G$ for the same or different k and G.

PROPOSITION 2. *Suppose that G is a k-proper group, and $f, g \in G$. Then the Archimedean classes $\mathcal{A}^k_f$ and $\mathcal{A}^k_g$ coincide if and only if $h = f \circ g^{-1} \in G_{\mathrm{rap}}$. In the language of formulas,*

$$\mathcal{A}^k_f \equiv \mathcal{A}^k_g \iff f \circ g^{-1} \in G_{\mathrm{rap}}.$$

PROOF. The germs $\varphi = \exp(-\exp^{[k]} \circ f)$ and $\psi = \exp(-\exp^{[k]} \circ g)$ are multiplicatively Archimedean equivalent if and only if the analogous equivalence holds for the germs

e:1.4-1
$$\exp(-A^{-k}h) \quad \text{and} \quad \exp(-\xi), \quad \text{where } h = f \circ g^{-1}. \tag{3.1}$$

Let $h \in G_{\mathrm{rap}}$. Then by the definition of the group G_{rap}, there exists $\lambda = \lambda_k(g)$ such that $A^{-k}h = (\lambda + o(1))\xi$.

This implies the multiplicative Archimedean equivalence of the germs in (3.1).

Conversely, if the germs in (3.1) are equivalent, then the germ $A^{-k}h$ does not increase more rapidly nor more slowly than a linear germ, that is, $h \in G_{\mathrm{rap}}$. $\square$

DEFINITION 6. Let G be a k-proper group, and let $f, g \in G$. We say that $f \prec\!\!\prec g$ in G if $f \circ g^{-1} \in G^-_{\mathrm{slow}}$.

PROPOSITION 3. *Suppose that G is a k-proper group, $f, g \in G$, $f \prec\!\!\prec g$, $\varphi \in \mathcal{A}^k_f$, $\psi \in \mathcal{A}^k_g$, and $\varphi \to 0$ and $\psi \to 0$ on $(\mathbb{R}^+, \infty)$. Then $|\psi| \prec |\varphi^\lambda|$ on $(\mathbb{R}^+, \infty)$ for any $\lambda > 0$.*

PROOF. By definition, the germs $|\varphi|$ and $|\psi|$ belong to the Archimedean classes of the germs $\tilde{\varphi} = \exp(-\exp^{[k]} \circ f)$ and $\tilde{\psi} = \exp(-\exp^{[k]} \circ g)$, respectively, and can be estimated from above and from below by positive powers of them. Therefore, it suffices to prove that for any $\lambda > 0$

$$\tilde{\psi} \prec \tilde{\varphi}^\lambda.$$

This is equivalent to the inequality $\exp(-\xi) \prec \exp(-\lambda A^{-k}h)$, where $h = f \circ g^{-1}$.

The last inequality follows from the fact that the germ h increases at infinity more slowly than any linear germ, by the definition of the semigroup G^-_{slow}. $\square$

PROPOSITION 4. *Suppose that G and $\tilde{G}$ are k- and m-proper groups, respectively, $k < m$, $f \in G$, $g \in \tilde{G}$, $\varphi \in \mathcal{A}_f^k$, $\psi \in \mathcal{A}_g^m$, and $\varphi \to 0$ and $\psi \to 0$ on $(\mathbb{R}^+, \infty)$. Then $|\psi| \prec |\varphi^\lambda|$ on $(\mathbb{R}^+, \infty)$ for any $\lambda > 0$.*

PROOF. Let $\tilde{\varphi} = \exp(-\exp^{[k]} \circ f), \quad \psi = \exp(-\exp^{[m]} \circ g)$.

As above, the germs φ and $\tilde{\varphi}$, as well as ψ and $\tilde{\psi}$, are multiplicatively Archimedean-equivalent. Therefore, as above, it suffices to prove that $\tilde{\psi} \prec \tilde{\varphi}^\lambda$ for any $\lambda > 0$.

This is equivalent to the inequality

e:1.4-2

$$\exp(-\sigma) \prec \exp(-\lambda\xi) \tag{3.2}$$

for any $\lambda > 0$, where

$$\sigma = \exp^{[m-k]} \circ A^{-k} h, \quad h = g \circ f^{-1}.$$

PROPOSITION 5. *The germ σ defined above increases on $(\mathbb{R}^+, \infty)$ more rapidly than any linear germ.*

PROOF. It suffices to prove the proposition for $m - k = 1$; further compositions with an exponential only increase the growth. Accordingly, let $\sigma = \exp \circ A^{-k} h$. We prove that for any $C > 0$,

e:1.4-3

$$A^{-k} h \succ C \ln \xi \tag{3.3}$$

This inequality can be proved by induction on k.

INDUCTION BASE: $k = 0$. Requirement 1 in the definition of a proper group is used here; it implies that the germs f and g differ from a linear germ by a bounded correction. Consequently, the germ h has the same property. Therefore, $h \succ \varepsilon\xi$ for some $\varepsilon > 0$. This gives the induction base: the inequality (3.3) for $k = 0$.

INDUCTION STEP. Suppose that the inequality (3.3) has been proved for some k. Let us prove it for $k + 1$. We have that

$$A^{-(k+1)} h \succ A^{-1}(2 \ln \xi) = \exp(2 \circ \ln \circ \ln \xi) = (\ln \xi)^2 \succ C \ln \xi$$

for arbitrary $C > 0$. The inequality (3.3) is proved.

Consequently, for arbitrary $C > 0$,

$$\sigma \succ \exp \circ C \ln \xi = \xi^C. \qquad \square$$

The inequality (3.2), and with it Proposition 4, follows immediately from Proposition 5. $\square$

sec:ax

S 1.4. Axiomatic description of functional cochains of class n

A. Strategy. Our goal is to decompose a non-identical monodromy transformation of an elementary polycycle of a planar analytic vector field (in what follows, simply monodromy transformation) into a sum

$$\Delta = \mathrm{id} + \varphi + \psi$$

such that ψ decreases faster than φ, and φ is not oscillating. The precise statement is the Additive Decomposition Theorem (ADT) below.

The terms φ and ψ are expressed through so called cochains of class n. As mentioned in Section 1.2, there are two types of these cochains denoted by $\mathcal{FC}_1^{n-1}$ and $\mathcal{FC}_0^n$. The explicit definitions of these cochains are lengthy; we postpone them until the second part of the chapter. In a few subsections to come we give an axiomatic definition of these cochains.

B. Shift lemmas. The cochain F that decreases exponentially fast in its domain (alwais included in the right half plane):

$$|F(\zeta)| \prec \exp(-\xi)$$

is called *rapidly decreasing*; the sets of these cochains in $\mathcal{FC}_1^{n-1}$ and $\mathcal{FC}_0^n$ are denoted by $\mathcal{FC}_{1+}^{n-1}$ and $\mathcal{FC}_{0+}^n$ respectively. Equalities with plus in brackets mean that the equality holds with plus (then with no brackets), as well as without plus.

LEMMA. $SL1_n$

$$a)\mathcal{FC}_{0(+)}^n \circ \exp^{[n]} \circ G_{rap}^{n-1} = \mathcal{FC}_{0(+)}^n \circ \exp^{[n]}.$$

$$b)\mathcal{FC}_{1(+)}^{n-1} \circ \exp^{[n-1]} \circ G_{rap}^{n-2} = \mathcal{FC}_{1(+)}^{n-1} \circ \exp^{[n-1]}.$$

Convention. Let n be fixed, and $1 \le m \le n$. Then $\mathcal{FC}^m$ stands for $\mathcal{FC}_1^m$ if $m \le n-1$, and $\mathcal{FC}_0^n$ if $m = n$.

Let $m \le n,\ g \in G^{n-1}$. Denote

$$\mathcal{F}_{(+)g}^m = \mathcal{FC}_{(+)}^m \circ \exp^{[m]} \circ g.$$

According to the above Convention, this means that

$$\mathcal{F}_{1(+)g}^m = \mathcal{FC}_{1(+)}^m \circ \exp^{[m]} \circ g,\ m \le n-1;$$

$$\mathcal{F}_{0(+)g}^n = \mathcal{FC}_{0(+)}^n \circ \exp^{[n]} \circ g.$$

Again, according to the above convention, $SL1_n$ takes the form

LEMMA. $SL1_n$ *Let $m = n-1$ or $n,\ g \in G_{rap}^{m-1}$. Then*

$$\mathcal{F}_{(+)g}^m = \mathcal{FC}_{(+)id}^m.$$

We precede the formulation of the next lemma by

DEFINITION 1. a. If f and g belong to G^k, then $f \prec\!\!\prec g$ in G^k if and only if $f \circ g^{-1} \in G_{\text{slow}}^{k^-}$.

b. $(k,f) \prec (m,g)$ if and only if $f \in G^{k-1}$, $g \in G^{m-1}$, and either $k < m$, or $k = m$ and $f \prec\!\!\prec g$ in G^{m-1}.

THE SECOND SHIFT LEMMA, SL 2_n. *Let $m = n-1$ or $m = n$, and suppose that $(k,f) \prec (m,g)$ and $\varphi \in \mathcal{F}_f^k$. Then*

$$\varphi \circ (\mathrm{id} + \mathcal{F}_{+g}^m) \subset \varphi + \mathcal{F}_{+g}^m.$$

THE THIRD SHIFT LEMMA, SL 3_n. a. *Let $m = n-1$ or $m = n$, and suppose that $f \succ\!\!\succ g$ in G^{m-1} or $f \circ g^{-1} \in G_{\text{rap}}^{m-1}$. Then*

$$\mathcal{F}_{(+)f}^m \circ (\mathrm{id} + \mathcal{F}_{+g}^m) \subset \mathcal{F}_{(+)f}^m.$$

b. $(\mathrm{id} + \mathcal{F}_{+g}^m)^{-1} = \mathrm{id} + \mathcal{F}_{+g}^m$ *for an arbitrary $g \in G^{m-1}$.*

THE FOURTH SHIFT LEMMA, SL 4_n. a. $J^{n-1} \subset \mathrm{Gr}(\mathrm{id} + \mathcal{F}_{1+}^{n-1})$.

b. $\mathcal{F}_{0(+)g}^n \circ J^{n-1} \subset \mathcal{F}_{0(+)g}^n$.

We emphasize once more that, by the induction hypothesis, all these lemmas are assumed to be proved for $1 \leq m \leq n-1$ (n a positive integer), as are Theorems MDT_m and ADT_m stated below. The induction base–proofs of the lemmas for $m = 1$–is contained in Part 1.ccc!

C. Weak realness and lower estimate.

Definition 2. A functional cochain is said to be weakly real if the corresponding partition contains the ray $(\mathbb{R}^+, \infty)$ in its boundary, the domains of the partition adjacent to $\mathbb{R}$ are mutually symmetric with respect to $\mathbb{R}$, and

$$F^u(\bar{\zeta}) = \overline{F^l(\zeta)}$$

A composition

$$\varphi \in \mathcal{F}_f^k,\ \varphi = F \circ \exp^{[k]} \circ f,\ F \in \mathcal{FC}^k,\ f \in G^{k-1},\ k \leq n$$

is said to be weakly real if F is weakly real.

If we replace a cochain by a holomorphic function, that is, F^u and F^l are analytic extensions of one another, then the previous definition simply means that $F^u \equiv F^l$ is real on $\mathbb{R}$.

Theorem 1 (Lower Estimate Theorem, LET_n). *Let $m = n-1$ or $m = n$, $F \in \mathcal{FC}^m$, and F is weakly real. Then there exists $\nu > 0$ such that*

$$|ReF| \succ \exp(-\nu\xi) \quad on \quad (\mathbb{R}^+, \infty).$$

Denote by S the symmetry operator $S : F \to SF,\ SF(\zeta) = \overline{F(\bar{\zeta})}$. Let $IF = F - SF$. A cochain F is weakly real iff $IF \equiv 0$ on $(\mathbb{R}^+, \infty)$.

Symmetry axiom. If $F \in \mathcal{FC}^k,\ k \leq n$, then $SF \in \mathcal{FC}^k$. Hence, if $\varphi \in \mathcal{F}_f^k$, then $S\varphi \in \mathcal{F}_f^k$.

D. Phragmen–Lindelof theorem for cochains. Let $m = n-1$ or $m = n$, $F \in \mathcal{FC}^m$. Let F decrease on $(\mathbb{R}^+, \infty)$ faster than any exponential:

$$|F(\xi)| \prec \exp(-\nu\xi) \text{ on } (\mathbb{R}^+, \infty) \forall \nu > 0.$$

Then $F^u \equiv 0,\ F^l \equiv 0$.

The theorem also holds for $G = IF$ for any $F \in \mathcal{FC}^m$.

E. Upper bound of the coboundary.

thm:cobound

Theorem 2. *Let $1 < m \leq n,\ F \in \mathcal{FC}^m$ Then*

$$|\delta F| \prec \exp(-\nu\xi) \ on \ (\mathbb{R}^+, \infty) \forall \nu > 0.$$

Moreover, the spaces $\mathcal{FC}^m_{(+)g}$ are linear (this statement will be made precise below).

The four Shift Lemmas above, as well as inclusion (1.2) and the statements of the three previous subsections are taken as axioms that hold for the functional cochains of classes $\mathcal{FC}^m, m < n$. These axioms imply the multiplicative and additive decomposition theorems stated below.

S 1.5. The multiplicative and additive decomposition theorems

Multiplicative Decomposition Theorem, MDT_n. 1°. $G^n = G^{n-1} \circ J^{n-1} \circ H^n$.

2^0. *Let Δ be a monodromy transformation of class n or, more generally, $\Delta \in G^n$. Then*

eqn:mdt

$$\Delta = a \circ \prod(\mathrm{id} + \varphi_j) \circ \prod(\mathrm{id} + \psi_l), \tag{5.1}$$

where

eqn:mdt1

$$\begin{gathered} a \in \mathcal{A}\!f\!f, \\ \varphi_j \in \mathcal{F}^{k_j}_{1+f_j},\ 0 \le k_j \le n-1, \\ \psi_j \in \mathcal{F}^{n}_{0+g_j},\ f_j \in G^{k_j-1},\ g_j \in G^{n-1}, \\ (k_j, f_j) \prec (k_{j+1}, f_{j+1}),\ g_j \prec\!\!\prec g_{j+1} \quad \textit{in } G^{n-1}; \end{gathered} \tag{5.2}$$

According to the Convention in Section 1.4 C, formulas (5.1), (5.2) take the form:

$$\Delta = a \circ \Pi(id + \varphi_j), a \in \mathcal{A}\!f\!f,\ \varphi_j \in \mathcal{F}^{k_j}_{+f_j},\ k_j \le n,\ (k_j, f_j) \prec (k_{j+1}, f_{j+1}).$$

thm:adt

Additive Decomposition Theorem, ADT_n. *Let Δ be a monodromy transformation of class n, or, more generally, $\Delta \in G^n$. Then the following expansion is valid:*

e:1.8-1

$$\begin{gathered} \Delta = a + \sum \varphi_j + \sum \psi_j \\ a \in \mathcal{A}\!f\!f,\ \varphi \in \mathcal{F}^{k_j}_{1+f_j},\ 0 \le k_j \le n-1, \\ \psi_j \in \mathcal{F}^{n}_{0+g_j},\ f_j \in G^{k_j-1},\ g_j \in G^{n-1}, \\ (k_j, f_j) \prec (k_{j+1}, f_{j+1}),\ g_j \prec\!\!\prec g_{j+1} \quad \textit{in } G^{n-1}. \end{gathered} \tag{5.3}$$

Moreover, all the terms given by the formula (5.3) *in the expansion for* $\Delta - \mathrm{id}$ *are weakly real.*

According to the Convention in Section 1.4 C, the ADT_n takes the form:

eqn:adddec

$$\Delta = a + \sum \varphi_j, a \in \mathcal{A}\!f\!f,\ \varphi_j \in \mathcal{F}^{k_j}_{+f_j},\ 0 \le k_j \le n,\ (k_j, f_j) \prec (k_{j+1}, f_{j+1}). \tag{5.4}$$

Remarks. 1. The theorem MDT_n enables us to represent an arbitrary monodromy transformation $\Delta \in G^n$ of class n as a composition

$$\Delta \in \mathcal{A}\!f\!f \circ J^0 \circ (H^1 \circ J^1) \circ \cdots \circ (H^{n-1} \circ J^{n-1}) \circ H^n.$$

The corrections of germs of class $H^k \circ J^k$ decrease no more slowly than $\exp(-\exp^{[k]} \mu\xi)$ on $(\mathbb{R}^+, \infty)$, where $\mu > 0$ depends on the germ.

2. The main theorem is the additive decomposition theorem; the multiplicative theorem is needed mainly in order to derive from it the additive theorem.

3. The assertion of the ADT_n about the weak realness of the terms in the expansion of the real correction $g - \mathrm{id}$ enables us to get a lower estimate for this correction, and prove that it is non-oscillating. This is done in the next section.

S 1.6. Reduction of the finiteness theorem to auxiliary results

Here we prove the Identity Theorem for the monodromy maps of class n. We recall its statement.

THEOREM. *Let Δ be a monodromy transformation of class n, and $\Delta \in Fix_\infty$. Then $\Delta = id$.*

PROOF. Suppose that $\Delta \neq \mathrm{id}$. Consider the decomposition (5.3) given by the ADT_n.

Suppose first that $a \neq \mathrm{id}$. In this case the correction $\Delta - \mathrm{id}$ does not vanish on $(\mathbb{R}^+, \infty)$, because $a - \mathrm{id}$ is bounded from zero, and all the other terms tend to zero on $(\mathbb{R}^+, \infty)$.

Suppose now that $a = \mathrm{id}$. Let φ be the first non-zero term after id in the decomposition (5.3). Let

$$\varphi = F \circ \exp^{[k]} \circ f,\ F \in \mathcal{FC}_+^k, f \in G^{k-1}.$$

By ADT_n, F is weakly real. Hence, by LET_n,

$$|\mathrm{Re}\ F| \succ \exp \nu\xi \text{ on } (\mathbb{R}^+, \infty)$$

for some $\nu > 0$. On the other hand,

$$|F| \prec \exp(-\varepsilon\xi) \text{ on } (\mathbb{R}^+, \infty)$$

for some $\varepsilon > 0$. Hence, $\mathrm{Re}\ \varphi \in \mathcal{A}_f^k$, an Archimedian class of $\exp(-\exp^{[k]} \circ f)$.

Any other term ψ in the decomposition (5.3) for Δ has the form

$$\psi = G \circ \exp^{[m]} \circ g,\ G \in \mathcal{FC}_+^m, g \in G^{m-1},$$

and $(k, f) \prec\!\prec (m, g)$. The cochain G is rapidly decreasing. Hence,

$$|G| \prec \exp(-\varepsilon\xi) \text{ on } (\mathbb{R}^+, \infty)$$

for some $\varepsilon > 0$.

Therefore, φ belongs to the Archimedian class $\mathcal{A}_f^k$, and ψ is majorized by a germ from an Archimedian class $\mathcal{A}_g^m$. By Propositions 3 and 4 in S1.2D, the germs of the second class decrease faster than the germs of the first one. This implies that

$$\Delta - \mathrm{id} \succ \frac{1}{2}|\mathrm{Re}\ \varphi| \succ \exp(-\exp^{[k]} \mu\xi)$$

for some $\mu > 0$. Hence, $\Delta \notin \mathrm{Fix}_\infty$, a contradiction. $\square$

The same method provides a proof of the following

prop:proper

PROPOSITION 1. *The group G^n is $(n+1)$-proper in sense of definition 1.*

PROOF. First, we need to prove that the group is ordered by the relation $\succ$. Consider two germs $g, \tilde{g} \in G$ and let g be decomposed as Δ in (5.4), and

$$\tilde{g} = \tilde{a} = \sum_1^M \psi_l,\ \psi_l \in \mathcal{F}_{g_l}^{m_l},\ (m_l, g_l) \prec (m_{l+1}, g_{l+1})$$

be a decomposition provided by the ADT_n. Consider the difference

$$g - \tilde{g} = a - \tilde{a} + \sum \varphi_j - \sum \psi_l.$$

If $a \neq \tilde{a}$, then $a \succ \tilde{a}$ or $a \prec \tilde{a}$ implies $g \succ \tilde{g}$ or $g \prec \tilde{g}$ respectively. If $a = \tilde{a}$, consider two sets

$$\{(k_j, f_j) | j = 1, \ldots, N\} \; \{(m_l, g_l) | l = 1, \ldots, M\}.$$

If $(k_1, f_1) \succ (m_1, g_1)$, then $g - \tilde{g} = \varphi_1(1 + o(1))$, and $g \succ g_1 \Leftrightarrow \varphi_1 \succ 0$. Let neither $(k_1, f_1) \succ (m_1, g_1)$ nor vise versa. Then $k_1 = m_1, \; f_1 \circ g_1^{-1} \in G_{\text{rap}}^{m_1 - 1}$. By Lemma $SL1_n$, in this case

$$\mathcal{F}_{+,f_1}^{m_1} = \mathcal{F}_{+,g_1}^{m_1}.$$

In more detail

$$\mathcal{FC}_+^{m_1} \circ A^{-m_1}(f_1 \circ g_1^{-1}) = \mathcal{FC}_+^{m_1}$$

by $SL1_n$. Hence, both $\varphi_1 - \psi_1 \in \mathcal{FC}_{+,f_1}^{m_1}$. If $\varphi_1 - \psi_1 \not\equiv 0$, then $g - \tilde{g} = (\varphi_1 - \psi_1)(1 + o(1))$, and the same arguments work.

Let $\varphi_1 \equiv \psi_1$. Then take the least j for which $\varphi_j \neq \psi_j$. If $(k_j, f_j) \prec (m_j, g_j)$, then

$$\text{Re }(g - \tilde{g}) = \text{Re }\varphi_j(1 + o(1)),$$

and is either positive or negative near infinity. If neither $f(k_j, f_j) \prec (m_l, g_l)$, nor $(k_j, f_j) \succ (m_l, g_l)$, then

$$\varphi_j - \psi_j \in \mathcal{F}_{+g_l}^{m_l} \setminus \{0\}.$$

Arguing as before, we get:

$$g - \tilde{g} = (\varphi_j - \psi_j)(1 + o(1)).$$

Hence, $g - \tilde{g}$ is comparable with 0 again.

Second, we have to prove that $A^{n+1}\mathcal{A}ff \subset G^n$. This was done in Proposition 3 of Section 0.6.C. $\square$

We now switch to the proof of the multiplicative and additive decomposition theorems.

S 1.7. Proof of the multiplicative and additive decomposition theorems, MDT$_n$ and ADT$_n$, modulo auxiliary facts

sec:pma

The above theorems are already proven for $n = 1$ in Part 1. We fix n and make an induction assumption that MDT$_m$ and ADT$_m$ are proven for all $m < n$. The induction step: proof of MDT$_n$ and ADT$_n$ occupies the rest of the book. In the next three subsections we deduce these theorems from the axioms above. Building a model for these axioms occupies the rest of Chapter 1. Justifying the model, that is, checking that the axioms hold for the model constructed, forms the rest of the book: Chapters 2 – 5.

A. Principle: shift – conjugacy. In this secton we prove conjugacy lemmas from which MDT$_n$ is deduced below. The general idea is that a shift property implies a corresponding conjugacy property, as is shown in the proof of the following lemma. The proof is presented in subsection C below.

Lemma (Conjugacy Lemma 1_n, CL1_n). *Let*

$$m \leq n, \; f \in G^{m-1}, \; \varphi \in \mathcal{F}_{+f}^m, \; \psi \in \mathcal{F}_{+g}^m, \; f \prec\!\prec g.$$

Then

$$Ad(id + \varphi)(id + \psi) \in id + \mathcal{F}_{+g}^m.$$

Proof. By SL3$_n$,

$$(\mathrm{id}+\varphi)^{-1} = (\mathrm{id}+\tilde{\varphi}),\ \tilde{\varphi} \in \mathcal{F}^m_{+f}.$$

Then

$$(\mathrm{id}+\tilde{\varphi}) \circ (\mathrm{id}+\psi) \circ (\mathrm{id}+\varphi) = (SL2_n)$$

$$(\mathrm{id}+\tilde{\varphi}+\tilde{\psi}) \circ (\mathrm{id}+\varphi)(\tilde{\psi} \in \mathcal{F}^m_g) = \mathrm{id}+\tilde{\psi} \circ (\mathrm{id}+\varphi) = (SL3_n)(\mathrm{id}+\hat{\psi})(\hat{\psi} \in \mathcal{F}^m_g).$$

□

Conjugacy Lemma 1_n allows us to order properly the terms inside J^{n-1} and H^n.

Lemma (Conjugacy Lemma 2_n, CL2_n).

$$Ad(J^{n-1})H^n = H^n.$$

Proof. Let us prove the conjugacy relation for the generators only; it will imply the lemma. By SL4_na, the group J^{n-1} is generated by germs of the form

$$\mathrm{id}+\varphi, \varphi \in \mathcal{F}^{n-1}_{+f},\ f \in G^{n-2}.$$

By definition, H^n is generated by the germs

$$\mathrm{id}+\psi, \psi \in \mathcal{F}^n_{+g},\ g \in G^{n-1}.$$

We have to prove that

$$Ad(\mathrm{id}+\varphi)(\mathrm{id}+\psi) \in \mathrm{id}+\mathcal{F}^n_{+g}.$$

The proof is the same as above; only the reference to SL3_n is replaced by a reference to SL4_nb. □

Lemma (Conjugacy Lemma 3_n, CL3_n).

$$Ad(G^{n-1})H^n = H^n.$$

Proof. Take a generator of H^n again:

$$\mathrm{id}+\psi, \psi \in \mathcal{F}^n_{+g},\ g \in G^{n-1}.$$

Let $f \in G^{n-1}$. We have to prove:

$$Ad(f)(\mathrm{id}+\psi) \in \mathrm{id}+\mathcal{F}^n_{+g}.$$

By the induction assumption, ADT$_{n-1}$ may be applied to f^{-1}:

$$f^{-1} = a + \sum \varphi_j,\ a \in \mathcal{A}\!f\!f, \varphi_j \in \mathcal{F}^{k_j}_{1,+f_j},\ k_j < n-1,\ \varphi_j \in \mathcal{F}^{k_j}_{0,+f_j},\ k_j = n-1.$$

By SL2_n,

$$\varphi_j \circ (\mathrm{id}+\psi) = \varphi_j + \psi_j,\ \psi_j \in \mathcal{F}^n_{+g}.$$

Let

$$\tilde{\psi} = \sum \psi_j \in \mathcal{F}^n_{+g}.$$

Then

$$f^{-1} \circ (\mathrm{id}+\psi) \circ f = (f^{-1}+\tilde{\psi}) \circ f = \mathrm{id}+\tilde{\psi} \circ f \in H^n$$

since $\tilde{\psi} \circ f \in \mathcal{F}^n_{+g\circ f}$ by definition. □

B. Proof of the MDT$_n$1°.

Proof. We have to prove that

$$Gr(H^n, J^{n-1}, G^{n-1}) = G^{n-1} \circ J^{n-1} \circ H^n.$$

For this it is sufficient to prove:

$$Ad(G^{n-1})J^{n-1} = J^{n-1},$$

$$Ad(J^{n-1})H^n = H^n,$$

$$Ad(G^{n-1})H^n = H^n.$$

The first equality is an immediate consequence of the definition of J^{n-1}; the second and third ones form the contents of the Lemmas CL2$_n$ and CL3$_n$ above respectively. □

C. Proof of the MDT$_n$2°. We have to prove that if $g \in G^n$, then

$$g = a \circ \prod(\mathrm{id} + \varphi_j) \circ \prod(\mathrm{id} + \psi_l),$$

where

$$a \in \mathcal{A}\!f\!f,$$

$$\varphi_j \in \mathcal{F}^{k_j}_{1+f_j},\ k_j \leq n-1,$$

$$\psi_j \in \mathcal{F}^n_{0+g_j},\ f_j \in G^{k_j-1},\ g_j \in G^{n-1},$$

and the factors are properly ordered, that is

$$(k_j, f_j) \prec (k_{j+1}, f_{j+1}),\ g_j \prec\!\!\prec g_{j+1} \quad \text{in } G^{n-1}.$$

By MDT$_n$1°,

$$g = \tilde{g} \circ j \circ h,\ \tilde{g} \in G^{n-1},\ j \in J^{n-1},\ h \in H^n.$$

By MDT$_{n-1}$2° that enters the induction hypothesis, the germ $\tilde{g}$ may be properly decomposed. On the other hand,

eqn:tg

$$\tilde{g} = \hat{g} \circ \hat{j} \circ \hat{h},\ \hat{g} \in G^{n-2},\ j \in J^{n-2},\ h \in H^{n-1}. \tag{7.1}$$

Moreover,

$$\hat{h} = \prod(\mathrm{id} + \tilde{\psi}),\ \tilde{\psi}_j \in \mathcal{F}^{n-1}_{+\tilde{f}_j},\ \tilde{f}_j \in G^{n-2}.$$

Formula (7.1) implies that these factors are the last in decomposition for $\tilde{g}$.

The germs j and h above are the products of generators of the groups J^{n-1} and H^n. Hence,

$$j = \prod(\mathrm{id} + \tilde{\varphi}_j),\ \varphi_j \in \mathcal{F}^{n-1}_{+\tilde{f}_j},\ \tilde{f}_j \in G^{n-2}.$$

$$h = \prod(\mathrm{id} + \psi_j),\ \psi_j \in \mathcal{F}^{n-1}_{+g_j},\ g_j \in G^{n-1}.$$

By CL1$_n$, the factors in the product $\hat{h} \circ j \circ h$ may be properly ordered. This completes the proof of MDT$_n$2°

Remarks. By the way, the factors in the product $\hat{h} \circ j$ may be shuffled.

D. Proof of the ADT$_n$.

PROOF. Let us deduce ADT$_n$ from MDT$_n$ and SL2$_n$. The proof goes by induction in the number of factors in the decomposition given by MDT$_n$.

INDUCTION BASE (case of one factor) is obvious.

INDUCTION STEP. Suppose now that the ADT$_n$ is proved for a product of $k-1$ factors. Let us prove it for k. Let g be the same as in the MDT$_n$:

$$g = a \circ \prod_1^k (\mathrm{id} + \varphi_j), \varphi_j \in \mathcal{F}^{m_j}_{+f_j}.$$

and the factors are properly ordered. Set

$$\tilde{g} = a \circ \prod_1^{k-1} (\mathrm{id} + \varphi_j).$$

By the induction assumption,

$$\tilde{g} = a + \sum \tilde{\varphi}_j, \varphi_j \in \mathcal{F}^{m_j}_{+f_j},\ 1 \le j \le k-1.$$

Then,

$$g = \tilde{g} \circ (\mathrm{id} + \tilde{\varphi}_k),\ \varphi_k \in \mathcal{F}^{m_k}_{+f_k},\ (m_j, f_j) \prec (m_k, f_k),\ 1 \le j \le k-1.$$

Hence,

$$g = \tilde{g} = a + \sum_1^{k-1} \tilde{\varphi}_j \circ (\mathrm{id} + \tilde{\varphi}_k).$$

By SL2$_n$,

$$g = a + \sum \tilde{\varphi}_j + \sum \tilde{\psi}_j,\ \tilde{\psi}_j \in \mathcal{F}^{m_k}_{+f_k}.$$

By linearity of $\mathcal{F}^{m_k}_{+f_k}$, we have

$$\sum \tilde{\psi}_j \in \mathcal{F}^{m_k}_{+f_k}.$$

This completes the proof of the required decomposition.

It remains to prove that all the terms in the decomposition (5.4) may be taken weakly real. Let S be the symmetry operator introduced above: $Sf(\zeta) = \overline{F(\bar{\zeta})}$. Replace all φ_j in (5.4) by $\tilde{\varphi}_j = \frac{1}{2}(\varphi_j + S\varphi_j)$. The latter composition belongs to $\mathcal{F}^{m_j}_{+f_j}$ by the symmetry axiom, and $\tilde{\varphi}_j = S\tilde{\varphi}_j$ on $(\mathbb{R}^+, \infty)$; hence, $\tilde{\varphi}_j$ is weakly real. On the other hand, $a + \sum \tilde{\varphi}_j = \frac{1}{2}(g + Sg)$. But g is real on $(\mathbb{R}^+, \infty)$, hence, $\frac{1}{2}(g + Sg) = g$ on $(\mathbb{R}^+, \infty)$. Hence, $g = a + \sum \tilde{\varphi}_j$, all $\tilde{\varphi}_j$ are weakly real. □

S 1.8. Strategy of the further proof of the Finiteness Theorem

sec:strategy

The first part of Chapter 1 is over. The second part is started by the general outlook of the further proof that may be illustrated by the diagram shown on Figure 1.

The blocks contain the names of the major auxiliary statements as well as that of the finiteness theorem itself. The arrows, as usual, show the implications. Solid arrows show the implications that are already proved. Dashed arrows show the ones to be proved. There are no arrows to enter the lower left block. It is included in the induction assumptions. The induction step – proof of the properties of special

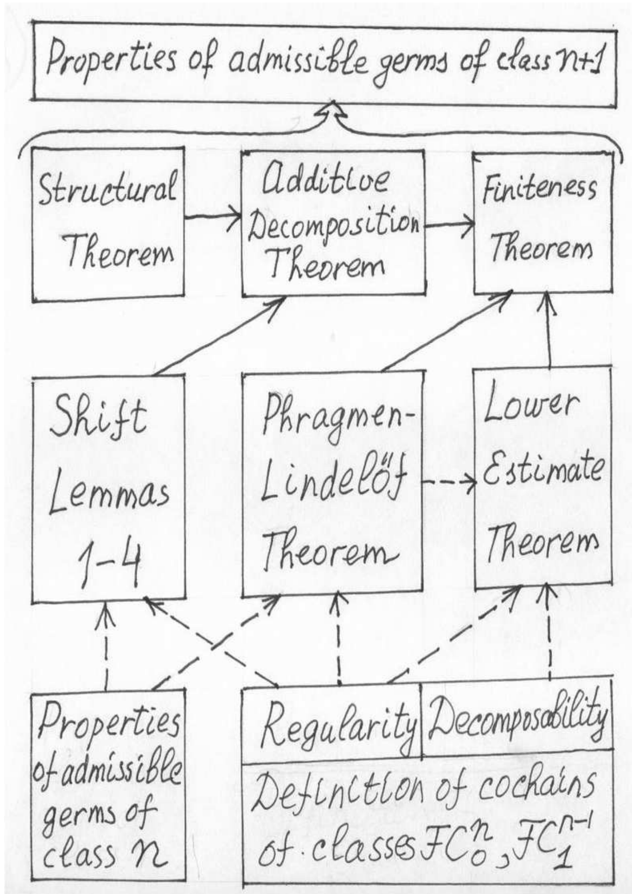

FIGURE 1 f1

admissible germs of the class $n+1$ relies upon all the previous results, and is carried on in Chapter 5. It is symbolized by the upper box.

The first step is to built the model for the axioms above. This is done in the second part of this chapter. By the way, admissible germs mentioned in the lower box, are defined. The cochains of classes $\mathcal{FC}_0^n$ and $\mathcal{FC}_1^{n-1}$ are characterized by two major properties: regularity and extendability. These properties correspond to the two upper boxes in the south-east corner of the scheme. The Phragmen-Lindelöf property relies upon the regularity only, whilst the lower estimate requires both regularity and extendability. The proofs of the Shift Lemmas and the Phragmen-Lindelöf theorem make use of some properties of the admissible germs. The induction assumption is that these properties hold for the admissible germs of class n. For this reason, no arrow comes to the south-west box on the scheme.

To complete the induction step, we prove the required properties for admissible germs of class $n+1$. This is done in Chapter 5. The proof makes use of the ADT_n and of LET_n. These implications are not shown on the scheme.

The Shift Lemmas are proved in Chapter 2 (regularity part), and in Chapter 4 (the expandability part). The Phragmen-Lindelöf theorem is proved in Chapter 3.

We turn to the explicit definitions of the cochains of class n.

S 1.9. A heuristic description of superexact asymptotic series

As shown in S0.2.C, ordinary asymptotic series are insufficient for the unique determination of monodromy transformations. Superexact asymptotic series are needed; the idea for constructing them goes as follows.

Suppose that a set M_1 of germs of mappings $(\mathbb{R}^+, \infty) \to (\mathbb{R}^+, \infty)$ is being investigated. Each of these germs can be expanded in an asymptotic series whose partial sums approximate the germ to within an arbitrary power of x, for example, in a Dulac series. Such series will be called ordinary series. However, it is desirable to expand the germs under study in series whose terms have not only a power order of smallness, but also an exponential order of smallness. At first glance this is impossible: an arbitrary remainder term of an ordinary series has power order of smallness, and it seems meaningless to take into account exponentially decreasing terms.

This difficulty is gotten around as follows. An intermediate class M_0 of functions is introduced; the functions in this class are expanded in ordinary series and are uniquely determined by them, that is, the zero function corresponds to the zero series. For example, M_0 can be taken to be the set of almost regular germs, given in a natural chart. Then the germs of the class M_1 are expanded in series of decreasing exponentials, and the coefficients in this series are no longer numbers but functions in the class M_0. The simplest example of a STAR has the appearance

e:0.5-1

$$\begin{gathered}\Sigma = a_0(\xi) + \sum a_j(\xi)\exp(-\nu_j \exp\xi), \\ a_j \in M_0, \quad 0 < \nu_j \nearrow \infty.\end{gathered} \tag{9.1}$$

A series Σ is said to be asymptotic for a germ f if for every $\nu > 0$ the series has a partial sum approximating the germ on $(\mathbb{R}^+, \infty)$ to within $o(\exp(-\nu\exp\xi))$. All information about the expansion of f in an ordinary series is included in the free term of the superexact series: the ordinary series for f and a_0 coincide.

By a simple example we show how to use superexact series to prove a simplified version of the identity theorem. Assume in addition to the preceding that the class M_0 (M_1) contains the germs of the functions 0 and x, and that the germs of this class can be expanded in Dulac series (respectively, in STAR (9.1)) and are uniquely determined by these series. Then we have the

THEOREM. $f \in M_1 \cap \mathrm{Fix}_\infty \Rightarrow f = \mathrm{id}$.

The theorem is proved according to the same scheme as the corrected Dulac lemma in S0.2B. Suppose that the theorem is false: there exists an $f \in M_1 \cap \mathrm{Fix}_\infty$, $f \neq \mathrm{id}$. Let (9.1) be a STAR for f. Assume first that $a_0 \neq \mathrm{id}$. Then the corresponding Dulac series $\hat{a}_0 - \mathrm{id}$ is not 0. Consequently, the germ of $a_0 - \mathrm{id}$ is equal to the principal term of its Dulac series, multiplied by $1 + o(1)$; in particular, for some $\nu > 0$,

$$|a_0 - \mathrm{id}| \succ \exp(-\nu\xi).$$

Further, it follows from the expandability of f in a STAR (9.1) that

$$|f - \mathrm{id}| \geq |a_0 - \mathrm{id}| + (|a_1|\exp(-\nu_1\exp\xi))(1 + o(1))$$

$$\succ \exp(-\nu\xi)(1 + o(1)).$$

Consequently, $f - \mathrm{id} \neq 0$ for small x, and hence $f \notin \mathrm{Fix}_\infty$, a contradiction.

Suppose now that $a_0 = \mathrm{id}$, $f \neq \mathrm{id}$. Then the STAR (9.1) is different from id; otherwise $f = \mathrm{id}$, since a germ in the class M_1 is uniquely determined by its series. We get from the definition of expandability that

$$f - \mathrm{id} = (a_1 \exp(-\nu_1 \exp \xi))(1 + o(1)).$$

Arguing as in the preceding paragraph, we get that $a_1 \neq 0$ for small x. The two other factors in the formula for $f - \mathrm{id}$ also do not vanish near zero. Consequently, $f \notin \mathrm{Fix}_\infty$, a contradiction.

Remark. Monodromy transformations of polycycles can be expanded in asymptotic series not only in simple but also in multiple exponentials of the type

$$\exp(-\nu \exp^{[n]} \xi).$$

The number n in this composition is the basic parameter used for proving the identity theorem by induction. In the first part, see also [**?**], we take the case $n = 1$; in the second part, we take arbitrary n. The second case is much more complicated technically, but all the basic ideas are used already in the case $n = 1$. An exception is Chapter V, an analogue of which is not needed for $n = 1$.

We now pass to the definitions of the cochains of class n, that is, to constructing the model for the axioms above.

S 1.10. Standard domains and admissible germs of diffeomorphisms

sec:stan

A. Standard domains: definition and examples.

def:standd

Definition 1. A standard domain is a domain that is symmetric with respect to the real axis, belongs to the right half-plane, and admits a real conformal mapping onto the right half-plane that has derivative equal to $1 + o(1)$ and extends to the δ-neighborhood of the part of the domain outside a compact set for some $\delta > 0$.

Remark. The correction of the conformal mapping in the previous definition increases more slowly than $\varepsilon|\xi|$ at infinity for each $\varepsilon > 0$.

Two half-strips are used repeatedly in the constructions to follow: right and standard. A right half-strip is defined by the formula

$$\Pi = \{\zeta \mid \xi \geq a,\, |\eta| < \pi/2\}, \quad a \geq 0.$$

A standard half-strip is defined by

$$\Pi_* = \Phi\Pi, \quad \Phi = \zeta + \zeta^{-2}.$$

An important example of a standard domain is given by

Proposition 1. *The exponential of a standard half-strip is a standard domain.*

Proof. Indeed,

$$\begin{aligned}\exp \Pi_* &= \exp \circ \Phi\Pi = \exp \circ \Phi \circ \ln(\mathbb{C}^+ \setminus K) \\ &= A^{-1}\Phi(\mathbb{C}^+ \setminus K),\end{aligned}$$

where K is the disk $|\zeta| \leq \exp a$, and a is the same as in the definition of the right half-strip Π. Further,

$$\begin{aligned}[A^{-1}(\zeta+\zeta^{-2})]' &= \exp\left(\ln\zeta + \frac{1}{\ln^2\zeta}\right)\cdot\left(1-\frac{2}{\ln^3\zeta}\right)\cdot\zeta^{-1} \\ &= \left(\exp\left(\frac{1}{\ln^2\zeta}\right)\right)\cdot(1+o(1)) \\ &= 1+o(1) \quad \text{in } (\mathbb{C}^+,\infty).\end{aligned}$$

Consequently, the real conformal mapping

$$A^{-1}\Phi\colon \mathbb{C}^+\setminus K \to \exp\Pi_*$$

on $(\mathbb{C}^+,\infty)$ has derivative of the form $1+o(1)$. The inverse mapping

$$\psi\colon \exp\Pi_* \to \mathbb{C}^+\setminus K$$

can be extended to the δ-neighborhood of the part of $\exp\Pi_*$ outside some compact set and also has derivative of the form $1+o(1)$. Further, there exists a conformal mapping $\psi_0\colon \mathbb{C}^+\setminus K \to \mathbb{C}^+$ with correction tending to zero as $\zeta\to\infty$ (it is given by the Zhukovskiĭ function if K is the unit disk). Therefore, the mapping $\psi_0\circ\psi$ is real, can be extended to the δ-neighborhood of the part of $\exp\Pi_*$ outside some compact set, and has derivative of the form $1+o(1)$, that is, it satisfies the requirements imposed in the definition of a standard domain. □

def:prop

DEFINITION 2. A class $\mathbf{\Omega}$ of standard domains is said to be proper if:

1°. For any $C>0$ an arbitrary domain of class $\mathbf{\Omega}$ contains a domain of the same class whose distance from the boundary of the first is not less than C.

2°. The intersection of any two domains of class $\mathbf{\Omega}$ contains a domain of the same class.

Usually, but not always, the classes $\mathbf{\Omega}$ of standard domains considered are proper.

B. Admissible germs.

def:adm

DEFINITION 3. Let $\mathbf{\Omega}$ be some set of standard domains. The germ of the diffeomorphism $\sigma_{\mathbb{R}}\colon (\mathbb{R}^+,\infty)\to(\mathbb{R}^+,\infty)$ is said to be admissible of class $\mathbf{\Omega}$, or $\mathbf{\Omega}$-admissible, if:

1°. the inverse germ ρ admits a biholomorphic extension to some standard domain, and for each standard domain $\Omega\in\mathbf{\Omega}$ there exists a standard domain $\tilde\Omega\in\mathbf{\Omega}$ such that ρ maps $\tilde\Omega$ biholomorphically into Ω, and, moreover,

2°. the derivative ρ' is bounded in $\tilde\Omega$,

3°. there exists a $\mu>0$ such that $\operatorname{Re}\rho<\mu\xi$ in $\tilde\Omega$,

4°. for each $\nu>0$,

$$\exp\operatorname{Re}\rho \succ \nu\xi \quad \text{in } \tilde\Omega.$$

The extension of the germ $\sigma_{\mathbb{R}}$ to the domain $\rho\tilde\Omega$ is denoted by σ and also called an $\mathbf{\Omega}$-admissible germ. To speak of an admissible and not an $\mathbf{\Omega}$-admissible germ means by definition that $\mathbf{\Omega}$ is understood to be the class of all standard domains.

def:non

DEFINITION 4. The germ of a diffeomorphism $\sigma_{\mathbb{R}}\colon(\mathbb{R}^+,\infty)\to(\mathbb{R}^+,\infty)$ is said to be nonessential of class $\mathbf{\Omega}$ if it admits a biholomorphic extension to a standard half-strip Π_* for some a and there exists a standard domain of class $\mathbf{\Omega}$ that belongs to $\sigma\Pi_*$.

Examples. 1. The germ $\sigma = \exp \circ \mu$ with $\mu > 1$ is nonessential. Indeed, the image contains the part of the right half-plane $\mathbb{C}^+$ outside some compact set.

2. The germ $\sigma = \exp$ is nonessential by Proposition 1.

3. The germ $\sigma = \exp$ is not admissible: requirement 4° of Definition 2 fails.

4. The germs of $\sigma = \exp \circ \mu$ with $0 < \mu < 1$, of $\sigma = \zeta^\mu$ with $\mu \geq 1$, and of $\sigma \in \mathcal{A}\!f\!f$ are admissible.

Remark. A standard half-strip has two opposing properties: it is not too broad and not too narrow. On the one hand, it is so narrow that the main function of each map-cochain of class $\mathcal{NC}$ extends to a standard half-strip for sufficiently large a dependent on the cochain; as a rule, it is impossible to implement such an extension to a right half-strip Π. On the other hand, it is so broad that the exponential of a standard half-strip is a standard domain. From this point of view, the strip Π is not enough narrow, and the strip $(1 - \varepsilon)\Pi$ is no broad enough for any $\varepsilon \in (0, 1)$.

sec:regc

S 1.11. Regular cochains

Two map-cochains or two functional cochains are said to be equivalent if there exists a standard domain in which they are defined and coincide. An equivalence class of map-cochains is called the germ of a map-cochains or a functional cochain. The representatives of a germ are considered in standard domains, by definition.

sub:regpar

A. Regular partitions. Let us first define regular partitions of standard domains. Chose and fix some class $\mathbf{\Omega}$ in a set of all standard domains.

Definition 1. Suppose that a partition Ξ is given in a standard domain $\Omega \in \mathbf{\Omega}$. The image of the partition Ξ under the action of an admissible germ of a diffeomorpism σ of class $\mathbf{\Omega}$ is the partition $\sigma_*\Xi$ of a standard domain $\tilde{\Omega} \in \mathbf{\Omega}$ in which a representative, carrying $\tilde{\Omega}$ into Ω, of the germ $\rho = \sigma^{-1}$ is defined and biholomorphic. The domains of the partition $\sigma_*\Xi$ are defined by the equalities

$$(\sigma_*\Xi)_j = \sigma(\Xi_j \cap \rho\tilde{\Omega}),$$

where the Ξ_j are the domains of the partition Ξ. The domain $(\sigma_*\Xi)_j$ are the domains of the partition Ξ. The domain $(\sigma_*\Xi)_j$, by definition, corresponds to the domain Ξ_j.

Example 1. Pictured in Figure 2 are the images of the standard partition under the action of the diffeomorpisms $\exp \circ \mu,\ \mu \in (0, 1);\ \zeta^\mu,\ \mu > 1;\ \mu\zeta,\ \mu > 0$. In the first case the domains of the image partition are ordinary sectors, in the second they are "parabolic sectors", and in the third they are horizontal half-strips.

Definition 2. The standard partition Ξ_{st} is the partition of a domain in $\mathbb{C}$ by the rays $\eta = \pi j,\ j \in \mathbb{Z}$. The strip $\eta \in [\pi(j-1), \pi j]$ is denoted by Π_j.

Consider an arbitrary domain Ω and a partition Σ of Ω. A boundary curve of a domain of this partition is called *exterior* if it belongs to $\partial\Omega$, and *interior* elsewhere. The union of all interior boundary curves of Σ is called *the boundary of* Σ and denoted $\partial\Sigma$.

def:reg

Definition 3. Consider a tuple

eqn:sigmanf

$$\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N) \tag{11.1}$$

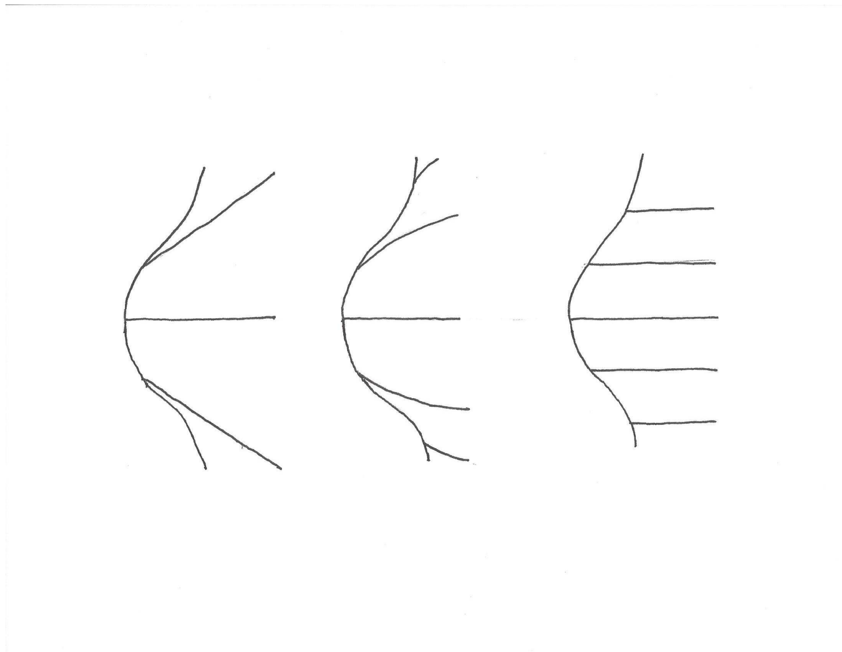

FIGURE 2 f12

of admissible germs. An $\mathbb{R}$-regular partition of a standard domain Ω is defined as a product

en:pxi
$$\Xi = \prod_1^N \sigma_{j*}\Xi_{\mathrm{st}}. \tag{11.2}$$

This partition is called *a partition of type* $\boldsymbol{\sigma}$ or $\boldsymbol{\sigma}_{\mathbb{R}}$ and of numerical type N.

The subscript $\mathbb{R}$ in the notation $\boldsymbol{\sigma}_{\mathbb{R}}$ recalls that the corresponding domain Ω is symmetric with respect to $(\mathbb{R}^+, \infty)$.

Together with ε-neighborhoods of the domains of partition (11.2) it is convenient to consider their generalized ε-neighborhoods defined as follows

def:gene

DEFINITION 4. Let (11.2) be an $\mathbb{R}$-regular partition of type (11.1) of a standard domain Ω. Let $\tilde{\Omega}$ be another standard domain such that for any $\sigma_j \in \boldsymbol{\sigma},\ \rho_j = \sigma_j^{-1}$, we have: $\rho_j\tilde{\Omega} \subset \Omega$. Let Ξ_j be the domains of the standard partition Ξ_{st} of Ω, and let

$$U = \Omega \cap_1^N \sigma_j(\Xi_{l_j})$$

be a domain of the partition (11.2) of $\tilde{\Omega}$. Here l_j are so chosen that $U \neq \emptyset$.

Let A^ε be an ε-neighborhood of A in $\mathbb{C}$. Then the generalized ε-neighborhood $U^{(\varepsilon)}$ of U is defined as

$$U^{(\varepsilon)} = \tilde{\Omega}^\varepsilon \cap_1^N \sigma_j(\Xi_{l_j}^\varepsilon).$$

def:boune

DEFINITION 5. For any boundary curve of the partition (11.2) which is a common boundary of two domains of the partition, the generalized ε-neighborhood is an intersection of the generalized ε-neighborhoods of the two domains of the partition mentioned above. The union of all these neighborhoods is denoted by $\partial\Xi^{(\varepsilon)}$.

DEFINITION 6. Any regular partition of type QQQ $\boldsymbol{\sigma}_{\mathbb{R}}$ generates a set of functional cochains defined in generalized ε-neighborhoods of the boundary of the partition, called "rigging cochains" and defined as follows. The function of the cochain given in a neighborhood of the boundary curve $\mathcal{L}$ is denoted by $m_{\boldsymbol{\sigma},C,\varepsilon,\mathcal{L}}$ and is equal to

eqn:rig (11.3)
$$m_{\boldsymbol{\sigma},C,\varepsilon,\mathcal{L}} = \sum \exp(-C \exp \ \operatorname{Re}\, \rho_j), \ \ \rho_j = \sigma_j^{-1}. \tag{11.3}$$

The summation is over all j such that $\mathcal{L}$ is an interior boundary curve of the partition $\sigma_{j*}\Xi_{\mathrm{st}}$. A partition considered together with the set of rigging cochains is said to be rigged.

rem:rpl REMARK 3. The ray $(\mathbb{R}^+, \infty)$ is a boundary line of a partition $\sigma_{j*}\Xi_{\mathrm{st}}$ for all j. Hence, a rigging cochain on this ray equals to the sum (11.3) over all $j = 1, \dots, N$.

sub:reg

B. Regular cochains. As always, $\xi = \operatorname{Re}\zeta, \ \zeta \in \mathbb{C}^+$.

def:reg DEFINITION 7. An ε-extendable germ of an $\mathbb{R}$-regular cochain of class $\boldsymbol{\Omega}$ is a germ with a representative (called an $\mathbb{R}$-regular functional cochain, and also a cochain of class $\sigma_{\mathbf{R}}$ defined in an ε-neighborhood of a standard domain of class $\boldsymbol{\Omega}$ depending on the germ such that:

1^0. The corresponding partition is an $\mathbb{R}$-regular partition of type $\boldsymbol{\sigma}_{\mathbf{R}}$, where $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)$, the germs σ_j are admissible germs of class $\boldsymbol{\Omega}$, and the partition is rigged;

2^0. For small ε, the functions forming the cochain extend holomorphically to generalized ε-neighborhoods of the domains of the partition that correspond to them;

3^0. The modulus of the cochain can be estimated from above by the function $\exp \nu\xi$ for some $\nu \in \mathbb{R}$ depending on the cochain, with the cochain called a rapidly decreasing cochain if ν can be taken < 0 in this estimate, and a weakly decreasing cochain if the modulus of the cochain can be estimated from above by the function $C|\zeta|^{-5}$ for some $C > 0$;

4^0. The functions forming the coboundary of the cochain admit analytic extension to the generalized ε-neighborhoods of the boundary lines of the partition and can be estimated in modulus there from above by the corresponding functions of the rigging cochain $\mathcal{M}_C$ for some $C > 0$ depending on the cochain.

We will call these requirements *partition, extendability, growth* and *coboundary* respectively.

The set of all regular functional cochains is denoted by $\mathcal{FC}_{\mathrm{reg}}$, and the rapidly decreasing regular functional cochains by $\mathcal{FC}^+_{\mathrm{reg}}$ ($\mathcal{FC}$ for functional cochains).

def:map DEFINITION 8. An ε-extendable germ of a regular map-cochain is a germ whose correction is the germ of an ε-extendable rapidly decreasing regular functional cochain. The germ of a weakly regular ε-extendable map-cochain is defined similarly: in the preceding definition the correction must decrease not rapidly, but weakly. The sets of all regular and weakly regular map-cochains are denoted by $\mathcal{MC}_{\mathrm{reg}}$ and $\mathcal{MC}_{\mathrm{wr}}$ ($\mathcal{MC}$ for map-cochains, and wr for weakly regular).

def:fcreg

DEFINITION 9. Let $\mathcal{D}$ be an arbitrary set consisting of germs of admissible diffeomorphisms. The germ of a functional cochain or map-cochain is regular of type $\mathcal{D}$ if the corresponding partition is of type $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)$, with $\sigma_j \in \mathcal{D}$. Notation:

$$\mathcal{FC}_{\mathrm{reg}}(\mathcal{D}),\ \mathcal{MC}_{\mathrm{reg}}(\mathcal{D}).$$

The sets of germs of rapidly or weakly decreasing functional cochains of class $\mathcal{D}$ and the set of weakly regular map-cochains of class $\mathcal{D}$ are denoted by

$$\mathcal{FC}^{+}_{\mathrm{reg}}(\mathcal{D}),\ \mathcal{FC}_{\mathrm{wr}}(\mathcal{D}),\ \mathcal{MC}_{\mathrm{wr}}(\mathcal{D})$$

respectively.

sub:regnon

C. Regular cochains in non-standard domains. Consider any connected domain $\hat{\Omega}$ that belongs to some standard domain Ω of some class $\boldsymbol{\Omega}$ and contains $(\mathbb{R}^+, \infty)$. Let $\hat{\Omega}^{(\varepsilon)}$ be any increasing family of domains:

$$\hat{\Omega}^{(\varepsilon)} \subset \hat{\Omega}^{(\varepsilon')} \text{ for } \varepsilon < \varepsilon'.$$

def:all

DEFINITION 10. A partition (11.2) of domain $\hat{\Omega}$ of the type (11.1) is the intersection of the partition (11.2) of Ω with $\hat{\Omega}$. More precisely, any domain W of this partition is an intersection of some domain U of partition (11.2) with $\hat{\Omega}$. A generalized ε-neighborhood of W is the intersection

$$W^{(\varepsilon)} = \hat{\Omega}^{(\varepsilon)} \cap U^{(\varepsilon)}.$$

Generalized ε neighborhoods of the boundary curves of the partition are defined in the same way as above. A rigging cochain for the partition (11.2) of $\hat{\Omega}$ is now defined by the formula (11.3) with the same summation rule accepted.

After that, the set of regular cochains of type (11.1) is well defined in any domain $\hat{\Omega}$ described above; only the extendability property should be modified.

Let the domain $\hat{\Omega}$ and the family $\hat{\Omega}^{(\varepsilon)}$ be the same as at the beginning of the subsection.

DEFINITION 11. Let $\boldsymbol{\Omega}$, $\Omega, \hat{\Omega}$ and $\hat{\Omega}^{(\varepsilon)}$ be the same as above. An ε-extendable germ of regular functional cochain corresponding to the family $\hat{\Omega}^{(\varepsilon)}$ is a cochain with the following properties:

1^0. *Partition* The corresponding partition has the form (11.2), where $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)$, the germs σ_j are admissible germs of class $\boldsymbol{\Omega}$, and the partition is rigged;

2^0. *Extendebility* For small ε, the functions forming the cochain extend holomorphically to generalized ε-neighborhoods of the domains of the partition that correspond to them; the generalized ε-neighborhoods are understood in sense of Definition 10

3^0. *Growth* The modulus of the cochain can be estimated from above by the function $\exp \nu\zeta$ for some $\nu \in \mathbb{R}$ depending on the cochain, with the cochain called a rapidly decreasing cochain if ν can be taken < 0 in this estimate, and a weakly decreasing cochain if the modulus of the cochain can be estimated from above by the function $C|\zeta|^{-5}$ for some $C > 0$;

4^0. *Coboundary* The functions forming the coboundary of the cochain admit analytic extension to the generalized ε-neighborhoods of the interior boundary curves of the partition and can be estimated in modulus there from above by the

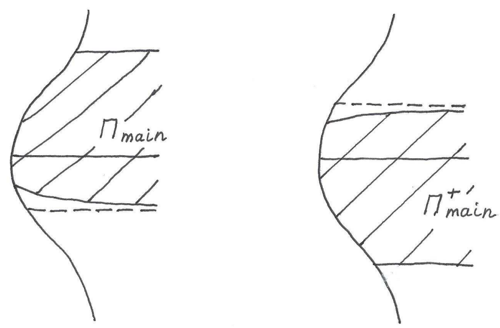

FIGURE 3 f12c

corresponding functions, that are the components of the rigging cochain $\mathcal{M}_C$, for some $C > 0$ and ε depending on the cochain.

An important example is given by *realizations of* $\mathbb{R}$*-regular cochains* defined in the next subsection.

sub:real

D. Realizations. Consider a proper class $\mathbf{\Omega}$ of standard domains, see Definition 2, and a class D of $\mathbf{\Omega}$-admissible germs.

We need the following definition.

def:we DEFINITION 12. Two admissible germs σ_1 and σ_2 are weakly equivalent if their composition quotient $\sigma_1^{-1} \circ \sigma_2$ has a bounded correction on $(\mathbb{R}^+, \infty)$. Notation $\sigma_1 \overset{w}{\sim} \sigma_2$.

Recall that $\Pi_j = \{\xi \geq a, \eta \in (\pi(j-1), \pi j)\}$. Let

eqn:pim
$$\Pi_{\text{main}} = \Pi_* \cup \Pi_1, \tag{11.4}$$

eqn:pimm
$$\Pi^+_{\text{main}} = \Pi_* \cup \Pi_0, \tag{11.5}$$

see Figure 3, and let

$$\Pi_* = \Phi\Pi, \quad \Pi_*^{(\varepsilon)} = \Phi_{1-\varepsilon}\Pi, \quad \Pi = \{\xi \geq a, |\eta| \leq \pi/2\},$$
$$\Phi = \zeta + \zeta^{-2}, \ \Phi_{1-\varepsilon} = \zeta + (1-\varepsilon)\zeta^{-2}, \quad a = a(\varepsilon), \quad \varepsilon \in [0, 1).$$

Let

eqn:pme
$$\Pi^{(\varepsilon)}_{\text{main}} = \Pi_*^{(\varepsilon)} \cup \Pi_1^{\varepsilon} \tag{11.6}$$

eqn:pmep (11.7) $$\Pi^{+(\varepsilon)}_{\text{main}} = \Pi^{(\varepsilon)}_{*} \cup \Pi^{\varepsilon}_{0},$$

where Π^{ε}_{j} are ε-neighborhoods of Π_j.

Suppose now that the class D has the following properties.

- *Ordering.* For any two germs $\sigma_1, \sigma_2 \in D$, $\sigma_1 - \sigma_2$ does not vanish on $(\mathbb{R}^+, \infty)$. Say that
$$\sigma_1 \succ \sigma_2 \Leftrightarrow \sigma_1 - \sigma_2 \succ 0 \Leftrightarrow \sigma_1 - \sigma_2 > 0 \text{ on } (\mathbb{R}^+, \infty).$$
- *Monotonicity.*
  1°. Suppose that $\sigma_1 \succ \sigma_2 \in D$, and the germs σ_1 and σ_2 are not weakly equivalent. Then

eqn:monot1 (11.8) $$(\sigma_1 \Pi_{\text{main}} \cap \Omega, \infty) \supset (\sigma_2 \Pi_{\text{main}} \cap \Omega, \infty).$$

  2°. Suppose that $\sigma_1, \sigma_2 \in D$ are weakly equivalent. Then these germs may be renumbered in such a way that for any $\varepsilon > \delta > 0$

eqn:monot3 (11.9) $$(\sigma_1 \Pi^{(\varepsilon)}_{\text{main}} \cap \Omega, \infty) \supset (\sigma_2 \Pi^{(\delta)}_{\text{main}} \cap \Omega, \infty).$$

REMARK 4. For the special classes of admissible germs defined below in Subsection F, these properties are included in the induction assumption in n. The induction step is proceeded in Chapter 5.

defin:rhd DEFINITION 13. Suppose that the germs g_1, g_2 either are not weakly equivalent and $g_1 \succ g_2$, or they are weakly equivalent and relation (11.9) holds. Then

$$\sigma_1 \rhd \sigma_2.$$

By symmetry, (7.9), (11.9) imply

eqn:monot6 (11.10) $$(\sigma_1 \Pi^{+}_{\text{main}} \cap \Omega, \infty) \supset (\sigma_2 \Pi^{+}_{\text{main}} \cap \Omega, \infty).$$

eqn:monot7 (11.11) $$(\sigma_1 \Pi^{+(\varepsilon)}_{\text{main}} \cap \Omega, \infty) \supset (\sigma_2 \Pi^{+(\delta)}_{\text{main}} \cap \Omega, \infty).$$

Consider any finite subset $\boldsymbol{\sigma}$ of a class D having the ordering and monotonicity properties. Let us first order this set by decreasing on $(\mathbb{R}^+, \infty)$:

$$\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N),\ \sigma_j \succ \sigma_{j+1}.$$

The weak equivalent germs in this set go by clusters; they are met in a row.

In any such cluster let us reorder the germs according to the relation $\rhd$. Note that if σ_1 and σ_2 are not weakly equivalent then relations $\sigma_1 \succ \sigma_2$ and $\sigma_1 \rhd \sigma_2$ are equivalent by assumption 1^0 of the monotonicity property. We achieved the following ordering of $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N),\ \sigma_j \rhd \sigma_{j+1};$$

if $j < k$, then for any $\varepsilon > \delta > 0$, (11.9) holds with $\sigma + 1 = \sigma_j$, $\sigma_2 = \sigma_k$:

eqn{}monot6 (11.12) $$(\sigma_j \Pi^{(\varepsilon)}_{\text{main}} \cap \Omega, \infty) \supset (\sigma_k \Pi^{(\delta)}_{\text{main}} \cap \Omega, \infty).$$

We will now define domains of realizations together with their generalized ε-neighborhoods .

Let

eqn:sigman5 (11.13) $$\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_N),\ \sigma_1 \rhd \sigma_2 \rhd ... \rhd \sigma_N$$

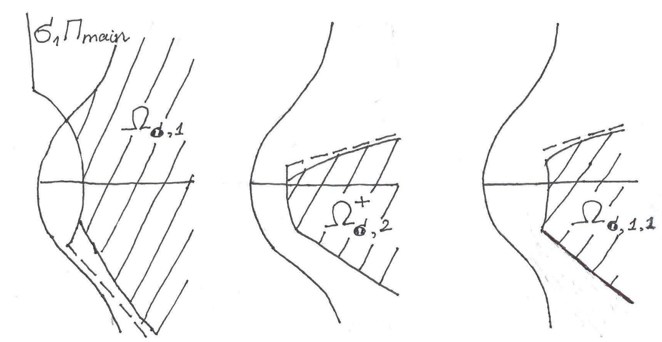

FIGURE 4 f12d

be a tuple of germs from D, and fix Ω from $\mathbf{\Omega}$. Define the following domains of type $(\boldsymbol{\sigma}, k), (\boldsymbol{\sigma}, k^+)$, $(\boldsymbol{\sigma}, k, l)$ and $(\boldsymbol{\sigma}, k, l)^+$ for $0 < k \le N, 0 < l \le N-k$.

eqn:omsk (11.14)
$$\Omega_{\boldsymbol{\sigma},k} = \sigma_k \Pi_{\text{main}} \cap \Omega,$$

eqn:omskp (11.15)
$$\Omega^+_{\boldsymbol{\sigma},k} = \sigma_k \Pi^+_{\text{main}} \cap \Omega,$$

eqn:omskl (11.16)
$$\Omega_{\boldsymbol{\sigma},k,l} = \Omega_{\boldsymbol{\sigma},k} \cap \Omega^+_{\boldsymbol{\sigma},k+l}, \ \Omega^+_{\boldsymbol{\sigma},k,l} = \Omega^+_{\boldsymbol{\sigma},k} \cap \Omega_{\boldsymbol{\sigma},k+l}.$$

Generalized ε-neighborhoods of these domains are defined as

eqn:omske (11.17)
$$\Omega^{(\varepsilon)}_{\boldsymbol{\sigma},k} = \sigma_k \Pi^{(\varepsilon)}_{\text{main}} \cap \Omega^\varepsilon,$$

eqn:omskpe (11.18)
$$\Omega^{+(\varepsilon)}_{\boldsymbol{\sigma},k} = \sigma_k \Pi^{+(\varepsilon)}_{\text{main}} \cap \Omega^\varepsilon,$$

eqn:omskle (11.19)
$$\Omega^{(\varepsilon)}_{\boldsymbol{\sigma},k,l} = \Omega^{(\varepsilon)}_{\boldsymbol{\sigma},k} \cap \Omega^{+(\varepsilon)}_{\boldsymbol{\sigma},k+l}.$$

and

$$\Omega^{+(\varepsilon)}_{\boldsymbol{\sigma},k,l} = \Omega^{+(\varepsilon)}_{\boldsymbol{\sigma},k} \cap \Omega^{(\varepsilon)}_{\boldsymbol{\sigma},k+l},$$

where $\Pi^{(\varepsilon)}_{\text{main}}$, $\Pi^{+(\varepsilon)}_{\text{main}}$ are the same as in (11.6), (11.7), see Figure 4.

An important consequence of the ordering (11.13) is: if $j < k$, then for any $\varepsilon > \delta > 0$:

eqn:monot4

$$(\Omega^{(\delta)}_{\boldsymbol{\sigma},k}, \infty) \subset (\Omega^{(\varepsilon)}_{\boldsymbol{\sigma},j}, \infty),\ (\Omega^{(\delta)}_{\boldsymbol{\sigma},k}, \infty) \subset (\Omega^{+(\varepsilon)}_{\boldsymbol{\sigma},j}, \infty). \tag{11.20}$$

Moreover, if $m > l$, then

eqn:monot5

$$(\Omega^{(\delta)}_{\boldsymbol{\sigma},k,m}, \infty) \subset (\Omega^{(\varepsilon)}_{\boldsymbol{\sigma},k,l}, \infty). \tag{11.21}$$

Note that $\Omega_{\boldsymbol{\sigma},k}$ and $\Omega^{+}_{\boldsymbol{\sigma},k}$, as well as $\Omega_{\boldsymbol{\sigma},k,l}$ and $\Omega^{+}_{\boldsymbol{\sigma},k,l}$ are pairwise symmetric.

def:mod

Definition 14. Partitions of type $(\boldsymbol{\sigma}, k), (\boldsymbol{\sigma}, k^+), (\boldsymbol{\sigma}, k, l), (\boldsymbol{\sigma}, k, l)^+$ for $\boldsymbol{\sigma}$ from (11.13) are

eqn:parsk

$$\text{partition } \prod_{k+1}^{N} \sigma_{j*}\Xi_{\text{st}} \text{ of } \Omega_{\boldsymbol{\sigma},k} \tag{11.22}$$

eqn:parskp

$$\text{partition } \prod_{k+1}^{N} \sigma_{j*}\Xi_{\text{st}} \text{ of } \Omega^{+}_{\boldsymbol{\sigma},k} \tag{11.23}$$

eqn:parskl

$$\text{partition } \prod_{k+l+1}^{N} \sigma_{j*}\Xi_{\text{st}} \text{ of } \Omega_{\boldsymbol{\sigma},k,l} \tag{11.24}$$

eqn:parsklp

$$\text{partition } \prod_{k+l+1}^{N} \sigma_{j*}\Xi_{\text{st}} \text{ of } \Omega^{+}_{\boldsymbol{\sigma},k,l}. \tag{11.25}$$

By monotonicity assumptions (11.20), (11.21), partitions of type $(\boldsymbol{\sigma}, k)$ have the following key property: they coincide with the partition of type $\boldsymbol{\sigma}$ of the domain $\Omega_{\boldsymbol{\sigma},k}$. Namely, for any $j \le k$, the only interior boundary line of the partition of type $\boldsymbol{\sigma}$ that belongs to $\sigma_j \partial \Xi_{\text{st}}$ is $(\mathbb{R}^+, \infty)$. Indeed, even for $j = k$, the line $\sigma_k(\eta = \pi)$ belongs to the upper boundary of $\Omega_{\boldsymbol{\sigma},k}$. For $j \ne 0, 1$, the lines $\sigma_k(\eta = \pi j)$ do not intersect $\Omega_{\boldsymbol{\sigma},k}$.

The rigging cochains $m_{\boldsymbol{\sigma}}$ and $m_{\boldsymbol{\sigma},k}$ corresponding to the partitions of type $\boldsymbol{\sigma}$ or $(\boldsymbol{\sigma}, k)$ of the domain $\Omega_{\boldsymbol{\sigma},k}$ coincide everywhere except for a generalized ε-neighborhood of $(\mathbb{R}^+, \infty)$. But on $(\mathbb{R}^+, \infty)$

$$m_{\boldsymbol{\sigma}} = \sum_{1}^{N} \exp(-C \exp \rho_j)$$

whilst

eqn:msigmak

$$m_{\boldsymbol{\sigma},k} = \sum_{k+1}^{N} \exp(-C \exp \rho_j). \tag{11.26}$$

By the ordering property of the germs σ_j in the tuple $\boldsymbol{\sigma}$, the rigging cochain $m_{\boldsymbol{\sigma}}$ on $(\mathbb{R}^+, \infty)$ is of the same multiplicative Archimedean class as its first term:

eqn:arch

$$\exp(-\mathcal{C} \exp \rho_1) \succ m_{\boldsymbol{\sigma}}|_{(\mathbb{R}^+,\infty)} \succ \exp(-C' \exp \rho_1) \tag{11.27}$$

for some $D < \mathcal{C}$. Indeed, let σ_1 be weakly equivalent to σ_j. Then $\sigma_1^{-1} \circ \sigma = id + \Phi$, Φ is bounded on $(\mathbb{R}^+, \infty)$. Hence, on $(\mathbb{R}^+, \infty)$:

$$\exp(-\mathcal{C} \exp \rho_1 \circ \rho_j^{-1}(\xi)) = \exp(-C \exp \sigma_1^{-1} \circ \sigma_j(\xi)) = \exp(-\mathcal{C} \exp(\xi + \Phi(\xi))) \succ$$

$$\exp(-(\mathcal{C}\exp(-D))\exp\xi),$$

and

$$\exp(-\mathcal{C}\exp\rho_1) \succ \exp(-D\exp\rho_j).$$

If σ_1 is not weakly equivalent to σ_j and $\sigma_1 \succ \sigma_j$, then the same calculation implies

$$\exp(-C\exp\rho_j) = o(1)\exp(-C\exp\rho_1).$$

This implies (11.27).

The same arguments yield:

eqn:arch1 (11.28) $$\exp(-\mathcal{C}\exp\rho_{k+1}) \succ m_{\boldsymbol{\sigma},k} \succ \exp(-D\exp\rho_{k+1}).$$

Hence, regular cochains corresponding to partitions of type (σ, k) in $\Omega_{\sigma,k}$ have "smaller coboundaries" and are "more holomorphic" than the restrictions of the cochains of type $\boldsymbol{\sigma}$ on $\Omega_{\boldsymbol{\sigma},k}$.

This motivates the following definitions based upon Definition 10 of Subsection C.

def:type DEFINITION 15. Let $\boldsymbol{\sigma}$ be as in (11.13), $0 < k \leq N$, $0 < l \leq N-k$. A regular cochain of type $(\boldsymbol{\sigma}, k)$ (or $(\boldsymbol{\sigma}, k^+)$, $(\boldsymbol{\sigma}, k, l)$) is a regular cochain defined in a domain (11.14) (or (11.15), (11.16) respectively) that corresponds to a partition of type $(\boldsymbol{\sigma}, k)$ (or $(\boldsymbol{\sigma}, k^+)$, $(\boldsymbol{\sigma}, k, l)$, $(\boldsymbol{\sigma}, k, l)^+$, respectively).

def:real DEFINITION 16. Let F be a regular functional cochain in a standard domain $\Omega, 0 < k \leq N$. A regular cochain $F_{(k)}$ of type $(\boldsymbol{\sigma}, k)$ is called a *k-realization* of F provided that

- It is of type $(\boldsymbol{\sigma}, k)$
- It coincides with F in the intersection of its domain with the upper half-plane; more precisely,

eqn:coink (11.29) $$F = F_{(k)} \text{ in } \sigma_k\Pi_1 \cap \Omega.$$

The cochain F itself is called its own 0-realization and defined $F_{(0)} : F = F_{(0)}$.

def:realp DEFINITION 17. Let F and k be the same as in previous definition. A regular cochain $F^+_{(k)}$ of type $(\boldsymbol{\sigma}, k^+)$ is called a k^+-realization of F provided that

- It is of type $(\boldsymbol{\sigma}, k^+)$
- It coincides with F in the intersection of its domain with the lower half-plane; more precisely,

eqn:coink (11.30) $$F = F^+_{(k)} \text{ in } \sigma_k\Pi_0 \cap \Omega.$$

def:realkl DEFINITION 18. Let F be the same as above, and $0 < l \leq N-k$. A regular cochain $F_{(k,l)}$ (or $F^+_{(k,l)}$) of type $(\boldsymbol{\sigma}, k, l)$ (or $(\boldsymbol{\sigma}, k, l)^+$) is called a (k,l) (or $(k,l)^+$)-realization of F provided that

- It is of type $(\boldsymbol{\sigma}, k, l)$ (or $(\boldsymbol{\sigma}, k, l)^+$)
- It coincides with $F_{(k)}$ in the "lower part" (coincides with $F_{(k)}$ in the "uuper part") of its domain; more precisely,

eqn:coink (11.31) $$F_{(k,l)} = F_{(k)} \text{ in } \Omega_{\boldsymbol{\sigma},k,l} \cap \{\eta < 0\}.$$

or

$$F^+_{(k,l)} = F^+_{(k)} \text{ in } \Omega^+_{\boldsymbol{\sigma},k,l} \cap \{\eta > 0\}$$

REMARK 5. (k,l) and $(k,l)^+$-realizations may be defined for cochains of numerical type $N > 1$ only.

Let us write explicitly what follows from Definitions 15 - 18. The k, k^+ and (k,l) realizations of a cochain of type σ defined in a standard domain Ω, are defined in the domains $\Omega_{\sigma,k,}, \Omega^+_{\sigma,k}, \Omega_{\sigma,k,l}$ respectively, see(11.14), (11.15), (11.16), (11.25).

def:alreal

DEFINITION 19. An $\mathbb{R}$-regular cochain of type (11.13) is

- *weakly realizable* if it has all k-realizations for $0 < k \le N$;
- *almost realizable* if it has all k and k^+-realizations for $0 < k \le N$;
- *absolutely realizable* if it has numeric type $N > 1$ and has all $k, k^+, (k,l)$ and $(k,l)^+$ realizations; or has numeric type 1 and is almost realizable; or is a holomorphic function.

Let $\mathbf{\Omega}$ be a proper class of standard domains, and D be any class of $\mathbf{\Omega}$-admissible germs with the ordering and monotonicity property from the beginning of this subsection.

def:classd

DEFINITION 20. Class $\mathcal{FC}_{\text{reg}}\ (D)$ is a class of all absolutely realizable regular cochains of type (11.13) with $\sigma_j \in D$.

EXAMPLE. Normalizing cochains are absolutely realizable.

Indeed, they are cochains of numerical type 1. Hence, for them absolute realizability means almost realizability. The latter requires the existence of two realizations: $F_{(1)}$ of type 1 and $F^+_{(1)}$ of type 1^+. These realizations exist by the Supplement to the Sectorial Normalization Theorem, and are holomorphic functions:

$$F_{(1)} = F^u,\ F^+_{(1)} = F^l.$$

This example is in a sense the main one.

sub:ssr

E. Substitutions, symmetries and realizations. We begin with a definition of a composition $F \circ \rho$, where F is a functional cochain , and ρ is a germ of a biholomorphic map at infinity that satisfies some requirements stated below.

Assumption A. Fix a class $\mathbf{\Omega}$ of standard domains. Let $\hat{\Omega}$ be a connected domain that contains $(\mathbb{R}^+, \infty)$ and belongs to some $\Omega \in \mathbf{\Omega}$. Let ρ be a conformal mapping of some standard domain $\tilde{\Omega} \subset \mathbf{\Omega}$ into $\hat{\Omega}$, and $\sigma = \rho^{-1}$. Let F be a cochain of type (11.1). Suppose that all the germs $\sigma \circ \sigma_j$ are $\mathbf{\Omega}$-admissible.

def:comp

DEFINITION 21. Under the assumption A a composition $F \circ \rho$ is a functional cochain defined in $\tilde{\Omega}$. If F is a tuple of functions F_j defined in the domains U_j of the corresponding partition, then $F \circ \rho$ is a tuple of functions $f_j \circ \rho$ defined in the domains $\tilde{U}_j = \tilde{\Omega} \cap \sigma_j U_j$, whenever this intersection is nonempty.

Under the assumptions above, the composition $F \circ \rho$ is a regular functional cochain of type $\sigma \circ \boldsymbol{\sigma} = (\sigma \circ \sigma_1, \dots, \sigma \circ \sigma_N)$. If F is weakly (almost, absolutely) realizable, then $F \circ \rho$ is weakly (almost, absolutely) realizable too.

This is proved in Chapter 2.

Assumption A^*. Let us now switch to a more general situation. Admit assumption A with the following change: suppose that not all the germs $\sigma \circ \sigma_1, \dots, \sigma \circ \sigma_N$ are $\mathbf{\Omega}$ admissible; on the contrary suppose that for some $k \in \{0, 1, \dots, N\}$ the germs $\sigma \circ g_{k+1}, \dots, \sigma \circ g_N$ are $\mathbf{\Omega}$-admissible and the germs $\sigma \circ \sigma_1 \dots, \sigma \circ \sigma_k$ are nonessential (for $k = 0$, this is a void assumption).

def:comp1

Definition 22. Under the assumption A^*, let us define a composition like $F\circ\rho$ above, that will be denoted $F_*\circ\rho$ for the reason explained right after the Definition. The definition is split in three cases.

Case 1. Let all the germs $\sigma\circ\sigma_j,\ j=1,\dots,N$, be $\mathbf{\Omega}$-admissible. Then let

$$F_*\circ\rho=F_{(0)}\circ\rho.$$

Case 2. Let $\sigma\circ\sigma_1,\dots,\sigma\circ\sigma_k$ be non-essential of class $\mathbf{\Omega}$, and $\sigma\circ\sigma_{k+1},\dots,\sigma\circ\sigma_N$ be $\mathbf{\Omega}$-admissible. Then

$$F_*\circ\rho=F_{(k)}\circ\rho.$$

This is a cochain of numeric type $N-k$.

Case 3. Let all the germs $\sigma\circ\sigma_j$ be non-essential of class $\mathbf{\Omega}$. Then

$$F_*\circ\rho=F_{(N)}\circ\rho.$$

This is a holomorphic function.

Remark 6. The notation F_* is chosen to stress that we do not know a priory, what realization of F to choose in order to define the substitution of ρ in F: the choice of the realization depends on ρ. In what follows, we omit the subscript $_*$ in the notations. Namely, when we write $F_{(k)}\circ\rho$ for the particular realization $F_{(k)}$ of a cochain F, we make use of Definition 21. When we write $F\circ\rho$, we mean $F_*\circ\rho$ in terms of Definition 22.

Remark 7. In Cases 2 and 3, the substitution $F_*\circ\rho$ is well defined in some standard domain $\tilde\Omega\in\mathbf{\Omega}$. Indeed, by Definition 4, there exists a standard domain $\tilde\Omega\in\mathbf{\Omega}$ such that

$$\sigma\circ\sigma_k(\Pi_*)\supset\tilde\Omega.$$

Hence,

$$\Omega_{(\boldsymbol{\sigma},k)}\supset\sigma_k(\Pi_*)\supset\rho\tilde\Omega.$$

Therefore, a substitution $F_{(k)}\circ\rho$ is well defined in $\tilde\Omega$ by Definition 21.

Remark 8. If the germ $\sigma\circ\sigma_1$ is $\mathbf{\Omega}$-non-essential, then a composition $F_{(0)}\circ\rho$ (that is well defined in $\tilde\Omega$) may not satisfy the *coboundary* requirement in the Definition 7. Thus, this will not be a regular cochain any more.

Under assumption A^*, the composition $F_*\circ\rho$ is a regular cochain of type $(\sigma\circ\sigma_{k+1},\dots,\sigma\circ\sigma_N)$ in Cases 1 or 2 ($k=0$ in Case 1); it is a holomorphic function in Case 3, and still a regular cochain in a standard domain of class $\mathbf{\Omega}$. If F is absolutely (weakly, almost) realizable, then $F_*\circ\rho$ also is.

This is proved in Chapter 2.

Let us now recall the definition of the symmetry operator. It is an operator on functions as: $(Sf)(\zeta)=\overline{f(\bar\zeta)}$. It also acts on cochains, as defined below.

Definition 23. For any cochain of class (11.1), where σ_j are admissible germs, let

eqn:sym

$$(SF)(\zeta)=\overline{F(\bar\zeta)}. \tag{11.32}$$

Let us now discuss the relations between the three features named in the title: substitutions, symmetry and realizability.

Note that if F is almost realizable, than SF also is: the k and k^+-realizations of SF are

$$(SF)_{(k)}=S(F^+_{(k)});\ (SF)^+_{(k)}=S(F_{(k)}).$$

If F is absolutely realizable, then SF also is:

$$(SF)_{(k,l)} = S(F^+_{(k,l)}),\ (SF)^+_{(k,l)} = S(F_{(k,l)}).$$

Consider now a cochain $S(F_* \circ \rho)$ for an absolutely realizable cochain F. In analogue to the above relations,

$$(SF_* \circ \rho)_{(l)} = S((F_* \circ \rho)^+_{(l)}).$$

But $F_* \circ \rho$ is in fact $F_{(k)} \circ \rho$ for some k. So,

$$(SF_* \circ \rho)_{(l)} = (SF_{(k)} \circ \rho)_l = S(F_{(k,l)} \circ \rho)$$

by definition 18.

In the same way we may prove that $F_* \circ \rho$ is absolutely realizable. But we stress here that for absolutely realizable cochain F, the cochain $SF_* \circ \rho$ is also absolutely realizable. This will be used several times when we apply the Phragmen-Lindelöf theorem to cochains of the form $SF_* \circ \rho$. This motivates the introduction of (k,l) and $(k,l)^+$ realizations.

sub:regcon

F. Standard domains, admissible germs and regular cochains of class n.

F.1. *Admissible germs of class n.* In this subsection we define regular cochains of class n and type 0 and 1. In the next section superexact asymptotic series of class n and type 0 and 1 ($STAR - n$) are constructed. Regular cochains of class n that may be decomposed in $STAR - n$ of types 0 and 1 respectively form the desired sets $\mathcal{F}C_0^n$ and $\mathcal{F}C_1^{n-1}$. Thus in this and the next section the model for the axioms stated above will be completed.

The definitions to follow are rather involved. They are motivated by the axioms stated above. These motivations will be presented in more details at the end of the section.

As mentioned in section 1.8, regularity properties imply the Phragmen-Lindelöf theorem, PL_n, and expandability, together with PL_n, implies the Lower Estimate Theorem, LET_n.

In this section we deal with regularity. We will define two sets of admissible germs of class n and types 0 and 1: D_0^n and D_1^{n-1}, and claim that they have the ordering and monotonicity properties. For any class D with these properties, a class of regular cochains $\mathcal{F}C_{\text{reg}}(D)$ was defined in 1.11B. Thus the sets

$$\mathcal{F}C^n_{0\text{reg}} = \mathcal{F}C_{\text{reg}}(D_0^n),\ \mathcal{F}C^{n-1}_{1\text{reg}} = \mathcal{F}C_{\text{reg}}(D_1^{n-1}),$$

occur. The sets that we plan to construct, $\mathcal{F}C_0^n$ and $\mathcal{F}C_1^{n-1}$ are subsets of $\mathcal{F}C^n_{0\text{reg}}$, $\mathcal{F}C^{n-1}_{1\text{reg}}$ respectively.

def:n0reg

Definition 24. The set of admissible germ of class n and type 0 is defined as

$$D_0^n = \{\exp \circ A^{1-n} g | g \in G^{n-1^-}_{\text{slow}}\}.$$

Example 2. For $n = 1$,

$$D_0^1 = \{\exp \mu\zeta | 0 < \mu < 1\}.$$

The set $\mathcal{F}C(D_0^1)$ is a set of all *regular sectorial* cochains.

The definition of D_1^{n-1} is more involved. Denote first by D_*^{n-1} a set

$$D_*^{n-1} = \{A^{1-n}g | g \in G_{\text{slow}}^{n-2^+} \cup G_{\text{rap}}^{n-2}\}.$$

For any $n \geq 3$ consider now a set

$$\mathcal{L}^{n-1} = \{A^{1-n}g | g \in G^{n-1}, g = Ad(f)A^{n-2}h, h \in \ln \circ \underline{\mathbf{TO}}, f \in G^{n-2}, \lambda_{n-2}(f) = 0\}$$

Note that this formula makes no sense for $n \leq 2$. Indeed, let $n \leq 2$. Then the set $\{f \in G^{n-2}, \lambda_{n-2}(f) = 0\}$ is empty. Let for $n \leq 3$, $\mathcal{L}^{n-1} = id$.

We can now define the set D_1^{n-1}:

$$D_1^{n-1} = D_0^{n-1} \cup D_*^{n-1} \circ \mathcal{L}^{n-1}.$$

EXAMPLE 3. For $n = 1$, $D_0^{n-1} = \emptyset$, $\mathcal{L}^{n-1} = id$ and $D^{n-1} = \mathcal{A}ff$. Hence, regular cochains of the class D_1^0 are regular simple cochains , see Part I. Recall that these cochains correspond to stretched standard partitions.

F.2. *Standard domains of class* n. Germs from the union $D_0^n \cup D_1^{n-1}$ are called *admissible germs of class* n. In order to discuss admissibility of these germs, we have to define standard domains of class n.

DEFINITION 25. A standard domain of class m corresponding to $\varepsilon > 0, C > 0$ is a domain

eqn:stn

$$\Omega_{m,\varepsilon,C} = \Phi_{m,\varepsilon}(\mathbb{C}_C^+) \tag{11.33}$$

where

eqn:phe

$$\Phi_{m,\varepsilon} : \zeta \mapsto \zeta(1 + (\ln^{[m-1]} \zeta)^{-\varepsilon}),\ \mathbb{C}_C^+ = \mathbb{C}^+ \setminus K_C, K_C = \{|z| \leq C\}. \tag{11.34}$$

PROPOSITION 1. *For any m, ε there exists $C(m, \varepsilon)$ such that for any $C > C(m, \varepsilon)$ the domain $\Omega_{m,\varepsilon,C}$ is standard in sense of Definition 1, and the map $\Phi_{m,\varepsilon}$ in the definition of this domain is conformal in $\mathbb{C}^+ \setminus K_C$.*

The proof is elementary.

DEFINITION 26. The set $\mathbf{\Omega}_m$ of standard domains of class m is the set of all domains $\Omega_{m,\varepsilon,C}$ with $C > C(m, \varepsilon)$.

In order to consider classes $\mathcal{FC}_{\text{reg}}$ for $D = D_0^n$ or D_1^{n-1} we need the following lemma:

lem:mon

LEMMA 1. 1^0. *Both classes D_0^n and D_1^{n-1} consist of $\mathbf{\Omega}_n$ and $\mathbf{\Omega}_{n-1}$ admissible germs respectively.*

2^0 *Each of the classes D_0^n and D_1^{n-1} has ordering and monotonicity properties defined in Subsection D.*

For $n = 1$, this lemma follows from Examples 2 and 3. For $1 < m \leq n$, we include this lemma into the induction assumption as a property of the germs of the classes D_0^m, D_1^{m-1}. For the classes D_0^{n+1}, D_1^n this lemma is proved in Chapter 5. This induction step (from n to $n + 1$) completes the proof of the lemma for any n.

Lemma 1 allows us to define weakly, almost and absolutely realizable cochains of classes D_0^n, D_1^{n-1}.

def:regn

DEFINITION 27. Regular cochains of class n type 0 or 1, are absolutely realizable cochains of classes $\mathcal{FC}_{\text{reg}}(D_0^n)$ and $\mathcal{FC}_{\text{reg}}(D_1^{n-1})$ respectively.

This completes the extandability part of the definition, the expandability part is given in the next section. Before passing to this part, let us give some motivations for the definitions given in this subsection.

G. Motivations: regularity. subsub:motiv

G.1. *Genesis of compositions with germs inverse to admissible ones.* As we will show in this subsection, shift lemmas imply that the sets of functional cochains of class n should be closed under the composition with certain germs: together with a germ of a cochain F, the set contains a germ $F \circ \rho$ for certain ρ. Compositions of this type are defined in subsection E. Note that if F is of type $\boldsymbol{\sigma}$, then $F \circ \rho$ is of type $\sigma \circ \boldsymbol{\sigma}$, where $\sigma = \rho^{-1}$.

G.2. *Genesis of the standard partition.* Normalizing cochains for parabolic germs of multiplicity 2 are cochains of numeric type 1 and type $\boldsymbol{\sigma} = id$. These cochains belong to the set $\mathcal{A}^0$. Recall that

$$J^{n-1} = Ad(G^{n-1})A^{n-1}\mathcal{A}^0,$$

and, by $SL_n a$,

$$J^{n-1} \subset \mathrm{Gr}(id + \mathcal{FC}^{n-1}_{1^+} \circ \exp^{[n-1]} \circ G^{n-2}).$$

This motivates a statement (that we will not prove, but use for motivations only): the set $\mathcal{FC}^{n-1}_{+1}$ contains cochains of the type $\boldsymbol{\sigma} = id$. These cochains correspond to standard partitions.

G.3. *Genesis of the sets D^n_0, D^{n-1}_0.* Occurrence of the set D^n_0 is motivated by $SL4_n b$. Together with Lemma $SL4_n a$, this lemma implies that for any $F_1 \in \mathcal{FC}^n_0$, $g \in G^{n-1}$, and some $F_2 \in \mathcal{FC}^{n-1}_{1^+}$, there exists $F \in \mathcal{FC}^n_0$ such that

$$F_1 \circ \exp^{[n]} \circ g \circ (id + F_2 \circ \exp^{[n-1]}) = F \circ \exp^{[n]} \circ g.$$

This implies

$$F = F_1 \circ \exp^{[n]} \circ g \circ (g^{-1} \circ \ln^{[n]} + F_2 \circ \rho),$$

where

$$\rho = A^{1-n} g \circ \ln .$$

We do not analyze here the whole expression for F, and prove that it belongs to $\mathcal{FC}^n_0$. We only note that this expression contains a term $F_2 \circ \rho$, and F_2 may be of type id. Then, according to Definition 21, $F_2 \circ \rho$ is a cochain of type $\sigma = \rho^{-1} = \exp \circ A^{1-n} g^{-1}$. We will prove that for $g^{-1} \in G^{n-1^+}_{\text{slow}} \cup G^{n-1}_{\text{rap}}$, the germ σ above is non-essential, and for $g^{-1} \in G^{n-1^-}_{\text{slow}}$, σ is admissible. This gives rise to the set D^n_0.

In the same way, the set D^{n-1}_0 arises, for n replaced by $n-1$.

G.4. *Genesis of $D^{n-1^+}_{slow}$.* Occurrence of this set is motivated by $SL3_n$ for $m = n-1$. According to this lemma, for any $F_1 \in \mathcal{FC}^{n-1}_1$, $F_2 \in \mathcal{FC}^{n-1}_{1^+}, f, g \in G^{n-2}$, $f \prec\prec g$, there exists $F \in \mathcal{FC}^{n-1}_1$ such that

$$F_1 \circ \exp_{[n-1]} \circ g \circ (id + F_2 \circ \exp^{[n-1]} \circ f) = F \circ \exp^{[n-1]} \circ g.$$

This implies:

$$F = F_1 \circ \exp^{[n-1]} \circ g \circ (g^{-1} \circ \ln^{[n]} + F_2 \circ \rho),$$

where

$$\rho = A^{1-n} h,\ h = f \circ g^{-1} \in G^{n-2^-}_{\text{slow}}.$$

The cochain F_2 may be of type id; hence $F_2 \circ \rho$ would be of type σ,

$$\sigma = A^{1-n} h^{-1},\ h^{-1} \in G^{n-2^+}_{\text{slow}},\ \sigma \in D^{n-1^+}_{\text{slow}}.$$

G.5. *Genesis of* D^{n-1}_{rap}. Occurrence of this set is motivated by $SL1_n$ for $m = n-1$. According to this lemma, for any $F_1 \in \mathcal{FC}^{n-1}_1$, $g \in G^{n-2}_{\mathrm{rap}}$, $f \prec\prec g$, there exists $F \in \mathcal{FC}^{n-1}_1$ such that

$$F_1 \circ \exp^{[n-1]} \circ g = F \circ \exp^{[n-1]}.$$

This implies:

$$F = F_1 \circ \rho,\ \rho = A^{1-n} g.$$

Hence, F is of type σ,

$$\sigma = A^{1-n} g^{-1},\ g^{-1} \in G^{n-2}_{\mathrm{rap}},\ \sigma \in D^{n-1}_{\mathrm{rap}}.$$

G.6. *Genesis of* $\mathcal{L}^{n-1}$. Occurrence of this set is motivated by $SL4_n a$, and the motivation is not as transparent as before: in our explanations we will skip many technical details presented later on in Chapters 2 and 5.

Lemma $SL4_n a$ claims:

$$J^{n-1} \subset \mathrm{Gr}(id + \mathcal{F}^{n-1}_{1\,+}).$$

To prove this lemma, we will need an analog of Lemma $SL4_n b$ for n and $\mathcal{FC}^n_0$ replaced by $n-1$ and $\mathcal{FC}^{n-1}_1$. Namely, for any $g \in G^{n-2}$,

$$\mathcal{F}^{n-1}_{1\ (+)g} \circ J^{n-2} = \mathcal{F}^{n-1}_{1\ (+)g}.$$

This implies that for any $F_1 \in \mathcal{FC}^{n-1}_1$, $j \in J^{n-2}$ there exists $F \in \mathcal{FC}^{n-1}_1$ such that

$$F_1 \circ \exp^{[n-1]} \circ g \circ j = F \circ \exp^{[n-1]} \circ g.$$

Note that $Ad(g)j = j_1 \in J^{n-2}$ by definition of J^{n-2}. Hence, we get:

$$F = F_1 \circ \rho, \rho = A^{1-n} j_1,\ j_1 \in J^{n-2}.$$

In Chapter 2 we will show that, in order that the set $\mathcal{FC}^{n-1}_1$ be closed under the compositions $F \circ \rho$ with $\rho \in A^{1-n} J^{n-2}$, the set D^{n-1}_* should be extended by the right compositions with the elements of the group $\mathcal{L}^{n-1}$.

G.7. *Genesis of standard domains of class* n. The necessity to change the class of standard domains together with n occurs in the proof of the admissibility of the germs of class D^{n-1}_*, and in the proof that the germs of the form $\exp A^{1-n} g$, $g \in G^{n-1}_{rap}$ are non-essential. Let us discuss the first statement only. Let $\sigma = A^{1-n} g, g \in G^{n-2}_*$. One of the requirements of $\mathbf{\Omega}_{n-1}$-admissibility of the germ σ claims that for any $\Omega \in \mathbf{\Omega}_{n-1}$ there exists $\tilde{\Omega} \in \mathbf{\Omega}_{n-1}$ such that

$$\rho\tilde{\Omega} \subset \Omega,\ \text{where } \rho = \sigma^{-1}.$$

For $g \in G^{n-2^+}_{\mathrm{slow}}$, the germ $\rho = A^{1-n} g^{-1}$ is contracting in a sense that

$$(\rho\Omega, \infty) \subset (\Omega, \infty)$$

for any $\Omega \in \mathbf{\Omega_{n-1}}$. In this case $\tilde{\Omega}$ may be obtained by deducting a compact set from Ω.

On the contrary, when $g \in G^{n-2^+}_{\mathrm{rap}}$, the germ ρ may be expanding in a sense that

$$(\Omega, \infty) \subset (\rho\Omega, \infty).$$

The size of the margin between Ω and $\rho\Omega \supset \Omega$ depends on n. The margin between Ω and $\tilde{\Omega}$ should be so large as to compensate the previous margin. The margin between Ω and $\tilde{\Omega}$ should therefore depend on n also. This motivates the necessity of dependence of the class $\mathbf{\Omega}_{n-1}$ on n.

To conclude, note that the margin between different domains of class $\mathbf{\Omega}_{n-1}$ in a sense grows with n.

This completes the discussion of the regularity requirements for the cochains of classes $\mathcal{FC}_0^n, \mathcal{FC}_1^{n-1}$. Let us pass to expendability.

S 1.12. Superexact asymptotic series

The series and cochains mentioned above, have class $n \geq 0$, rank $r \geq 0$ and type 0 or 1. They are defined by double induction: the exterior one in n and the interior one in r. Notations: STAR$-(n-1,r)_1$, $\mathcal{FC}_1^{n-1}$, STAR$-(n,r)_0$, $\mathcal{FC}_0^n$. Series and cochains of class 1 are defined in the first part. Below we briefly repeat the definition. We start with the STAR of type 1.

A. Base of induction. For $n = 0$, the rank is not considered, and the type is zero. Superexact series of class 0 and type 0 are the *Dulac exponential series*. Functional cochains of class zero type zero are almost regular germs. They are trivial cochains: the coboundary is zero. The Dulac exponential series have the form that will be used all over the book:

eqn:super

$$\Sigma = \sum a_j \exp \mathbf{e}_j, \tag{12.1}$$

where $\mathbf{e}_j$ are called *exponents*, and a_j are *the coefficients*. In order to define the new class of series we should define the set E of exponents, and the set $\mathcal{K}$ of the coefficients. For the Dulac exponential series $E = \mathbb{R}$, and $\mathcal{K}$ is the set of all real polynomials.

Step of induction in n from 0 to 1 is proceeded in Part 1. The description of this step may be repeated if we substitute $n = 1$ into the following text. Here we briefly recall this description. The series of class 1 type 1, STAR-0_1, are the Dulac exponential series again. The cochains of class 1 type 1, $\mathcal{FC}_1^0$, are simple cochains $\mathcal{FC}^0$ defined in Part 1. They are decomposed in STAR-0_1.

The series of class 1 type 0 rank 0, STAR$-(0,0)_1$ are of the form (12.1) with $\mathbf{e}_j \in E^1$, $a_j \in \mathcal{K}_1^{0,0}$, where the sets of exponents E^1, and the set of coefficients $\mathcal{K}_1^{0,0}$ are defined as follows.

The set E^1 does not depend on the rank. It is a set of all the partial sums of the exponential Dulac series with non-negative exponents. Namely,

$$E^1 = \{\mathbf{e} | \mathbf{e} = \sum P_j(\zeta) \exp \mu_j \zeta\},$$

where the sum is finite, P_j are real polynomials, and μ_j are positive. The set of coefficients $\mathcal{K}_1^{0,0}$ of rank 0 is the set $\mathcal{FC}_1^0$ of all simple functional cochains. In Part 1 this set was denoted as $\mathcal{FC}^0$. This defines the set of all STAR of class 1, type 1 and rank 0. For the higher rank, the definition is given in Part 1, and may be deduced from the general definition given below.

B. Step of induction from $n-1$ to n: definition of STAR$-(n-1)_1$, and cochains of class $\mathcal{FC}_1^{n-1}$. Now the step of exterior induction in n comes. Fix n and suppose that the STARs and functional cochains of class $m < n$ are already defined. This implies, in particular, that the set of exponents E^{n-1} is defined.

We define first the series and cochains of class n and type 1, named in the heading. The definition goes by induction in r.

Base of induction in r is the definition of the set of coefficients $\mathcal{K}_1^{n-1,r}$ of STAR$-(n-1,r)_1$ for $r=0$.

def:coef0

DEFINITION 1.

$$\mathcal{K}_1^{n-1,0} = \mathcal{L}(\mathcal{FC}_1^{n-2} \circ \exp^{[n-2]} \circ g | g \in G^{n-3}),$$

the set $\mathcal{FC}_1^{n-2}$ is defined by the "exterior" induction assumption in n; $\mathcal{L}(\cdot)$ is the linear hull.

Step of induction in r.

def:ser

DEFINITION 2. Suppose that the set $\mathcal{K}_1^{n-1,r}$ is already defined. Let the set $\mathcal{E}_1^{n-1,r}$ of STAR$-(n-1,r)_1$ be the set of all formal series:

$$\Sigma = \sum a_j \exp \mathbf{e}_j, \ \mathbf{e}_j \in E^{n-1}, \ a_j \in \mathcal{K}_1^{n-1,r}.$$

def:decomp

DEFINITION 3. A functional cochain F is said to be expandable in its domain Ω in a STAR$-(n-1,r)_1$ denoted by Σ, if there exists for every $\nu > 0$ a partial sum of the series Σ approximating $F \circ \exp^{[n-1]}$ with accuracy $o(\exp(-\nu \text{ Re } \exp^{[n-1]}))$ in the domain $\ln^{[n-1]} \Omega$.

def:classn

DEFINITION 4. A functional cochain F is of class $n-1$, rank r, and type 1, if it has the following properties: regularity and expandability.

Regularity: F is an absolutely realizable regular cochain of class $\mathcal{FC}_{\text{reg}}(\mathcal{D}_1^{n-1})$ in the sense of the definition in Section 1.11.

Expandability: the composition $F \circ \exp^{[n-1]} = \varphi$ can be expanded in a STAR$-(n-1,r)_1$. Moreover, the compositions $F \circ \exp^{[n-1]} = \varphi$ may be expanded in the same asymptotic series for all the k, k^+ and (k,l)-realizations of F. Here F is considered in an appropriate standard domain of class $n-1$, and its realizations are considered in some generalized neighborhoods of the the corresponding domains, see (11.17), (11.18), (11.19).

Notation: recall that the set of all functional cochains of class $n-1$, rank r, and type 1 is denoted by $\mathcal{FC}_1^{n-1,r}$.

The following definition completes the induction step:

$$\mathcal{K}_1^{n-1,r+1} = \mathcal{L}(\mathcal{K}_1^{n-1,r}, \ \mathcal{FC}_1^{n-1,r} \circ \exp^{[n-1]} \circ g | g \in G_{\text{slow}}^{n-2^-}).$$

Namely, we define the set of series $\mathcal{E}_1^{n-1,r+1}$ by the use of Definition 2, replacing r by $r+1$.

The set of all STAR$-(n-1)_1$ is denoted by $\mathcal{E}_1^{n-1}$ and defined as

$$\mathcal{E}_1^{n-1} = \cup_{r \geq 0} \mathcal{E}_1^{n-1,r}.$$

The set of all functional cochains of class $n-1$ and type 1 is denoted by $\mathcal{FC}_1^{n-1}$ and defined as

$$\mathcal{FC}_1^{n-1} = \cup_{r \geq 0} \mathcal{FC}_1^{n-1,r}.$$

Summarizing, we give the following

def:classn1

DEFINITION 5. Functional cochains of class n type 1 are absolutely realizable functional cochains of type D_1^{n-1} that may be decomposed to STAR$-(n-1)_1$ in the generalized ε-neighborhoods of their domains, together with all their realizations.

This completes the definition of superexact asymptotic series and functional cochains of class n type $1 : \mathcal{FC}_1^{n-1}$. Let us pass to type 0.

C. Definition of STAR$-(n)_0$, and cochains of class $\mathcal{FC}_0^n$. We can now define the set of series and cochains of level n and type 0: STAR$-n_0$, $\mathcal{FC}_0^n$. For this we first define the set E^n of exponents of STAR$-n_0$.

def:expn

DEFINITION 6. The set E^n of exponents of STAR$-n_0$ is the set of partial sums of STAR$-(n-1)_1$ with the following properties:

1^0. The sum $\mathbf{e} \in E^n$ is weakly real and has the form:

$$\mathbf{e} = \sum_1^N a_j \exp \tilde{\mathbf{e}}_j, \ \tilde{\mathbf{e}}_j \in E^{n-1},$$

and

$$\nu_{n-1}(\tilde{\mathbf{e}}_j) := \lim_{(\mathbb{R}^+,\infty)} \frac{\tilde{\mathbf{e}}_j}{\exp^{[n-1]}} \geq 0;$$

2^0. The real limit

$$\nu(\mathbf{e}) = \lim_{(\mathbb{R}^+,\infty)} \frac{\mathbf{e}}{\exp^{[n]}}$$

exists; it is called the principal exponent of the term with exponent $\mathbf{e}$;

3^0. There exists a $\mu > 0$ and a standard domain of class n in which

$$|\ \mathrm{Re}\ \mathbf{e} \circ \ln^{[n]} | < \mu\xi, \ |\mathbf{e} \circ \ln^{[n]} | < \mu|\zeta|;$$

4^0.

$$\mathrm{Im}\ \mathbf{e} \to 0 \text{ on } (\mathbb{R}^+, \infty).$$

Requirement 4^0 follows from 1^0 by the criterion of being weakly real; it is included for the future references.

The mapping $\nu :\to \mathbb{R}, \ \mathbf{e} \mapsto \nu(\mathbf{e})$, is called the principal exponents mapping.

Let us now pass to the definition of STARs and cochains of class n, type 0. They have rank $r \geq 0$ and are defined in the same way as those of class $n-1$, type 1; the only difference is that standard domains of class $n-1$ are replaced by those of class n. The definition goes by induction in r.

Base of induction: $r = 0$. Let

$$\mathcal{K}_0^{n,0} = \mathcal{FC}_1^{n-1} \circ \exp^{[n-2]} \circ G^{n-2}.$$

After that the induction goes exactly as in the previous case, and results in the definition of the sets $\mathcal{E}_0^n$ and $\mathcal{FC}_0^n$ of series and cochains of class n and type 0.

Summarizing, we give the following

def:classn0

DEFINITION 7. Functional cochains of class n type 0 are absolutely realizable functional cochains of type D_0^n defined in some standard domain of class n that may be decomposed to STAR$-n_0$ in the generalized ε-neighborhoods of their domains, together with all their realizations.

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Proposition Theorem Remark Important Remark Lemma Corrolary Problems Problem Definition

CHAPTER 2

Function-Theoretic Properties of Regular Functional Cochains

Here we prove the "function-theoretic" part of the lemmas in S1.10. The corresponding formulations are obtained if cochains of class $\mathcal{FC}^m$ are replaced everywhere in these lemmas by cochains of class $\mathcal{FC}_{\mathrm{reg}}(\mathcal{D}^m)$, keeping thereby the regularity requirement and waiving the decomposability requirement. The lemmas obtained are denoted by SLl-$4_{n,\mathrm{reg}}$; they are stated in Subsection "5.1" CCC

The chapter splits in two parts. In the first one we present the general theory of regular functional cochains. This part requires very few information on the admissible germs with the help of which the cochains are defined. In other words, we study cochains of type $\mathcal{F}C_{\mathrm{reg}}(D)$ with minimal requirements on the set D.

In the second part we study the cochains of types $\mathcal{FC}_{\mathrm{reg}}(D_0^n)$ and $\mathcal{FC}_{\mathrm{reg}}(D_1^{n-1})$. Their main property is that they satisfy the shift lemmas $SL1-4n,\mathrm{reg}$, stated in Section **??**. QQQ In Subsection **??** of Chapter 1, we have shown that the cochains that satisfy these lemmas *should be* of classes (denote them by S_0^n, S_1^{n-1} for a while) that contain the classes defined in Section **??**: QQQ

$$S_0^n \supset D_0^n,\ S_1^{n-1} \supset D_1^{n-1}.$$

The rest of the chapter, except for the last section, is devoted to the proofs of Shift Lemmas $1_n - 4_{n,\mathrm{reg}}$. In particular we show that the classes defined in **??** QQQ are sufficient; one may take:

$$S_0^n = D_0^n,\ S_1^{n-1} = D_1^{n-1}.$$

In the last section we prove that partial sums of $STAR - n_0$ and $STAR - (n-1)_1$ may be expressed through regular cochains of classes D_0^n, D_1^{n-1} respectively.

S 2.1. Differential algebras of cochains

sec:difalg

Recall the following

def:proper

Definition 1. A class $\mathbf{\Omega}$ of standard domains is said to be proper if:

(1°) for any $C > 0$ an arbitrary domain of class $\mathbf{\Omega}$ contains a domain of the same class whose distance from the boundary of the first is not less than C;

(2°) the intersection of any two domains of class $\mathbf{\Omega}$ contains a domain of the same class.

In the first part of this chapter: Sections 2.1 -2.5 we consider a fixed proper class $\mathbf{\Omega}$ of standard domains. We also consider a fixed set D of $\mathbf{\Omega}$-admissible germs. We require that D satisfy the ordering and monotonicity properties from F below. QQQ As soon as these two sets are fixed, a class $\mathcal{FC}_{\mathrm{reg}}(D)$ of germs of absolutely (respectively, weakly or almost) realizable functional cochains occurs according to the definitions in Section 1.11. QQQ Let us study the properties of the class of cochains $\mathcal{FC}_{\mathrm{reg}}(D)$.

sub:difst

A. Cochains in standard domains.

lem:difalg3

LEMMA 1. *Let $\mathbf{\Omega}$ be a proper class of standard domain such that $|\zeta| \prec \xi^3$ in any domain of this class. Let D be any set of $\mathbf{\Omega}$-admissible germs, and $\mathcal{FC}_{reg}(D)$ be the corresponding set of germs of regular functional cochains of class D. Then this set forms a differential algebra.*

sub:sump

A.1. *Sums and products.*

PROOF. Recall that arithmetic operations on cochains are defined at the end of (**??**). Namely, let F_1 and F_2 be two cochains of type $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ respectively; let $\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 \cup \boldsymbol{\sigma}_2$:

eqn:ssigmanf (1.1)
$$\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_K),\ \sigma_j \rhd \sigma_{j+1};$$

$\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are ordered subsets of $\boldsymbol{\sigma}$.

Let two cochains $F_l,\ l = 1, 2$ be defined in a domain $\Omega_l \in \mathbf{\Omega}$, and let $\Omega \in \mathbf{\Omega}$ belong to the intersection $\Omega_1 \cap \Omega_2$. Such a domain Ω exists because the class $\mathbf{\Omega}$ is proper.

By definition of the sum, $F_1 + F_2$ is a cochain of type $\boldsymbol{\sigma}$ that corresponds to the product Ξ of the partitions

$$\Xi_l = \prod_{\sigma_j \in \boldsymbol{\sigma}_l \sigma_{j*} \Xi_{\mathrm{st}},\ l=1,2,}$$

of the domain Ω. This product is the partition

eqn:ppxi (1.2)
$$\Xi = \prod_{\sigma_j \in \boldsymbol{\sigma}} \sigma_{j*} \Xi_{\mathrm{st}}.$$

Any domain U of this partition is an intersection of two domains U_l of the partitions $\Xi_l,\ l = 1, 2$, intersected with Ω. A function F_U of the tuple F corresponding to this domain is

$$F_U = F_{U_1} + F_{U_2} \text{ in } U = U_1 \cap U_2 \cap \Omega.$$

Let us check the assumptions from B of Chapter 1. QQQ

Partition: guarantied by (1.2).

Extendability: the functions F_{U_l} are extendable to the generalized ε-neighborhoods $U_l^{(\varepsilon)}$ of the domains U_l. The intersections of these neighborhoods form a generalized ε-neighborhoods $U^{(\varepsilon)}$ of U by definition 4. QQQ Hence, F_U may be extended to $U^{(\varepsilon)}$ as required.

Growth: exponential growth of the terms implies exponential growth of the sum.

Coboundary: the rigging cochain for the product of two regular partitions is, by Definition **??**, QQQ the sum of corresponding rigging cochains. The magnitude of the coboundary of the sum is estimated by the sum of magnitudes of the coboundaries for the terms. This verifies the coboundary condition.

This proves that the set $\mathcal{FC}_{\mathrm{reg}}(D)$ is closed in Ω with respect to addition and deduction. Let us prove that it is closed with respect to multiplication.

The cochain $F = F_1 \cdot F_2$ is a tuple of holomorphic functions that correspond to the domains of the partition (**??**). In the above notations

$$F_U = F_{U_1} \cdot F_{U_2}.$$

This allows us to prove the *Partition* and *Expendability* properties for products as well as for sums.

The *Growth* property follows from the remark that the product of two exponentials is an exponential again.

The *Coboundary* property requires a minor calculation based on the fact that "a rigging cochain beats an exponential". Namely, we estimate the coboundary of the product:

$$|\delta(\tilde{F}_1\tilde{F}_2)| < |\tilde{F}_1||\delta\tilde{F}_2| + |\tilde{F}_2||\delta\tilde{F}_1| \leq (\exp\nu\xi)(m^1 + m^2).$$

Here m^1 and m^2 are rigging cochains of the partitions Ξ_1 and Ξ_2 that majorize the coboundaries $\delta\tilde{F}_1$ and $\delta\tilde{F}_2$.

But the product of a rigging cochain of a regular partition by an exponential is majorized by some other rigging cochain corresponding to the same partition. Indeed, by requirement 4° in the definition of admissible germs (Definition 2 in S1.5), QQQ for an arbitrary admissible germ there exists a standard domain $\Omega \in \mathbf{\Omega}$ (this domain is also of class n for germs of class n) such that for arbitrary $\nu > 0$,

$$\operatorname{Re}\exp\rho \succ \nu\xi \text{ in } \Omega.$$

Consequently,

$$|\exp(-C\exp\rho + \nu\xi)| \prec \left|\exp\left(-\frac{C}{2}\exp\rho\right)\right| \text{ in } \Omega.$$

sub:deriv This ends the proof of Lemma 1 for sums, differences, and products.

A.2. *Derivatives.* The proof that derivatives are regular is based on the Cauchy estimate and uses Proposition 1 below.

Let us prove that a derivative of a regular functional cochain of class $\mathcal{FC}_{\text{reg}}(D)$ in a standard domain $\Omega \in \mathbf{\Omega}$ is again a regular functional cochain of class $\mathcal{FC}_{\text{reg}}(D)$.

Properties *partition and extendability* are immediate. Let us prove the *growth* property. Let $F \in \mathcal{FC}_{\text{reg}}(D), F$ is defined in $\Omega \in \mathbf{\Omega}$. Then

$$|F(\zeta)| \prec \exp\nu\xi \text{ in } \Omega^\varepsilon$$

for some $\nu > 0$ by the *extendability* and *growth* requirement for F. We will use the Cauchy inequality, and the extendability property of the cochain.

def:marg DEFINITION 2. The margin between the domains imbedded one in the other is a function in the smaller domain equal to the distance from the point of this domain to the boundary of the larger domain.

prop:marg PROPOSITION 1. *Consider an arbitrary domain Ω and a partition (1.2) of Ω. Then for any $\varepsilon > \delta > 0$, the margin between generalized ε and δ neighborhoods of ant domain of the partition is bounded from below by a value $(\varepsilon - \delta)C$ for some $C > 0$ depending on the partition only.*

This proposition is proved in Section D below.

Now, let the components of F be estimated as above in generalized ε-neighborhoods of their domains. Then, for $\delta < \varepsilon$, their derivatives are estimated in the generalized ε-neighborhoods of their domains in the following way:

$$|F'(\zeta)| \prec C\varepsilon^{-1}\exp\nu\xi \prec \exp\nu'\xi$$

in Ω for any $\nu' > \nu$. This implies the *growth* property.

Let us now prove the *coboundary* property for F'.

By Proposition 1, the margins between the generalized ε and δ neighborhoods of a partition (1.2) are bounded from below. Our arguments are the same as above, but presented much shorter. By the Cauchy inequality, the coboundary of F' is bounded from above by a rigging cochain of the partition (1.2) that majorizes the coboundary of F multiplied by a constant. This is a rigging cochain of the partition (1.2) again.

This proves Lemma 1. □

sub:nonstan

B. Differential algebras in non-standard domains. Let Ω_ε be a family of connected domains in $\mathbb{C}^+$ that contain $(\mathbb{R}^+, \infty)$. Let this family have the following two properties:

Monotonicity:

$$\Omega_\varepsilon \supset \Omega_\delta \text{ for } \delta < \varepsilon;$$

Margin:

For any $\delta < \varepsilon$, consider a function $M_{\delta,\varepsilon}$ on the boundary $\partial\Omega_\sigma$:

$$M_{\delta,\varepsilon}(\zeta) = \text{ dist } (\zeta, \partial\Omega_\varepsilon).$$

Suppose that there exists $C > 0$ such that

eqn:mar

$$M_{\delta,\varepsilon}(\zeta) > C\xi^{-3}, \ \xi = \text{ Re } \zeta, \ \zeta \in \partial\Omega_\delta. \tag{1.3}$$

This inequality is called *the margin condition.*

Remark 9. Below we prove that the generalized neighborhoods of domains of realizations of regular cochains (**??**), QQQ (**??**), QQQ (**??**) QQQ have this property, see Proposition 2.

Let $\hat{\Omega} = \Omega_0$. Fix any set D of admissible germs. Definitions of C, B of Chapter 1 provide a set of regular cochains of class D in $\hat{\Omega}$, $\mathcal{FC}_{\text{reg}}(D, \hat{\Omega})$, provided that ε-extendability requires, in particular, holomorphic extendability of cochains of this class to Ω_ε.

lem:difalgnon

Lemma 2. *The set of cochains $\mathcal{FC}_{reg}(D, \hat{\Omega})$ defined above, forms a differential algebra.*

Proof. The statement about sums and products is proved exactly as in the previous lemma. The differentiability follows from the Cauchy inequality, and the margin assumption (1.3). □

sub:difreal

C. Realizations. We now strengthen Lemma 1.

lem:difalgreal

Lemma 3. *For Ω and D as in Lemma 1, absolutely realizable cochains of class $\mathcal{FC}_{reg}(D)$ and $\mathcal{FC}^+_{reg}(D)$ form a differential algebra.*

sub:deriv1

C.1. *Derivatives in the domains of realizations.*

Proof. Sums and products in nonstandard domains are considered exactly as in standard domains.

Consider the derivatives of realizations. Take an absolutely realizable cochain $F \in \mathcal{FC}_{\text{reg}}(D)$ and let G be any of its realizations $(k, k^+$ or $(k, l))$. We claim that G' is the corresponding realization of F'. The two properties to be justified are the *Growth* and the *Coboundary* conditions. As shown in the proof of Lemma 2, it follows from the margin condition (1.3) for the domains where the cochains are defined. Hence, Lemma 3 for derivatives follows from the Proposition:

prop:marg1

Proposition 2. *Suppose that σ is an admissible germ. Then for some compact set K and any small $0 < \delta < \varepsilon$, the margin between $\sigma(\Pi_*^{(\varepsilon)} \setminus K^{(\varepsilon)})$ and $\sigma(\Pi_*^{(\delta)} \setminus K^{(\delta)})$ is no less than $C(\varepsilon - \delta)|\zeta|^{-3}$.*

Proposition 2 implies Lemma 3 for derivatives.

This implies that the germs of regular cochains defined in the domains CCC where the realizations are defined, and corresponding to the partitions CCC, form a differential algebra.

subsub:sumpreal

C.2. *Sums and products revisited.* Realizations of sums and products are sums and products of "corresponding realizations", to be specified. Sums and products of realizations are regular cochains again by Lemma 2. The body of the proof of Lemma 3 for sums and products is the choice of proper realizations of the entries in order to produce the required realization of the outcome.

Consider an ordered finite set A and its subset B:

$$A = \{a_1, \dots, a_K\},\ B = \{b_1, \dots, b_N\}.$$

When these sets will consist of admissible germs, the order relation will be $\rhd$. So, in the general context, we use the same relation. Let us define a map $i(A, B)$:

$$i(A, B) = i : \{1, \dots, K\} \to \{0, 1, \dots N\} :$$

$$i(k) = \begin{cases} 0 \text{ for } a_k \rhd b_1 \\ j \in \{1, \dots, N-1\} \text{ for } b_j \trianglerighteq a_k \succ b_{j+1} \\ N \text{ for } b_N \trianglerighteq a_k. \end{cases}$$

Let now

$$\boldsymbol{\sigma}_1 = \{\sigma_1, \dots, \sigma_N\},\ \boldsymbol{\sigma}_2 = \{\tilde{\sigma}_1, \dots, \tilde{\sigma}_M\},\ \boldsymbol{\sigma} = \boldsymbol{\sigma}_1 \cup \boldsymbol{\sigma}_2 = (\hat{\sigma}_1, ..., \hat{\sigma}_K)$$

Denote by i, j the maps

$$i = i(\boldsymbol{\sigma}, \boldsymbol{\sigma}_1) : \{1, \dots, K\} \to \{0, 1, \dots, N\};\ j = i(\sigma, \sigma_2) : \{1, \dots, K\} \to \{0, 1, \dots, M\}.$$

Set

$$s(k, l) = i(k + l) - i(k),\ t(k, l) = j(k + l) - j(k).$$

We can now define the realizations for the sum (for the difference and product they are defined in a similar way, only summation is replaced by deduction or multiplication). We have:

$$(F_1 + F_2)_{(k)} = F_{1(i,(k))} + F_{2(j(k))}$$

$$(F_1 + F_2)_{(k,l)} = F_{1(i(k),s(k,l))} + F_{2(j(k),t(k,l))}.$$

These cochains satisfy all the requirement of Definitions 16, 17, 18. QQQ Checking of these requirement are routine, similar to what was done in this context in part 1, and we skip it.

Definitions of k^+ and $(k, l)^+$-realizations are obtained from those for k and (k, l)-realizations by the symmetry with respect to the real axis. So we discuss the existence of k and (k, l)-realizations only. The existence of k^+ and $(k, l)^+$-realizations follows authomatically. $\square$

sub:margins

D. Margins. Here Propositions 1 and 2 are proved.

Proof of Proposition 1. We begin with a remark. Let σ be a biholomorphic mapping of a planar domain, and assume that the inverse mapping has bounded derivative: $\sigma^{-1} = \rho, |\rho'| < C$. Then σ decreases the distance between points no more than by a factor of C. Indeed, if ζ_1 and ζ_2 are arbitrary points in the domain of σ and $\omega_j = \sigma(\zeta_j)$, for $j = 1, 2$, then $\zeta_j = \rho(\omega_j)$ and $|\zeta_1 - \zeta_2| < C|\omega_1 - \omega_2|$. Therefore,

$$|\sigma(\zeta_1) - \sigma(\zeta_2)| = |\omega_1 - \omega_2| > |\zeta_1 - \zeta_2|/C.$$

This immediately implies the first assertion of the proposition, since the derivative of the germ inverse to an admissible germ is bounded according to Definition 2 in S1.5.

Proof of Proposition 2. It follows from the definition of the generalized ε-neighborhood of the half-strip Π_{main} (Definition 6 in S1.6) that there exists a constant $C_1 > 0$ such that $\mathrm{dist}(\zeta, \partial\Pi^{(\varepsilon)}_{\mathrm{main}}) \geq C_1(\varepsilon - \delta)|\zeta|^{-3}$ for $z \in \Pi^{(\delta)}_{\mathrm{main}}$. Consequently, by the remark at the beginning of the proof,

$$\mathrm{dist}(\sigma(\zeta), \partial\Pi^{(\varepsilon)}_{\mathrm{main}}) \geq C_1(\varepsilon - \delta)/|\rho(\zeta)|^3 \sup|\rho'|.$$

But since the derivative $|\rho'|$ is bounded, the function $|\rho(\zeta)|$ increases no more rapidly than $|\zeta|$; consequently, there exists a constant $C_2 > 0$ such that $|\rho(\zeta)| < C_2|\zeta|$. This implies the proposition, and with it Lemma 3. □ □

S 2.2. Completeness

We fix a standard domain Ω or a nonstandard domain $\hat{\Omega}$, that belongs to Ω and contains $(\mathbb{R}^+, \infty)$. Let Ω_ε be an increasing family of domains that satisfies the margin condition (1.3).

Consider a space $\mathcal{F}$ of regular cochains of one and the same type $\boldsymbol{\sigma}$, see (9.2), in Ω or $\hat{\Omega}$ that are ε-extendable with the same ε and are extimated from above by the exponential $a \cdot \exp \nu\xi$ with a common exponent ν and coefficient a depending on the cochain.

Moreover, the coboundaries of the cochains $F \in \mathcal{F}$ can be estimated from above in modulus by one and the same rigging cochain m for the partition Ξ, multiplied by a constant depending on the cochain, $|\delta F| \leq C_F m$.

Let $\Xi^{(\varepsilon)}$ be the domain of the rigging cochain m. We introduce in the space $\mathcal{F}$ the norm

eqn:norm

$$\|F\| = \sup_{\Omega} |F \exp(-\nu\xi)| + \sup_{\Xi^{(\varepsilon)}} |F/m|. \tag{2.1}$$

lem:comp4

Lemma 1. *The space $\mathcal{F}$ is complete in the norm introduced.*

Proof. This is an immediate consequence of the Weierstrass theorem on completeness of the space of holomorphic functions with the C-norm. □

Let us now state and prove the completeness of the space of absolutely realizable regular functional cochains.

The formula (2.1) defines the norm of a cochain F in a standard domain, as well as the norms of all its realizations. The exponent $\exp \nu\xi$ is the same for all the realizations. On the contrary, the rigging cochains in the deminator depend on realization as explained in Section 11.D of Chapter 1. Denote by $\mathbf{F}$ the set of all realizations G of F:

$$\mathbf{F} = \{\mathbf{F}_{(\mathbf{k})}, \mathbf{F}^+_{(\mathbf{k})}, \mathbf{F}_{(\mathbf{k},\mathbf{l})}, \mathbf{F}^+_{(\mathbf{k},\mathbf{l})}\}.$$

Now let

eqn:normreal

$$||F||_r = ||F|| + \Sigma_{G \in \mathbf{F}} ||G||. \tag{2.2}$$

lem:comp5

Lemma 2. *The space of absolutely realizable regular cochains with the norm* (2.2) *is complete.*

Proof. The proof is the same as for Lemma 1. □

In the corollaries below, if the cochains in the right hand side are absolutely realizable, then so are the cochains in the left hand side .

Corrolary 1. *Let $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function. Then $\varphi \circ \mathcal{FC}^+_{\mathrm{reg}}(\mathcal{D}) \subset \mathcal{FC}^+_{\mathrm{reg}}(\mathcal{D})$.*

Proof. Let $F \in \mathcal{FC}^+_{\mathrm{reg}}(\mathcal{D})$. Then $\varphi \circ F = \sum \varphi^{(j)}(0) F^j / j!$. By Lemma 1, the partial sums of this series are cochains in $\mathcal{FC}_{\mathrm{reg}}(\mathcal{D})$. Since the Taylor series for φ converges, and F is bounded (and even decreasing), the partial sums form a Cauchy sequence in $\mathcal{F}$. □

Corrolary 2.

$$\exp \mathcal{FC}^+_{\text{reg}}(\mathcal{D}) \subset 1 + \mathcal{FC}^+_{\text{reg}}(\mathcal{D}).$$

Proof. The function $\varphi : \zeta \mapsto \exp \zeta - 1$ satisfies the condition of Corollary 1. □

Corrolary 3.

$$\ln(1 + \mathcal{FC}^+_{\text{reg}}(\mathcal{D})) \subset \mathcal{FC}^+_{\text{reg}}(\mathcal{D}),$$
$$1/(1 + \mathcal{FC}^+_{\text{reg}}(\mathcal{D})) \subset 1 + \mathcal{FC}^+_{\text{reg}}(\mathcal{D}).$$

Proof. The proof is analogous to the preceding proof. □

S 2.3. Group property of map cochains $id + \mathcal{FC}_{wr}$

sec:grp
lem:grp

Lemma 1. *The set of germs at infinity of the map cochains $id + \mathcal{FC}_{wr}$ forms a group with the operation composition.*

To prove the lemma, we have to prove that the set $id + \mathcal{F}^{m^+}_{wr}$ is closed under multiplication and inversion (multiplication and inversion properties respectively). Let us prove the second one.

A. Inversion property of the set $id + \mathcal{FC}_{wr}$. A map cochain $id + F \in id + \mathcal{F}^{m^+}_{wr}$ is a collection of functions $id + f_j$ defined in the generalized ε-neighborhoods of the corresponding domains U_j. The map-cochain $(id + F)^{-1}$ is a collection of functions $(id + f_j)^{-1}$ defined in generalized $\frac{\varepsilon}{2}$-neighborhoods of the same domains U_j. Let us check that the Ω of the cochain F may be so chosen that the cochain $(id + F)^{-1} - id$ satisfies all the requirements of the definition of weakly decreasing regular cochains .

0^0. *Domain.* Let us chose the domain Ω of F in such a way that

- components of F are extendable to the generalized ε-neighborhoods of the corresponding domains,
- the margins between the generalized $\frac{\varepsilon}{2}$ and ε-neighborhoods of these domains are greater than $\delta < \varepsilon$,
- $|F(\zeta)| < \exp(-\nu\xi)$ in Ω^ε for some $\nu > 0$,
- $|F| < \frac{\delta}{2}$ in Ω^ε,
- $\xi > C$ in Ω^ε, C is to be chosen later.

3^0. *Growth.* It is convenient at this spot to justify the growth requirement for

$$\tilde{F} = (id + F)^{-1} - id.$$

The correction $\tilde{F}$ of the map-cochain $(id + F)^{-1}$ satisfies the following equation:

$$\tilde{F} = F \circ (id - \tilde{F}).$$

This implies the following apriori estimate for $\tilde{F}$:

$$|\tilde{F}| < \frac{\delta}{2}$$

in $\Omega^{\frac{\varepsilon}{2}}$. We now obtain a sharper estimate for $\tilde{F}$. We have:

$$|F(\zeta - \tilde{F}(\zeta))| < \exp(-\nu\xi + \nu\delta/2) < \exp(-\nu\xi/2)$$

if $\zeta > C$ in Ω^ε for C large enough. This justifies the growth condition for $\tilde{F}$.

1^0. *Partition* for $\tilde{F}$ is the same as for F.

2^0. *Extendability.* The cochain $\tilde F$ is $\frac{\varepsilon}{2}$-extendable because

$$|\tilde F(\zeta)| < \exp(-\nu\xi/2),$$

if $\zeta > C$ in Ω^ε for C large enough so that

$$\exp(-\nu\xi/2) < \delta.CCC$$

4^0. *Coboundary.* Finally, we get an upper estimate of the coboundary of the cochain $\tilde F$. In place of the difference coboundary of $\tilde F$ we consider the correction of the composition coboundary of the cochain $\mathrm{id} + \tilde F$. The correction of the composition coboundary of this map-cochain and its difference coboundary vanish simultaneously and differ by a factor $1 + o(1)$. The composition coboundaries of mutually inverse map-cochains are mutually inverse. Consequently, the correction $-G$ of the composition coboundary of $\mathrm{id} - \tilde F$ satisfies the equation

$$G = \delta_o F \circ (\mathrm{id} - G),$$

where $\mathrm{id} + \delta_o F$ is the composition coboundary of the cochain $\mathrm{id} + F$. The subscript o indicates composition. Therefore, F can be estimated from above by rigging functions differing from the rigging functions estimating δF from above only by a shift of the argument:

$$\exp(-c \exp \operatorname{Re} \rho_j(\xi_j + \tilde c)), \qquad |\tilde c| < \varepsilon.$$

These functions are majorized by rigging functions of the same class, but corresponding to another value of c, since the derivatives ρ_j'. are bounded. This finishes the verification of all the requirements of Definition 10 in S1.6 CCC and proves the inversion property of $\mathbb{R}$-regular cochains of class $\mathbf{\Omega}$. In order to prove it for the absolutely realizable cochains of this class, we have to prove the inversion property for the realizations. This is done in the same way as above, only the margin requirement in the choice of Ω should be replaced by: the margins between the generalized $\frac{\varepsilon}{2}$ and ε-neighborhoods of these domains is greater than $\delta|\zeta|^{-3}$, and the estimate

$$\exp(-\nu\xi/2) \prec \delta|\zeta|^{-3}$$

should be used.

Similar arguments work for realizations. CCC

B. Multiplication property of the set $id + \mathcal{FC}_{wr}(\mathcal{D})$**.** The property in the title follows from the lemma:

lem:simsh

LEMMA 2. *Suppose that $\mathbf{\Omega}$ is a proper class of standard domains, and $\mathcal{D}$ is the set of admissible germs of class $\mathbf{\Omega}$. Let F_1 and F_2 be the germs of regular functional cochains, the second weakly decreasing:*

$$F_1 \in \mathcal{FC}_{\mathrm{reg}}(\mathcal{D}), \quad F_2 \in \mathcal{FC}^+_{\mathrm{wr}}(\mathcal{D}).$$

Then

$$F = F_1 \o (\mathrm{id} + F_2) \in \mathcal{FC}_{\mathrm{reg}}(\mathcal{D}).$$

If in addition F_1 is a weakly or rapidly decreasing cochain, then F is a weakly or rapidly decreasing cochain, respectively.

PROOF. 0^0. **Domain.** Let F_1 be 2ε-extendable in a $\Omega \in \mathbf{\Omega}$.

Let the margin between a generalized ε and δ-neighborhoods of the domains of the partition of F_1 in Ω_1 be greater than $C(\varepsilon - \delta)$. Such C exists by Proposition **??**. QQQ Let $|F_2| < C\varepsilon$ in a domain

Om_2^ε, $\Omega_2 \in \mathbf{\Omega}$. Let us choose a domain $\Omega \subset \Omega_1 \cap \Omega_2$, $\Omega \in \mathbf{\Omega}$. This is possible because the class $\mathbf{\Omega}$ is proper. The domain Ω is the desired one.

$1^0, 2^0$. **Partition and extendability.** Let F_1 and F_2 be two cochains of type $\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}$ respectively; let $=_1 \cup_2$,

$$= (\sigma_1, \dots, \sigma_k), \ \sigma_j \rhd \sigma_{j+1}.$$

Any domain U of the of type is an intersection of two domains U_1 and U_2 of partitions of types $\boldsymbol{\sigma_1}$ and $\boldsymbol{\sigma_2}$ respectively. Let f_1 and f_2 be two functions of the cochains F_1 and F_2 corresponding to domains U_1 and U_2. Then a function

$$f = f_1 \circ (id + f_2)$$

corresponds to the domain U and may be extended to its generalized ε-neighborhood .

Indeed, the margin between $U_1^{(\varepsilon)}$ and $U_1^{2\varepsilon}$ is greater than $C\varepsilon$, and $U^{(\varepsilon)} = U_1^{(\varepsilon)} \cap U_2^{(\varepsilon)} \subset U_1^{(\varepsilon)}$. Hence, the composition

$$f = f_1 \circ (id + f_2)$$

is well defined and holomorphic in $U^{(\varepsilon)}$. Hence, the cochain F corresponds to the $=_1 \cup_2$, and is ε-extendable.

3^0. *Growth* We estimate the modulus of the germ F. By a condition of the lemma, $|F_1| < C_1 \exp \nu\xi$ in the domain Ω_1^δ, and $|F_2| < \delta = C\varepsilon$ QQQ in Ω_2^ε, Consequently, in Ω_1^ε

$$|F| < C' \exp \nu\xi, \quad C' = C_1 \exp \nu\delta.$$

If F_1 is a weakly or rapidly decreasing cochain, then in Ω_1^ε we have the respective inequalities

$$|F_1| < C_1 |\zeta|^{-5}$$

or $|F_1| < C_1 \exp(-\nu\xi)$, $\nu > 0$. Then, respectively,

$$|F_1| < C' |\zeta|^{-5}, \quad C' = C \sup_{\Omega_2^\delta} (|\zeta| - \delta)^5 |\zeta|^{-5},$$

or

$$|F| < C' \exp(-\nu\xi), \quad C' = C_1 \exp \nu\delta.$$

This concludes the estimate of the modulus of F.

4^0. *Coboundary* The coboundary is estimated just as in Lemma 1 below; the argument used in part 4 of the proof of this lemma goes through with minimal changes for the difference

$$\begin{aligned} &f_1 \emptyset (\mathrm{id} + g_1) - f_2 \emptyset (\mathrm{id} + g_2) \\ &\qquad = (f_1 \emptyset (\mathrm{id} + g_1) - f_2 \emptyset (\mathrm{id} + g_1)) + (f_2 \emptyset (\mathrm{id} + g_1) - f_2 \emptyset (\mathrm{id} + g_2)). \end{aligned}$$

□

□

Lemma 3. *QQQ If, in assumptions of Lemma* **??**, *QQQ F_1 and F_2 are absolutely realizable, then $F = F_1 \circ (id + F_2)$ also is.*

PROOF. The only addition to the previous proof should be checking of the extendability condition for the realizations. The generalized ε-neighborhoods of the domain of the for cochains and their realizations differ only near the boundary lines of the domain of realization. Let L be the boundary curve of the latter domain of the form σl, where l is a boundary curve of

$$\Pi_* : l = (\zeta - \zeta^{-2})(\xi - i\frac{\pi}{2}),\ \xi > 0.$$

Without entering the details, note that the margin between $\sigma\Pi_*^{(\varepsilon)}$ and $\sigma\Pi_*^{(2\varepsilon)}$ is no smaller then $C\varepsilon\xi^{-3}$, whilst $|F_2| \prec |\zeta|^{-5}$. Moreover, $|\zeta|^{-5} \prec C\xi^{-5}$ in any horizontal strip. This implies ε-extendability of realizations of F. □

sec:shifts

S 2.4. Shifts of cochains by slow germs

In this section we deal with compositions of cochains with germs of holomorphisms growing at infinity in a sense no faster than linear ones.

A. First general shift lemma. Recall the definition.

def:shift

DEFINITION 1. Let F be a functional cochain in some domain Ω that contains $(\mathbb{R}^+, \infty)$.Let F be a tuple of holomorphic functions: $F = \{F_j\}$. Let ρ be a germ $(\mathbb{R}^+, \infty) \to (\mathbb{R}^+, \infty)$ that is biholomorphically extendable to a domain $\tilde{\Omega}$ such that $\rho\tilde{\Omega} \subset \Omega$. Then the functional cochain $F \circ \rho$ is a tuple functions $\{F_j \circ \rho\}$, taken on the domain $\tilde{\Omega}$.

REMARK 10. If the cochain F is of type $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$, then the cochain $F \circ \rho,\ \rho = \sigma^{-1}$, is of type

eqn:sigmod

$$\sigma \circ \boldsymbol{\sigma} = (\sigma \circ \sigma_1, \ldots, \sigma \circ \sigma_N). \tag{4.1}$$

The cochain $F \circ \rho$ may not be regular, even if F is. The regularity of the composition requires the use of realizations.

Recall that a germ σ_0 is $\boldsymbol{\Omega}$-nonessential provided that there exists a $\Omega \in \boldsymbol{\Omega}$ such that

$$\sigma_0\Pi_* \supset \Omega.$$

aaa

lem:gens1

LEMMA 1. *Let $\boldsymbol{\Omega}$ be a proper class of standard domains; let D_1 and D_2 be two sets of $\boldsymbol{\Omega}$-admissible germs having the ordering and monotonicity properties from Subsection D of Chapter* 1. *Let σ be a germ of a biholomorphism such that*

1^0.

eqn:combinn

$$\sigma D_1 \subset D_2 \cup \{\textit{nonessential germs}\} \tag{4.2}$$

2^0. *Let $\sigma_1 \rhd \sigma_2$ in D_1. Then:*

- if the germ $\sigma \circ \sigma_1$ is admissible, then $\sigma \circ \sigma_2$ is admissible

- if the germ $\sigma \circ \sigma_2$ is nonessential, then $\sigma \circ \sigma_1$ is nonessential.

3^0. *There exist two $\Omega, \tilde{\Omega}$ such that*

eqn:rhoom

$$\rho\tilde{\Omega} \subset \Omega,\ \rho = \sigma^{-1}. \tag{4.3}$$

4^0. *There exists $\mu > 0$ such that*

$$Re\ \rho \prec \mu\xi\ \textit{in}\ \tilde{\Omega}.$$

Then for any germ $F \in \mathcal{FC}_{reg}(D_1)$ there exists k such that the germ

eqn:simples

$$F_{(k)} \circ \rho \in \mathcal{FC}_{reg}(D_2). \tag{4.4}$$

Moreover, if there exists $\varepsilon > 0$ such that

$$Re\ \rho \succ \varepsilon\xi \text{ in } \tilde{\Omega},$$

and F is rapidly decreasing, then $F_{(k)} \circ \rho$ also is.

Remark 11. Assumption 2^0 holds for Π_1 replaced by Π_0 by symmetry.

sub:choice

B. Choice of realization and domain.

Proof. Let F be of type

$$\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N),\ \sigma_j \in D_1,\ \sigma_j \rhd \sigma_{j+1}.$$

Consider a tuple

$$\sigma \circ \boldsymbol{\sigma} = (\sigma \circ \sigma_1, \dots, \sigma \circ \sigma_N).$$

By assumption 1^0, this tuple contains either admissible germs from D_2, or nonessential germs. By assumption 2°, nonessential germs (if any) follow first, and in a row; admissible germs (if any) follow last, and in a row too.

We will now find k such that (4.4) holds. Set

$$k = \begin{cases} 0 \text{ if } \sigma \circ \sigma_1 \text{ is admissible (Case 1)} \\ j \text{ if } \sigma \circ \sigma_j \text{ is nonessential, and } \sigma \circ \sigma_{j+1} \text{ is admissible (Case 2)} \\ N \text{ if } \sigma \circ \sigma_N \text{ is nonessential.} \end{cases}$$

The value of k in Case 2 is well defined by assumption 2^0 of the lemma.

This completes the choice of k.

Consider the composition $F_{(k)} \circ \rho$; in the Case 1, $F_{(0)} = F$. We will find a Ω_0 of class $\boldsymbol{\Omega}$ such that $F_{(k)} \circ \rho$ is well defined in Ω_0.

Case 1. $\Omega_0 = \tilde{\Omega}$ from assumption 3^0 of the lemma.

Cases 2 **and** 3. In these cases, the germ $\sigma \circ \sigma_k$ is nonessential. By definition recalled before the statement of the lemma, there exists a $\Omega_0 \in \boldsymbol{\Omega}$ such that

$$(\sigma \circ \sigma_k)^{-1}\Omega_0 \subset \Pi_*.$$

This implies that

$$\rho\Omega_0 \subset \sigma_k\Pi_*,\ \rho = \sigma^{-1}.$$

But

$$\sigma_k\Pi_* \subset \Omega_{\boldsymbol{\sigma},k},$$

the domain of the realization $F_{(k)}$. Hence, the composition $F_{(k)} \circ \rho$ is well defined in Ω_0. This completes the choice of the realization and domain.

sub:part

C. Partition. The trace of

eqn:sigmakk

$$\Xi = \Pi_{\sigma_j \in \boldsymbol{\sigma}} \sigma \circ \sigma_{j*}\Xi_{\text{st}} \tag{4.5}$$

on the domain $\Omega_{\boldsymbol{\sigma},k}$ is a of type $(\boldsymbol{\sigma}, k)$:

eqn:sigmap

$$\Xi_k = \Pi_{k+1}^N \sigma \circ \sigma_{j*}\Xi_{\text{st}}. \tag{4.6}$$

Indeed, no interior boundary line of $\sigma_{j*}\Xi_{\rm st}$ for $j \le k$, except for $(\mathbb{R}^+, \infty)$, crosses the domain $\Omega_{\boldsymbol{\sigma},k}$. The interior boundary lines mentioned above belong to the boundary of the domain $\sigma \circ \sigma_j(\Pi_1) \cup \sigma \circ \sigma_j(\Pi_0)$ or lie outside this domain. But for any $j < k$,

eqn:inchome (4.7) $$\Omega_{\boldsymbol{\sigma},k} \subset \sigma \circ \sigma_k(\Pi_1) \cup \sigma \circ \sigma_k(\Pi_0)$$

For $j \in J(p)$ this follows from the choice of the "smaller" domain $\sigma \circ \sigma_j(\Pi_1)$. For $j \notin J$, $j < p$, the germs $\sigma \circ \sigma_j$ and $\sigma \circ \sigma_k$ are not weakly equivalent, and $\sigma \circ \sigma_j \succ \sigma \circ \sigma_k$. Then (4.7) follows from the assumption 2^0.

Hence, the cochain $F_{(k)} \circ \rho$ is of type $(\boldsymbol{\sigma}, p)$; the partition (4.6) coincided with the partition

$$\Pi_{k+1}^N \sigma \circ \sigma_{j*}\Xi_{\rm st} = \Xi_k$$

in the domain $\Omega_{\boldsymbol{\sigma},k}$. This completes the description of the partition in Case 2.

In Case 1, the partition for $F_{(0)} \circ \rho$ coincides with (4.5).

In Case 3, the composition $F_{(N)} \circ \rho$ is a holomorphic function, and the corresponding partition is void.

This completes checking the *partition* requirement for the composition $F_{(k)} \circ \rho$.

D. Checking regularity assumptions. Let us check the other requirements of the Definition **??** QQQ of regular cochains.

Extension. The property of ε-extendability of cochains $F_{(k)} \circ \rho$ above follows from ε-extendability of cochains $F_{(k)}$ and the following remark.

REMARK 12. Generalized ε-neighborhood of a domain U of a partition of type $(\sigma \circ \sigma_1, \dots, \sigma \circ \sigma_N)$ is the image of the generalized ε-neighborhood of the domain ρU under the map σ:

$$U^{(\varepsilon)} = \sigma(\rho U)^{(\varepsilon)} \cap \tilde{\Omega}_{(\varepsilon)}.$$

Note that the generalized ε-neighborhood $(\rho U)^{(\varepsilon)}$ in the right hand side corresponds to the partition of type $(\sigma_1, \dots, \sigma_N)$, whilst in the left hand side it corresponds to $(\sigma \circ \sigma_1, \dots, \sigma \circ \sigma_N)$.

Growth. By assumption, $|F| \prec \exp \nu\xi$ in Ω for some $\nu > 0$. Consequently, $|F\o\rho| \prec \exp \nu \operatorname{Re} \rho$ in $\tilde{\Omega}$. But by assumption, the domain $\tilde{\Omega}$ can be chosen so that for some $\mu > 0$

$$\operatorname{Re} \rho < \mu\xi \text{ in } \tilde{\Omega}.$$

Consequently,

$$|F\o\rho| \exp \nu\mu\xi \text{ in } \tilde{\Omega}.$$

If $\nu < 0$, and $\operatorname{Re} \rho \succ \varepsilon\xi$ in $\tilde{\Omega}$, then $|F\o\rho| \exp(-|\nu\varepsilon|\xi)$ in $\tilde{\Omega}$: the cochain $F\o\rho$ is rapidly decreasing.

Coboundary. We estimate the coboundary $\delta F\o\rho$. The formula for a rigging cochain is precisely suited to accommodate a shift by a germ ρ such that the condition (**??**) CCC holds with $\sigma = \rho^{-1}$. Suppose that in the (σ, ε)-neighborhood of a standard boundary line $\mathcal{L}$ of the partition Ξ the corresponding rigging function is equal to

$$m_{\mathcal{L}} = \sum \exp(-C \exp \operatorname{Re} \rho_j).$$

Then in the $(\tilde{\sigma}, \varepsilon)$-neighborhood of the boundary line $\sigma(\mathcal{L} \cap \rho\tilde{\Omega})$ of the partition $\sigma_*\Xi$ the function

$$m_{\mathcal{L}}\o\rho = \sum \exp(-C \exp \operatorname{Re} \rho_j \o\rho)$$

is, by definition, a rigging cochain function corresponding to the partition $\sigma_*\Xi$. Therefore, the estimate $|\delta F| < m$ implies that $|\delta F \o \rho| < m \o \rho$, and $m \o \rho$ is a rigging cochain of the partition $\sigma_*\Xi$.

This completes checking all the necessary assumptions for the cochain $F \circ \rho$, and thus of the prof of the lemma. □

sub:realrho

E. Realizations of shifted cochains.

lem:gens2

LEMMA 2. *Suppose that in assumptions of Lemma 1, the cochain F is absolutely realizable. Then the regular cochains $F_{(k)} \circ \rho$ are absolutely realizable as well.*

PROOF. Without entering full detail, let us simply write down the formulas for m, m^+ and (m, l) realizations of $F_{(k)} \circ \rho$. We have:

$$\left(F_{(k)} \circ \rho\right)_{(m)} = F_{(k+m)} \circ \rho$$

$$\left(F_{(k)} \circ \rho\right)^+_{(m)} = F_{(k,m)} \circ \rho$$

$$\left(F_{(k)} \circ \rho\right)_{(m,l)} = F_{(k+m,l)} \circ \rho.$$

$$\left(F_{(k)} \circ \rho\right)^+_{(m,l)} = F^+_{(k+m,l)} \circ \rho.$$

We skip the routine checking of the corresponding definitions. □

sub:equiv

F. Equivalence. The diversity of admissible germs may be reduced with the use of the following notion.

def:equiv

DEFINITION 2. Let $\mathbf{\Omega}$ be a class of . Consider two germs of diffeomorpisms on $(\mathbb{R}^+, \infty)$, σ and $\tilde\sigma$. The germ $\tilde\sigma$ is called $\mathbf{\Omega}$-equivalent to σ if the following holds. There exist two domains $\Omega, \tilde\Omega$ of class $\mathbf{\Omega}$ such that the germs $\rho = \sigma^{-1}$, $\tilde\rho = \tilde\sigma^{-1}$ may be biholomorphically extended to Ω and $\tilde\Omega$ respectively. Moreover:

- $\tilde\Omega^\varepsilon \subset \Omega$ for some $\varepsilon > 0$;
- The germ $N = \rho \circ \tilde\sigma$ is negligible in $\tilde\rho\tilde\Omega$.

Remarks. (1). 1. A germ $\tilde\sigma\mathbf{\Omega}$-equivalent to an $\mathbf{\Omega}$-admissible germ σ is admissible itself. QQQ

2. $\mathbf{\Omega}$-equivalence is not an equivalence relation: it is not symmetric. It is easy to make it symmetric by the requirement that σ is $\mathbf{\Omega}$-equivalent to $\tilde\sigma$ as well, but we will not need this.

prop:equiv

PROPOSITION 1. *Let $\mathbf{\Omega}$ be a class of standard domains, and $= (\sigma_1, \ldots, \sigma_N)$ be a set of $\mathbf{\Omega}$-admissible germs. Let$\tilde{}$ $= (\tilde\sigma_1, \ldots, \tilde\sigma_N)$ be a set of germs such that $\tilde\sigma_j$ is $\mathbf{\Omega}$-equivalent to σ_j. Let F be a regular cochain of type . Then at the same time it is a cochain of type$\tilde{}$.*

PROOF. We may assume that the domains Ω and $\tilde\Omega$ from Definition (2) are the same, and F is defined in Ω^ε. We will prove that F satisfies all the requirements of the definition of regular cochain of type$\tilde{}$ in the domain $\tilde\Omega$.

0^0. *Domain.* This requirement is already satisfied because F is defined in $\tilde\Omega \in \mathbf{\Omega}$.

1^0. *Partition and extendability.* By definition of a cochain of type , F is a collection of functions that are in a bijection with the domains of the

$$\Xi = \Pi\sigma_{j*}\Xi_{\mathrm{st}},$$

and are holomorphic in the corresponding domains. A domain of aΞ has the form

$$U = \cap \sigma_j(\Pi_{l_j} \cap \rho_j \Omega).$$

Its generalized ε-neighborhood has the form

$$U^{(2\varepsilon)} = \cap \sigma_j(\Pi_{l_j}^{2\varepsilon} \cap \rho_j \Omega^{2\varepsilon}).$$

The margin between $U^{(\varepsilon)}$ and $U^{(\delta)}$, $\delta < \varepsilon$ is no less than $C(\varepsilon - \delta)$, C does not depend on the choice of domains. Consider a domain

$$\tilde{U}^{(\delta)} = \cap \tilde{\sigma}_j(\Pi_{l_j}^{2\varepsilon} \cap \tilde{\rho}_j \tilde{\Omega}^{\delta}).$$

We have: $\tilde{\sigma}_j = \sigma_j \circ N_j$, where N_j is negligible. We may assume that $|N_j - id| < \varepsilon$ in $\tilde{\rho}_j \tilde{\Omega}$ for all j. Then

$$N_j(\Pi_{l_j}^{\varepsilon} \cap \tilde{\rho}_j \tilde{\Omega}^{\varepsilon}) \subset \Pi_{l_j}^{2\varepsilon} \cap \rho_j \Omega^{2\varepsilon},$$

Indeed, inclusion

$$N_j \Pi_{l_j}^{\varepsilon} \subset \Pi_{l_j}^{2\varepsilon}$$

follows from the estimate: $|N_j - id| < \varepsilon$, and the inclusion

$$N_j \tilde{\rho}_j \tilde{\Omega}^{\varepsilon} \subset \rho_j \Omega^{2\varepsilon}$$

follows from $\sigma_j N_j \tilde{\rho}_j = \tilde{\sigma}_j \tilde{\rho}_j = id$ and $\tilde{\Omega}^{\varepsilon} \subset \Omega$. □

Growth. The growth estimates for the functions in the tuple F in the $(\tilde{}, \varepsilon/2)$-neighborhoods of the domains of the of type˜are the same as in the $(, \varepsilon)$-neighborhoods of the domains of the of type .

Coboundary. The rigging cochains for the of types and˜are comparable, that is, for each rigging cochain of one there exists a rigging cochain of the other that is defined possibly in more narrow neighborhoods of the boundary lines of the and majorizes the first rigging cochain . This follows from the equality

$$\text{Re } \tilde{\rho}_j = \text{Re } (\tilde{\rho}_j \circ \rho_j^{-1}) \circ \rho_j = \text{Re } \rho_j + o(1);\ \rho_j = \sigma_j^{-1},\ \tilde{\rho}_j = \tilde{\sigma}_j^{-1}.$$

□

prop:equiv1

PROPOSITION 2. *In assumptions of Proposition 1, let F be a germ of an absolutely realizable regular cochain of type $\boldsymbol{\sigma}$. Then, at the same time, F is a germ of an absolutely realizable regular cochain of type˜.*

PROOF. The only difference with the previous proof is the justification of ε-extendability across the boundary of the domain of the realization. For this we need to prove that

eqn:incluse

$$\sigma_j(\Pi_*^{(2\varepsilon)}) \cap \tilde{\Omega} \supset \tilde{\sigma}_j(\Pi_*^{(\varepsilon)}) \cap \tilde{\Omega}. \tag{4.8}$$

Recall that

$$^{(\varepsilon)}_{*} = (\zeta + (1-\varepsilon)\zeta^2)(\Pi),\ \Pi = \{\xi + i\eta | \xi > 0,\ |\eta| \leq \frac{\pi}{2}\}.$$

The margin between $\Pi_*^{(\varepsilon)}$ and $\Pi_*^{(2\varepsilon)}$ decreases as $|\zeta|^{-3}$. On the other hand, the correction of the quotient $\sigma_j^{-1} \circ \tilde{\sigma}_j$ in $\rho_j \tilde{\Omega}$ decreases more rapidly that $|\zeta|^{-5}$. This implies the inclusion

$$\Pi_*^{(2\varepsilon)} \cap \rho_j \tilde{\Omega} \supset \sigma_j^{-1} \circ \tilde{\sigma}_j(\Pi_*^{(\varepsilon)}) \cap \rho_j \tilde{\Omega},$$

which, in turn, implies inclusion (4.8). □

lem:gens3

LEMMA 3. *Assume all the conditions of Lemma 1 except one: instead of* (4.2) *we suppose that the set* $\sigma\mathcal{D}_1$ *consists of nonessential germs and germs* equivalent *to germs of the set* $\mathcal{D}_2$. *Then the conclusion of Lemma 1 holds.*

PROOF. Suppose that F is an absolutely realizable cochain of type $= (\sigma_1, \ldots, \sigma_M)$, the germs $\sigma\emptyset\sigma_1, \ldots, \sigma\emptyset\sigma_k$ are nonessential, and the germs $\sigma\emptyset\sigma_j$ with $k+1 \leq j \leq M$ are equivalent to germs $\tilde{\sigma}_j \in \mathcal{D}_2$. We claim that $F \circ \rho$ is a regular functional cochain of type $\tilde{} = (\tilde{\sigma}_{k+1}, \ldots, \tilde{\sigma}_M)$.

It follows from Lemma 1 that $F \circ \rho$ is an absolutely realizable cochain of type $(\boldsymbol{\sigma}, k) = (\sigma_{k+1}, \ldots, \sigma_M)$ that is ε-extendible for some $\varepsilon > 0$. By Propositions 1, 2 it is at the same time an absolutely realizable functional cochain of type $\mathcal{D}_2$. This proves the lemma. □

S 2.5. Combined shifts of cochains by cochains .

sec:cs

In this section we prove for functional cochains a general lemma that, together with Lemma 1, lyes at the basis of the proof of the function-theoretic shift lemmas $\mathrm{SL}\,2_{n,\mathrm{reg}}$ and $\mathrm{SL}\,3_{n,\mathrm{reg}}$ formulated in Section 2.6.

sub:cs1

A. Statement and sketch of the proof of the second shift lemma. In the statement below we consider a cochain of the form $H = F \circ (\rho + G)$. Such a composition requires a special definition.

Let F be defined and ε-extendable in a domain Ω, and $\tilde{\Omega}$ b such that:

$$\rho\tilde{\Omega} \subset \Omega.$$

Let Ξ^F and Ξ^G be partitions $\Omega, \tilde{\Omega}$ corresponding to F and G. Let U and V be two domains of partitions Ξ^F and Ξ^G respectively such that the following holds:

$$W := \sigma U \cap V \cap \tilde{\Omega} \neq 0, \ \sigma = \rho^{-1}.$$

Let F_U, G_V be the cochains F and G corresponding to the domains U and V respectively. Then a component of the cochain H corresponding to the domain W is defined as:

$$H_W = F_U \circ (\rho + G_V).$$

The assumptions of the lemma below guarantee that compositions of this kind considered in the lemma are well defined and δ-extendable for some δ depending on ε, F and G.

lem:gens2

LEMMA 1. *Let* $\boldsymbol{\Omega}, D_1, D_2$ *and* σ *be the same as in Lemma 1, and all the assumptions of this lemma hold. Namely,*

1^0.

$$\sigma D_1 \subset D_2 \cup \{\textit{nonessential germs}\}$$

2^0. *Let* $\sigma_1 \rhd \sigma_2$ *in* D_1. *Then:*

- *if the germ* $\sigma \circ \sigma_1$ *is admissible, then* $\sigma \circ \sigma_2$ *is admissible*
- *if the germ* $\sigma \circ \sigma_2$ *is nonessential, then* $\sigma \circ \sigma_1$ *is nonessential.*

3^0. *There exist two* $\Omega, \tilde{\Omega}$ *such that*

eqn:rhoom

$$\rho\tilde{\Omega} \subset \Omega, \ \rho = \sigma^{-1}. \tag{5.1}$$

4^0. *Moreover, let* $\rho = \sigma^{-1}$, *and in some domain* $\Omega \in \boldsymbol{\Omega}$

eqn:slowm

$$Re\rho \prec \alpha\xi \textit{ for any } \alpha > 0. \tag{5.2}$$

Then for any two germs: $F_1 \in \mathcal{FC}_{reg}(D_1)$ *and* $F_2 \in \mathcal{FC}_{+reg}(D_2)$, *there exists* k *such that the germ*

eqn:fshifted

$$F = F_{1(k)} \circ (\rho + F_2) - F_{1(k)} \circ \rho \in \mathcal{FC}_{+reg}(D_2). \tag{5.3}$$

The proof goes in the following way. First we choose the proper k in the formula (5.3) for F. Then we choose the domain Ω of F. After that we check all the requirements of the definition of regular functional cochain for F: *partition, extendability, growth* and *coboundary*. All these requirements will be stated explicitly before their proof.

B. Choice of the realization.

Proof. We construct a $\tilde{\Omega} \subset \mathbf{\Omega}$ in which the cochain F is defined and satisfies the requirements listed above. Let k be chosen as above, and $\boldsymbol{\sigma}_1$ be the type of the for F_1. Let $\Omega_k := \Omega_{\boldsymbol{\sigma}_1,k}$ be a domain of the realization $F_{1(k)}$ in which $F_{1(k)}$ is ε-extendable and satisfies the inequality

$$|F_{1(k)}| < \exp \nu\xi$$

for some $\nu > 0$. Let $|F_2| \prec \exp(-\mu\xi)$, and α be so small that $\alpha\nu - \mu < 0$. Let us take $\delta < \varepsilon C^{-1}$ where $C = \sup |\rho'|$ in $\Omega_0 \in \mathbf{\Omega}$. CCC Let us take a domain $\tilde{\Omega} \in \mathbf{\Omega}$ such that :

$$\tilde{\Omega} \subset \Omega_0,$$
$$\rho\tilde{\Omega}^{2\delta} \subset \Omega_{(\sigma^1,k)},$$
$$|F_2| < \exp(-\mu\xi) \text{ in } \tilde{\Omega},$$
$$\text{Re } \rho < \alpha\xi \text{ in } \tilde{\Omega}.$$

The domain $\tilde{\Omega}$ is the desired one.

C. Partition and extendability. We will check the assumptions and *extendability* for thr cochain

$$\tilde{F} = F_1 \circ (\rho + F_2).$$

This will imply the same assumptions for F.

Let $\boldsymbol{\sigma}_2$ be the type of the for F_2, and $(\boldsymbol{\sigma}_1, k)$ be the type of the for $F_{1(k)}$. Consider a of the type

$$\boldsymbol{\sigma} = \sigma \circ (\boldsymbol{\sigma}_1, k) \cup \boldsymbol{\sigma}^2$$

of the domain $\tilde{\Omega}$. We will prove that F is defined in the domain $\tilde{\Omega}$, corresponds to the $\boldsymbol{\sigma}$ and is 2δ-extendable.

Let us $U_1 \subset \tilde{\Omega}$ and $U_2 \subset \tilde{\Omega}$ be domains of partitions $\sigma \circ (\boldsymbol{\sigma}_1, k)$ and $\boldsymbol{\sigma}_2$ respectively. Let σ_j and $\tilde{\sigma}_l$ be the maps of the tuples $(\boldsymbol{\sigma}_1, k)$ and σ_2 respectively. By definition,

$$U_1 = (\cap_{j \in J} \sigma \circ \sigma_j \Pi_{s_j}) \cap \tilde{\Omega}$$
$$U_2 = (\cap_{l \in L} \tilde{\sigma}_l \Pi_{t_l}) \cap \tilde{\Omega},$$

where Π_{s_j}, Π_{t_l} are some strips of the standard partition, the sets J and L are some subsets of $\mathbb{Z}$.

By definition of the generalized neighborhoods ,

$$U_1^{(2\delta)} = (\cap_{j \in J} \sigma \circ \sigma_j \Pi_{s_j}^{2\delta}) \cap \tilde{\Omega}^{2\delta}$$

$$U_2^{(2\delta)} = (\cap_{l\in L}\tilde{\sigma}_l\Pi_{t_l}^{2\delta}) \cap \tilde{\Omega}^{2\delta}.$$

In the same way U_1^δ and U_2^δ are defines. By the choice of δ, the margin between U_1 and $U_1^{(\delta)}$, as well as U_2 and $U_2^{(\delta)}$ is no greater than $\frac{\varepsilon}{2}$. Hence, the cochain F is 2δ-extendable.

sub:cs3

D. Growth. Let us prove the the cochain

$$F = F_1 \circ (\rho + F_2) - F_1 \circ \rho$$

is rapidly decreasing.

We get an upper estimate of the modulus of the cochain $\tilde{F}$: we show that there exist $c > 0$ and $\varkappa > 0$ such that in Ω_2^δ

$$|\tilde{F}| < c\exp(-\xi).$$

By a condition of the lemma, there exist positive constants c_1, c_2, μ, and ν such that $|F_1| < c_1 \exp \nu\xi$ in Ω_1, and $|F_2| < c_2 \exp(-\mu\xi)$ in Ω_2^δ. In the δ-neighborhood of Ω_2^δ we have the estimate

$$|F| \le \max_{\theta[0,1]} |F_1'\text{ø}(\rho + \theta F_2)||F_2|.$$

The domain $\rho\Omega_2^\delta$ belongs to the domain Ω_1 together with its 2δ-neighborhood. Moreover, $|F_2| \le \delta$ and $0 \le \theta \le 1$. Therefore, $(\rho + \theta F_2)\Omega_2^\delta$ belongs to the domain Ω_1, together with its δ-neighborhood. Consequently, by Cauchy's inequality,

$$|F_1'\text{ø}(\rho + F_2)| < \delta^{-1}C_1 \exp(\nu \operatorname{Re}(\rho + F_2)) \le \delta^{-1}C_1' \exp \nu \operatorname{Re}\rho$$

for arbitrary $\in [0,1]$, where $C_1' = C_1 \exp \nu\delta$; recall that $|F_2| < \delta$ in Ω_2^δ. Next, by a condition of the lemma, for any $\alpha > 0$ the domain Ω_2^δ can be chosen so that in it $\operatorname{Re}\rho < \alpha\xi$. Take α small enough that $\nu\alpha - \mu = - < 0$. Then $|F| < C\exp(-\xi)$, and requirement 4 is verified.

sub:cs4

E. Coboundary. *Estimate of the coboundary.* Let $\mathcal{L}$ be a standard boundary line of the partition Ξ^F. Three cases are possible: $\mathcal{L}$ is a standard boundary line of both partitions of type $\sigma\text{ø}\sigma^{\mathbf{F_1}}$ and $\sigma^{\mathbf{F_2}}$—the first case; of only one of these partitions—the other two cases. We consider the first case. The other two can be reduced to this one if it is assumed that one of the pairs of functions considered below does not have a jump on $\mathcal{L}$. Let f_1 and f_2 be the functions in the tuple $F_1\text{ø}\rho$ that are defined in the domains of the partition $\sigma_*\Xi^{F_1}$ adjacent to $\mathcal{L}$; g_1 and g_2 are the analogous functions in the tuple F_2. Then the function in δF defined in the $(\sigma, \varepsilon/\mathbf{2})$-neighborhood of $\mathcal{L}$ has the form

$$\begin{aligned} h &= (f_1\text{ø}(\rho + g_1) - f_1\text{ø}\rho) - (f_2\text{ø}(\rho + g_2) - f_2\text{ø}\rho) \\ &= [(f_1 - f_2)\text{ø}(\rho + g_1) - (f_1 - f_2)\text{ø}\rho] + [f_2\text{ø}(\rho + g_1) - f_2\text{ø}(\rho + g_2)]. \end{aligned}$$

We estimate separately each of the square brackets; denote the first by h_1, and the second by h_2. The difference $f_1 - f_2$ is a function in the tuple δF_1. By definition, there exists a $C > 0$ such that in the $(\delta^{F_1}, \varepsilon)$-neighborhood of the line $\rho\mathcal{L}$

$$|f_1 - f_2| < \sum \exp(-C \exp \operatorname{Re}\rho_0),$$

where the summation is over those j such that $\mathcal{L}$ is a standard boundary line of the partition $\sigma_{j*}\Xi_0$; the diffeomorphisms σ_j are in the tuple σ^{F_1}. In the $(\sigma, \varepsilon/\mathbf{2})$-neighborhood of the line $\mathcal{L}$

$$|(f_1 - f_2)'| < C' \sum \exp(-C'' \exp \operatorname{Re}\rho_j)$$

for some $C' > 0$ and $C'' \in (0, C)$. This follows from Proposition 1 in S2.1 and Cauchy's inequality. Consequently, if δ is sufficiently small, then in the $(\sigma, \mathbf{e/3})$-neighborhood of $\mathcal{L}$

$$\begin{aligned} |h_1| &< C' \left| \sum \exp(-C'' \exp \operatorname{Re} \rho_j \o \rho) \right| |g_1| \\ &< C' \sum \exp(-C'' \exp \operatorname{Re} \rho_j \o \rho) \end{aligned}$$

since $|g_1| < \delta < 1$. This is the required estimate on the function of the tuple forming the coboundary, but the estimate is only on the first term. Let us estimate the second term, omitting the details; the arguments are analogous to the preceding arguments. Using the exponential estimate for regular cochains and the Cauchy inequality, we obtain $|f_2'| < \exp \nu\xi$ in Ω_2^δ for some $\nu > 0$. The same holds for $f_2' \o \rho$, because $\operatorname{Re} \rho < \alpha \xi$ in Ω_2^δ Consequently, we have that

$$|h_2| < C \exp(\nu\xi) |g_1 - g_2| < C \sum \exp \o C_2 (\nu\xi - \exp \operatorname{Re} \rho_j),$$

where the summation is over those j such that $\mathcal{L}$ is a boundary line of the partition $\sigma_{j*}\Xi_0$; this time the functions σ_j are from the tuple $\sigma^{\mathbf{F_2}}$. By the definition of an admissible diffeomorphism (part 4 of Definition 4 in 1.4),

$$\nu\xi - \varepsilon \exp \operatorname{Re} \rho_j \to -\infty$$

for arbitrary $\varepsilon > 0$. Consequently, there exist C' and C'' such that

$$|h_2| < \sum C' \exp(-C'' \exp \operatorname{Re} \rho_j).$$

The final estimate $|h| \le |h_1| + |h_2|$ is what is required in Definition 10 in S1.6. $\square$

sub:cs5

F. Nonstandard domains.

lem:scnon

LEMMA 2. *Consider two families $\Omega_{1,\varepsilon}$ and $\Omega_{2,\varepsilon}$ of nonstandard domains that satisfy the same properties as in B. Suppose that they all belong to a Ω of class $\mathbf{\Omega}$. Consider two sets D_1, D_2 of $\mathbf{\Omega}$-admissible germs and a germ of a diffeomorpism $\sigma : (\mathbb{R}^+, \infty) \to (\mathbb{R}^+, \infty)$ satisfying the following conditions:*

1. σ can be extended to a complex domain in such a way that

eqn:d12non

$$\sigma \circ D_1 \subset D_2 \cup (\Omega_1, \Omega_2 - \textit{nonessential germs}\ ); \tag{5.4}$$

2. for any compact set $K_1 \subset \mathbb{C}^+$ there exists a compact set $\mathcal{K}_2 \subset \mathbb{C}^+$ such that

$$\rho(\Omega_2 \setminus K_2) \subset \Omega_1 \setminus K_1;$$

3. in the domains $\Omega_{1,\varepsilon}$, ρ' is bounded and

$$\frac{\textit{Re}\ \rho}{\xi} \to 0 \ \textit{in}\ (\Omega_{1,\varepsilon}, \infty).$$

Then for arbitrary $F_j \in \mathcal{FC}_{reg}(D_j, \Omega_j)$, which are $\Omega_{j\varepsilon}$-extendable,

eqn:fc

$$F \stackrel{def}{=} F_1 \circ (\rho + F_2) - F_1 \circ \rho \in \mathcal{FC}^+_{reg}(D_2). \tag{5.5}$$

The proof follows the same lines as for Lemma 1.

sub:cs6

G. Realizations of combined shifts.

lem:screal

LEMMA 3. *Suppose that in assumptions of lemma 2, the cochains F_1, F_2 are absolutely realizable. Then so F from* (5.5) *is.*

PROOF. Let us give explicit formula for the realizations of F. Fix the notations:

$$(\sigma^1, k) = (\sigma^1_{k+1}, \dots, \sigma^1_N),\ \sigma^2 = (\sigma^2_1, \dots, \sigma^2_M),\ \sigma = (\sigma^1, k) \cup \sigma^2 = (\sigma_1, \dots, \sigma_K).$$

Recall that for any ordered set A and its subset B a map $i(A, B)$ is well defined, see CCC. Let $i = i((\sigma^1, k), \sigma),\ j = i(\sigma^2, \sigma)$.

Let

$$s(m, l) = i(m + l) - i(m),\ t(m, l) = j(m + l) - j(m).$$

Take

$$F^{(+)}_m = F^{(+)}_{1,(k+i(m))} \circ (\rho + F^{(+)}_{2,(j(m))}) - F^{(+)}_{1,(k+i(m))} \circ \rho,$$

$$F_{(m,l)} = F_{1,(k+i(m),s(m,l))} \circ (\rho + F_{2,(j(m),t(m,l))}) - F_{1,(k+i(m),s(m,l))} \circ \rho$$

CCC

Checking of the assumptions of the definition of realizations is routine, and we skip it. $\square$

This is the end of the first pa4rt of Chapter 1. All the necessary lemmas from the general theory of regular functional cochains are proved. We now switch to the study of the regular functional cochains of class n and to the proof of shift lemmas for these cochains.

S 2.6. Regular versions of the shift lemmas

sec:classn

This section starts the second part of the chapter: the study of the regular cochains that occur in the description of the composition of Dulac's maps of depth n. We prove here the regular versions of the shift lemmas $SL1_n - SL4_n$. These versions occur when in the statements the lemmas named above we keep the regularity assumption and waive the decomposability ones. Explicit statements of the lemmas follow.

Recall the following convention. The set D^m means D^m_1 for $m < n$, and D^n_0 for $m = n$. Denote for brevity $\mathcal{FC}^{m(+)}_{\text{reg}} \circ \exp^{[m]} \circ g = \mathcal{F}^{m(+)}_{reg,g}$

lem:1sl

LEMMA SL1$_n$. QQQ reg ili net? Let $m = n - 1$ or $m = n$. Then, for any $g \in G^{m-1}_{\text{rap}}$,

$$\mathcal{FC}^{(+)}_{\text{reg}}(\mathcal{D}^m) \circ A^{-m} G^{m-1}_{\text{rap}} \subset \mathcal{FC}^{(+)}_{\text{reg}}(\mathcal{D}^m).$$

Recall the notation

$$f \prec\prec g \text{ in } G^k$$

means

eqn:lamk0

$$\lim_{\pm} \frac{A^{-(k+1)} h}{\xi} = 0,\ h = f \circ g^{-1}. \tag{6.1}$$

Moreover,

eqn:gek

(6.2)

$$G^{k^-}_{\text{slow}} = \{g \in G^k / g \prec\prec \text{id in } G^k\};\ G^{k^+}_{\text{slow}} = {G^{k^-}_{\text{slow}}}^{-1},\ G^k_{\text{rap}} = G^k \setminus (G^{k^-}_{\text{slow}} \cup G^{k^+}_{\text{slow}}).$$

Recall that

$$(k, f) \prec (m, g),\ f \in G^{k-1},\ g \in G^{m-1}$$

eqn:2sl means that either $k < m$, or $k = m$, and $f \prec\prec g$ in G^{m-1}.

lem:2sl LEMMA SL $2_{n,\text{reg}}$. *Suppose that $m = n-1$ or $m = n$, $(k,f) \prec (m,g)$, and $\varphi \in \mathcal{F}^{k}_{\text{reg},f}$. Then*

eqn:2sl
$$\varphi \circ (\text{id} + \mathcal{F}^{m^+}_{\text{reg},g}) - \varphi \subset \mathcal{F}^{m^+}_{\text{reg},g}. \tag{6.3}$$

Comment. (1). to $SL2_{n,\text{reg}}$. A shift of a slower decreasing composition by a faster decreasing one results in adding some other faster decreasing composition to the original slow decreasing one.

LEMMA SL$3_{n,\text{reg}}$. a. *Suppose that $m = n-1$ or $m = n$ and either $f \succ\succ g$ in G^{m-1} or $f \circ g^{-1} \in G^{m-1}_{\text{rap}}$. Then*

$$\mathcal{F}^{m^{(+)}}_{\text{reg},f} \circ (\text{id} + \mathcal{F}^{m^+}_{\text{reg},g}) \subset \mathcal{F}^{m^{(+)}}_{\text{reg},f}.$$

b. *For any $g \in G^{m-1}$ the set of germs For any $g \in G^{m-1}$, the set of germs*

$$id + \mathcal{F}^{m^+}_{reg,g}$$

forms a group with the operation "composition".

Comment. (2). For any two rapidly decreasing cochains F, G, and $f \prec\prec g$ in G^{m-1}, the composition $F \circ \exp^{[m]} \circ g$ decreases on $(\mathbb{R}^+, \infty)$ faster than $G \circ \exp^{[m]} \circ f$. Let us call these compositions "compositions of level (m,g)". Statement a claim that a shift of a of level (m,g) by a slower decreasing of level (m,f) preserves the class (m,g) of the composition.

Let $\mathcal{F}^{m}_{\text{reg}} = \mathcal{F}\mathcal{C}^{+}_{\text{reg}}(\mathcal{D}^{m}_{1}) \circ \exp^{[m]} \circ G^{m-1} = \bigcup_{g \in G^{m-1}} \mathcal{F}^{m^+}_{\text{reg},g}$

lem:4sl LEMMA SL $4_{n,\text{reg}}$ *a.*

eqn:sl4a
$$J^{n-1} \subset \text{Gr}(\text{id} + \mathcal{F}^{n-1^+}_{\text{reg}}). \tag{6.4}$$

b. Let $m = n$ or $m = n-1$. Then for any $g \in G^{m-1}$,

eqn:sl4b
$$\mathcal{F}^{m^{(+)}}_{\text{reg},g} \circ J^{m-1} = \mathcal{F}^{m^{(+)}}_{\text{reg},g}. \tag{6.5}$$

Decoding of assertion *b*. For $m = n,\ g \in G^{n-1}$

eqn:sl4c
$$\mathcal{F}\mathcal{C}_{\text{reg}} \circ (D^{n}_{0}) \exp^{[n]} \circ g \circ J^{n-1} \subset \mathcal{F}\mathcal{C}_{\text{reg}}(D^{n}_{0}) \circ \exp^{[n]} \circ g. \tag{6.6}$$

For $m = n-1,\ g \in G^{n-2}$

eqn:sl4d
$$\mathcal{F}\mathcal{C}_{\text{reg}}(D^{n-1}_{1}) \circ \exp^{[n-1]} \circ g \circ J^{n-2} \subset \mathcal{F}\mathcal{C}_{\text{reg}}(D^{n-1}_{1}) \circ \exp^{[n-1]} \circ g. \tag{6.7}$$

The inclusion (6.7) is a property of the set J^{n-2} which appears on the level $n-1$. But the set D^{n-1}_{1} occurs for the compositions of class n only; so, (6.7) is included in the statement of the lemma.

This lemma is stronger than the regular version of the Shift Lemma 4 stated in Chapter 1. Namely, the regular part of the latter lemma claims statement "b" for $m = n$ only. Yet in the proof of Lemma 2.6 statement "b" for $m = n-1$ is the first to be proved, and the rest of the proof of the lemma is based on this statement.

sub:admisn

S 2.7. Properties of standard domains and admissible germs of class n

The properties in this section will have their own names.

sub:propor

A. Properness. Recall the definition.

DEFINITION 1. The set of standard domains of class n is defined by the formula
:

eqn:omegan

$$\mathbf{\Omega}_n = \{\Omega = \Omega_{n,\varepsilon} \mid \Omega_{n,\varepsilon} = \left(\zeta + \frac{\zeta}{(\ln^{[n-1]}\zeta)^{\varepsilon}}\right)\mathbb{C}^{+}\} \tag{7.1}$$

DEFINITION 2. A class $\mathbf{\Omega}$ of standard domains is said to be proper if:

(1°) for any $C > 0$ an arbitrary domain of class $\mathbf{\Omega}$ contains a domain of the same class whose distance from the boundary of the first is not less than C;

(2°) the intersection of any two domains of class $\mathbf{\Omega}$ contains a domain of the same class.

lem:stan LEMMA 1. *The class $\mathbf{\Omega}_n$ of standard domains of class n is proper.*

B. Properties of the germs from $A^{-m}G^{m-1}$. Let us say that a function σ satisfies the logarithm-exponential estimate in a domain $D \supset (\mathbb{R}^{+}, \infty)$ if the following inequalities hold:

eqn:le

$$|\ln\zeta|^{C} \prec |\sigma| \prec \exp|\zeta|^{\varepsilon}, \tag{7.2}$$

eqn:lep

$$|\ln\zeta|^{C}|\zeta^{-1}| \prec |\sigma'| \prec \exp|\zeta|^{\varepsilon} \tag{7.3}$$

in D for arbitrary $C > 0,\ \varepsilon > 0$.

lem:conj LEMMA 2. *Suppose that $m < n$, and $g \in G^{m}$. Then:*

1°. The germ $A^{-m}g$ extends biholomorphically from $(\mathbb{R}^{+}, \infty)$ to the germ of any half-strip $(\Pi^{\forall}, \infty)$ and carries it into the germ of any sector $(S^{\forall}, \infty)$;

2°. The germ $A^{-(m+1)}g$ extends biholomorphically from $(\mathbb{R}^{+}, \infty)$ to the germ of any sector $(S_{\alpha}, \infty), \alpha \in (\pi/2, \pi)$;

3°. For any $\varepsilon > 0$ and any $C > 0$

the germ $A^{-m}g$ satisfies the logarithm-exponential estimate in $(\Pi^{\forall}, \infty)$;

the germ $A^{-(m+1)}g$ satisfies the logarithm-exponential estimate in .

sub:genmult

C. Generalized multipliers.

lem:genmult LEMMA 3. *For any $g \in G^{n-1}, \sigma = A^{-n}g, \Omega \in \mathbf{\Omega}_n$ there exist the limits*

eqn:genmult

$$\lambda_{n-1}(g) = \lim_{(\Pi^{\forall},\infty)} \frac{A^{1-n}g}{\zeta}. \tag{7.4}$$

eqn:genmult1

$$\lambda_n(g) = \lim_{\Omega} \frac{\sigma}{\zeta} = \lim_{\Omega} \sigma'. \tag{7.5}$$

subsub:genprow

D. Generalized powers.

lem:genprow LEMMA 4 ($L5.5_n$). *For any $\sigma \in D^{n}$ there exists a limit*

eqn:lss

$$l(\sigma) = \lim_{(\mathbb{R}^{+},\infty)} \frac{\xi\sigma'}{\sigma} \in [1, \infty]. \tag{7.6}$$

for any $\sigma \in D^{n}_{0},\ l(\sigma) = \infty$.

If the limit (7.6) is finite, then

$$\sigma = \zeta^{(l(\sigma)+0(1))} \text{ on } (\mathbb{R}^+, \infty).$$

Thus $l(\sigma)$ is called a *generalized power* of σ.

In particular, if $l(\sigma) = 1$,

$$\sigma = \zeta^{1+o(1)}.$$

sub:genexp

E. Generalized exponents.

lem:logder

LEMMA 5 ($L5.4_n$). *a. For any $\sigma \in D_0^n \cup D_1^{n-1}$ there exists a limit*

eqn:llss

$$L(\sigma) = \lim_{(\Pi^\forall, \infty)} \frac{\sigma'}{\sigma} \in [0, 1]. \tag{7.7}$$

b. If $\sigma \in D_1^{n-1}$, then $L(\sigma) = 0$
c. If $L(\sigma) = 0$, then

eqn:arg11

$$\arg \sigma' \to 0 \text{ in } (\Pi^\forall, \infty) \tag{7.8}$$

If the value $L(\sigma)$ is positive finite, then σ has the form $\exp(L(\sigma) + o(1))\sigma$ in $(\Pi^\forall, \infty)$. Thus $L(\sigma)$ is called a *generalized exponent* of σ.

subsub:monot

F. Ordering and monotonicity.

lem:monat

LEMMA 6. *Let $m = n - 1$ or $m = n$. Then the germs of class D^m are ordered by the relation $\succ$. Moreover, the following monotonicity property holds.*

a. Suppose that $\Omega \subset \mathbf{\Omega_m}$, $\sigma_1 \succ \sigma_2 \in D^m$, and the germs σ_1 and σ_2 are not weakly equivalent. Then for any $\varepsilon \in (0, 1)$,

eqn:monot1

$$(\sigma_1 \Pi_{main} \cap \Omega^{(\varepsilon)}, \infty) \supset (\sigma_2 \Pi_{main} \cap \Omega, \infty). \tag{7.9}$$

b. Let the germs σ_1 and σ_2 be weakly equivalent but not equal. Then σ_1 and σ_2 can be renumbered so that for arbitrary ε and δ with $0 < \delta < \varepsilon < 1$

eqn:monot2

$$(\sigma_2 \Pi^{(\delta)}_{main} \cap \Omega, \infty) \subset (\sigma_1 \Pi^{(\varepsilon)}_{main} \cap \Omega, \infty). \tag{7.10}$$

sub:adm

G. Admissibility.

lem:adm

LEMMA 7. *a. The germs of classes D_0^n and D_1^{n-1} are $\mathbf{\Omega}_n$-admissible. Germs of the sets D_0^n and D_1^{n-1} are ordered by the relation $\sigma_1 \succ \sigma_2$ on $(\mathbb{R}^+, \infty)$.*

b. Let $\sigma = A^{-n}g$, $g \in G^{n-1^+}_{\text{slow}}$ and let $\rho = \sigma^{-1}$. Then there exists a standard domain of class n such that in this domain $\operatorname{Re} \rho \prec \varepsilon\xi$ for every $\varepsilon > 0$.

Recall that admissible germs are defined in Section 1.5. QQQ

REMARK 13. All the germs ρ inverse to the special admissible germs σ, except for germs inverse to $\sigma \in D^{n-1}_{\text{rap}} \circ \mathcal{L}^{n-1}$, map the of class one (or 2, or any fixed k) non depending on n, strictly inside:

$$\rho(\Omega, \infty) \subset (\Omega, \infty).$$

The exceptional case is the germs of class $D^{n-1}_{\text{rap}} \circ \mathcal{L}^{n-1}$. The germs of these classes may grow slower than identity, and the inverse germs may grow faster. Such an inverse germ ρ may bring a Ω to a larger one. The size of the margin between Ω and $\rho\Omega$ depends on n. Formula (7.1) is adjusted to getting the inclusion:

$$\forall \Omega \in \mathbf{\Omega}_n, \ \rho \in D^{n-1}_{\text{rap}} \circ \mathcal{L}^{n-1} \exists \ \tilde{\Omega} \in \mathbf{\Omega}_n : \rho\tilde{\Omega} \subset \Omega.$$

This statement is a part of the proof of Lemma 7. It motivates the necessity of consideration of the domains of class $\mathbf{\Omega}_n$.

subsub:allin

H. Almost identical admissible germs. Admissible germs of class $\mathcal{L}^{n-1}$ are almost identical in a sense that $\sigma(\zeta) = \zeta(1 + o(1))$. Moreover, the following lemma holds.

Let S_α be a sector

$$S_\alpha = \{\zeta||\arg\zeta| < \alpha\} \ \ \frac{\pi}{2} < \alpha < \pi.$$

lem:ln

LEMMA 8. *Any germ l of class $\mathcal{L}^{n-1}$ is biholomorphic in the germ at infinity of any sector S_α, and $l' \to 1$ there.*

This implies that the generalized multiplier of any germ from $\mathcal{L}^{n-1}$ equals 1.

sub:noness

I. Nonessential germs.

lem:noness

LEMMA 9 ($L5.3_n$). *a. The germs of the form $\sigma = A^{-n} g \circ \exp$ with $g \in G^{n-1^+}_{slow} \cup G^{n-1}_{rap}$ are nonessential of class $\mathbf{\Omega}_n$.*

b. The product of two germs of class D^n_0 is a nonessential germ of class $\mathbf{\Omega}_n$.

c. Let $m = n$ or $m = n - 1$, and $\sigma = \exp^{[m]} \circ h \circ \ln^{[k]}$, $\sigma_0 \in D^k$, $k \le m$. Suppose that a composition $\sigma \circ \sigma_0$ contains at least by two more exponentials than logarithms. Then this composition is a nonessential germ of class $\mathbf{\Omega}_m$.

J. Properties of the groups $A^{-m}J^{m-1}$.

In Section 2.11 we will need a description of the groups named in the title. The stateement of the corresponding lemma is postponed until there.

These and only these properties are used in the current chapter. In the next chapter we use some extra properties stated there. All these properties are proved in Chapter 5.

sub:adm1

K. Admissible germs of class 1. Here we present two examples: classes D^n_0 and D^{n-1}_1 for $n = 1$.

prop:class10

PROPOSITION 1. *Admissible germs of class D^1_0 are equivalent to sectorial germs $\sigma = \exp \circ \mu$, $\mu \in (0, 1)$ multiplied by a constant in quadratic standard domains.*

PROOF. Admissible germs of class D^1_0 have the form $\exp \circ g$ where $g \in G^{0^-}_{\text{slow}}$. Recall that G^0 is the group of all almost regular germs: $G^0 = \mathcal{R}$. The semigroup $G^{0^-}_{\text{slow}}$ has the form $\{g \in \mathcal{R}|\lambda_0(g) := \lim_{(\mathbb{R}^+,\infty)} g' < 1\}$. Let $g = a \circ g_0$, $g_0 \in \mathcal{R}_0$. Then the germs $\sigma = \exp \circ g$ and $\tilde\sigma = \exp \circ a$ are equivalent. Indeed, $N = \tilde\sigma^{-1} \circ \sigma = expg_0$ is negligible. $\square$

It is easy to see that the multiplications by positive constants do not change neither the partitions of the class $\exp \circ Aff\Xi_{\text{st}}$, not the class of corresponding rigging cochains . Hence, cochains of class $\exp \circ \mathcal{A}ff$ are in fact sectorial.

prop:class01

PROPOSITION 2. *Admissible germs of class D^0_1 are the affine ones.*

PROOF. The group G^{-1} consists, by definition, of affine maps. This group consists of rapid germs only, because $\lambda_0(g) = \lim \frac{g}{\xi}$. The semigroups $G^{0\pm}_{\text{slow}}$ are empty. Hence, $D^0_0 = \emptyset$. On the other hand, $G^{-1} = G^{-1}_{\text{rap}}$ and $D^0_{\text{rap}} = A^0 G^{-1}_{\text{rap}} = \mathcal{A}ff$. Hence, $L^{-1} = id$, and

$$D^0_1 = A^{1-1}G^{-1} = G^{-1} = \mathcal{A}ff.$$

$\square$

It is easy to see that the shifts do not change neither the partitions of the class $Aff\Xi_{\mathrm{st}}$, not the class of corresponding rigging cochains . Hence, cochains of class $\mathcal{A}ff$ are in fact simple ones.

Lemmas of this section are supposed to be true. This is the induction hypothesis for the induction in n. The base of induction is the proof of the above lemmas for the special admissible germs of class 1, namely for simple and sectorial ones. This proof is trivial and we skip it here. It is scattered along the Part I. The step of induction is proceeded in Part II. Namely, the same lemmas with n replaced by $n+1$ are proved in Chapter5, where the induction step is completed.

S 2.8. The first shift lemma, regularity: $\mathbf{SLl}_{n,\mathrm{reg}}$

sec:ls1

A. Formulation of the lemma. Recall that $\mathcal{D}^m = \mathcal{D}_0^n$ for $m = n$, and $\mathcal{D}^m = \mathcal{D}_1^m$ for $m \leq n-1$.

Lemma SL1$_n$. *Suppose that $m = n$ or $m = n-1$. Then*

$$\mathcal{FC}_{\mathrm{reg}}^{(+)}(\mathcal{D}^m) \circ A^{-m} G_{\mathrm{rap}}^{m-1} \subset \mathcal{FC}_{\mathrm{reg}}^{(+)}(\mathcal{D}^m).$$

The plus in parentheses means that the assertion is true both with the plus (without the parentheses) and without it.

Proof. We reduce $SL1_{n,\mathrm{reg}}$ to the first general shift lemma, Lemma 1.

It is required to prove that for any $F_1 \in \mathcal{FC}_{\mathrm{reg}}^{(+)}(D^m)$ and any $g \in G_{\mathrm{rap}}^{m-1}$ there exists $F \in \mathcal{FC}_{\mathrm{reg}}^{(+)}(D^m)$ and such that

$$F_1 \circ A^{-m} g = F.$$

Let $\rho = A^{-m} g$. We will prove that the assumptions of Lemma 1 hold for $\sigma = \rho^{-1}$ and $D_1 = D_2 = D^m$.

sub:combsl1

B. Checking the combinatorial assumptions of Lemma 1. Assumption1^0 of Lemma 1 will be checked in a stronger form. Let $m = n-1$ or $m = n$. Then for any $\sigma \in D_{\mathrm{rap}}^m = A^{-m} G_{\mathrm{rap}}^{m-1}$,

eqn:combin2

$$\sigma \circ D^m = D^m. \tag{8.1}$$

Recall the definitions of the sets of special admissible germs of class n:

$$D_0^n = D_{\mathrm{slow}}^{n^-} \circ \exp, \; D_{\mathrm{slow}}^{n^-} = A^{-n} G_{\mathrm{slow}}^{n-1^-}.$$

$$D_1^{n-1} = D_0^{n-1} \cup D_*^{n-1^+} \circ \mathcal{L}^{n-1},$$

where

$$D_*^{n-1^+} = \{A^{1-n} g | g \in G_{\mathrm{slow}}^{n-2^+} \cup G_{\mathrm{rap}}^{n-2}\},$$

$$\mathcal{L}^{n-1} = \{A^{1-n} g | g \in G^{n-1}, g = Ad(f) A^{n-2} h, h \in \ln \circ \underline{\mathbf{TO}}, f \in G^{n-2}, \lambda_{n-2}(f) = 0\}.$$

In the proof below we will not need the definition of $\mathcal{L}^{n-1}$. Equality (8.1) QQQ is equivalent to

prop:n1

Proposition 1.

$$D_{rap}^{n-1} \circ D_1^{n-1} = D_1^{n-1}.$$

$$D_{rap}^{n} \circ D_0^{n} = D_0^{n}.$$

PROOF. It is sufficient to prove that

eqn:drs (8.2)
$$D^{n-1}_{\text{rap}} \circ D^{n-1^+}_* = D^{n-1^+}_*.$$

eqn:drs0 (8.3)
$$D^{n}_{\text{rap}} \circ D^{n^-}_{slow} = D^{n^-}_{slow}.$$

□

Recall that

$$D^{n-1}_{\text{rap}} = A^{1-n} G^{n-2}_{\text{rap}},$$

$$D^{n^-}_{\text{slow}} = A^{-n} G^{n-1^-}_{\text{slow}}.$$

Moreover,

$$G^{n-2}_{\text{rap}} = \{g \in G^{n-2} | \lambda_{n-1}(g) \in (0, \infty)\},$$

where

$$\lambda_{n-1}(g) = \lim_{(\mathbb{R}^+,\infty)} \frac{A^{1-n} g}{\xi}.$$

Similarly,

$$G^{n-1^-}_{\text{slow}} = \{g \in G^{n-2} | \lambda_{n-1}(g) = 0\}$$

$$G^{n-1^+}_{\text{slow}} = \{g \in G^{n-2} | \lambda_{n-1}(g) = \infty\}.$$

Proposition 1 now follows from the definitions recalled above, and from the following group property of the map λ_m: if $g_0 \in G^{m-1}_{\text{rap}}$, $g \in G^{m-1}$, then

$$\lambda_m(g_0 \circ g) = \lambda_m(g_0) \cdot \lambda_m(g).$$

□

REMARK 14. The arguments in the proofs of Proposition 1 are purely combinatorial; they are reduced to decoding of the corresponding definitions. Similar arguments are used in the proof of lemmas $SL2_{n,\text{reg}}$ and $SL3_{n,\text{reg}}$ below, when we satisfy assumption (5.4). They are purely combinatorial again. More involved analytic arguments occur in the proof of the Fourth Shift Lemma, when we verify assumption (**??**).

sub:otherrap

C. Checking other assumptions. Let us first check assumption 2^0 of Lemma 1. As it was proved above, all the germs of the form $\sigma \circ D_1^{n-1}$, $\sigma \circ D_0^n$ belong to D_1^{n-1}, D_0^n respectively. By Lemma 7, all these germs are admissible. This justifies assumption 2^0 of Lemma 1.

We will now prove assumptions 3^0 and 4^0 of this lemma. Note that $\sigma \in D^{n-1}_{\text{rap}}$ or to D^n_{rap}. Thus σ is admissible by Lemma 7. Now, assumptions 3^0 and 4^0 of Lemma 1 follow from items 1^0 and 3^0 in the definition of admissible germ, see 3 QQQ of Chapter 1.

All the assumptions of Lemma 1 are checked for σ, D_1, D_2 from Lemma $SL1_{n,\text{reg}}$. Hence, the first of these lemmas implies the second one. □

sec:sl2

S 2.9. The second shift lemma and the conjugation lemma, function-theoretic variant: SL $2_{n,\mathrm{reg}}$ and CL$_{n,\mathrm{reg}}$

A. Formulations. We recall the conventions in S1.10:

$$\mathcal{D}^m = \begin{cases} \mathcal{D}_1^m & \text{for } m \le n-1, \\ \mathcal{D}_0^m & \text{for } m = n. \end{cases}$$

Let also

$$\mathcal{F}^{m^{(+)}}_{\mathrm{reg},g} = \mathcal{F}\mathcal{C}^{(+)}_{\mathrm{reg}}(\mathcal{D}^m) \circ \exp^{[m]} \circ g, \quad g \in G^{m-1}$$

The plus in parentheses means that the equality is true both with the plus (without parentheses) and without it.

Lemma SL $2_{n,\mathrm{reg}}$. *Suppose that $m = n-1$ or $m = n$, $(k, f) \prec (m, g)$, and $\varphi \in \mathcal{F}^k_{\mathrm{reg},f}$. Then*

$$\varphi \circ (\mathrm{id} + \mathcal{F}^{m^+}_{\mathrm{reg},g}) - \varphi \subset \mathcal{F}^{m^+}_{\mathrm{reg},g}.$$

Lemma CL$_{n,\mathrm{reg}}$. *Suppose that $m = n-1$ or $m = n$, $1 \le k \le m-1$, and $f \in G^k$. Then*

$$\mathrm{Ad}(f)(\mathrm{id} + \mathcal{F}^{m^+}_{\mathrm{reg},g}) \subset \mathrm{id} + \mathcal{F}^{m^+}_{\mathrm{reg},gf}.$$

The assertions of these lemmas for fixed k are denoted by SL $2_{k,n,\mathrm{reg}}$, and CL$_{k,n,\mathrm{reg}}$, respectively. Both lemmas can be proved simultaneously by induction on k. The induction base is CL$_{-1,n,\mathrm{reg}}$, and the induction step is carried out according to the scheme

eqn:sheme

$$\mathrm{CL}_{k-1,n,\mathrm{reg}} \overset{\mathrm{A}}{\Rightarrow} \mathrm{SL}\,2_{k,n,\mathrm{reg}} \overset{\mathrm{B}}{\Rightarrow} \mathrm{CL}_{k,n,\mathrm{reg}}. \tag{9.1}$$

The implication A is proved for $k \le m$ in this and the next three subsections, and the implication B is proved for $k \le m-1$ in E.

sub:impla

B. Induction step: implication A. It is required to prove that for each pair of germs

$$F_1 \in \mathcal{F}\mathcal{C}_{\mathrm{reg}}(\mathcal{D}^k) \text{ and} F_2 \in \mathcal{F}\mathcal{C}_{\mathrm{reg}}(\mathcal{D}^m)$$

with $k \le m$ and $m = n-1$ or n and each pair of germs $f \in G^{k-1}$, $g \in G^{m-1}$ with $f \prec\!\prec g$ in G^{m-1} for $k = m$ there exist a germ $F \in \mathcal{F}\mathcal{C}^+_{\mathrm{reg}}(\mathcal{D}^m)$ and a standard domain Ω of class m such that

$$F_1 \circ \exp^{[k]} \circ f \circ (\mathrm{id} + F_2 \circ \exp^{[m]} \circ g) - F_1 \circ \exp^{[k]} \circ f = F \circ \exp^{[m]} \circ g;$$

the equality holds in the domain $g^{-1} \circ \ln^{[m]} \Omega$. It is equivalent to the equality

$$F_1 \circ [\mathrm{Ad}(f^{-1} \circ \ln^{[k]}) \circ (\mathrm{id} + F_2 \circ \exp^{[m]} \circ g)] - F_1 = F \circ \sigma,$$

where

eqn:ssigma

$$\sigma = \exp^{[m]} \circ h \circ \ln^{[k]}, \ h = g \circ f^{-1}; \tag{9.2}$$

the equality holds in the domain $\rho\Omega$, where

eqn:rrho

$$\rho = \sigma^{-1} = \exp^{[k]} \circ h^{-1} \circ \ln^{[m]}; \ h \in G^{m-1}; \ \text{ if } k = m, \text{ then } h \in G^{m-1^-}_{\mathrm{slow}}. \tag{9.3}$$

We investigate the composition in square brackets.

By the CL$_{k-1,n,\mathrm{reg}}$, which appears in the induction hypothesis, there exists a germ $F_3 \in \mathcal{F}\mathcal{C}^+_{\mathrm{reg}}(\mathcal{D}^m)$ such that

$$\mathrm{Ad}(f^{-1})(\mathrm{id} + F_2 \circ \exp^{[m]} \circ g) = \mathrm{id} + F_3 \circ \exp^{[m]} \circ h.$$

Induction base: $k = -1$.

LEMMA 1 ($CL_{-1,n,\mathrm{reg}.}$). *Let $m = n-1$ or $m = n$, and $f = \alpha\zeta + \in \mathcal{A}ff := G^{-1}$. Then*

$$Ad(f)(id+^{m^+}) = id + \mathcal{F}^{m^+}_{reg, g\circ f}.$$

PROOF. Let

$$\varphi \in^{m^+},\ \varphi = F\circ \exp^{[m]}\circ g,\ F \in \mathcal{FC}^{m^+}_{\mathrm{reg}}.$$

Then

$$Ad(f)(id+\varphi) = f^{-1}\circ(f+\varphi\circ f) = id + \alpha^{-1}\cdot\varphi\circ f.$$

□

Note further that $\exp^{[k[} \in \mathcal{F}^{k-1}_{\mathrm{reg}}$. By the SL $2_{k-1,n,\mathrm{reg}}$, which appears in the induction hypothesis, there exists a germ $F_4 \in \mathcal{F}\,\mathcal{C}^+_{\mathrm{reg}}(\mathcal{D}^m_1)$ such that

$$\exp^{[k]}\circ(\mathrm{id} + F_3\circ\exp^{[m]}\circ h) = \exp^{[k]} + F_4\circ\exp^{[m]}\circ h.$$

Consequently,

$$A^{-k}\circ(\mathrm{id} + F_3\circ\exp^{[m]}\circ h) = \mathrm{id} + F_4\circ\sigma.$$

Accordingly, it is required to prove that in some standard domain Ω of class m there exists a representative of the germ $F \in \mathcal{F}\,\mathcal{C}^+_{\mathrm{reg}}(\mathcal{D}^m)$ such that in $\rho\Omega$

$$F_1\circ(\mathrm{id}+F_4\circ\sigma) - F_1 = F\circ\sigma,$$

or, equivalently,

$$F_{1(k)}\circ(\rho+F_4) - F_{1,(k)}\circ\rho = F \qquad \boxed{\texttt{eqn:fshift2}}(*)(9.5)$$

for appropriate k.

$\boxed{\texttt{rem:sl3}}$ REMARK 15. This computation does not use the inequalities $f \prec\!\prec g$ for $f, g \in G^{m-1}$; the result of it will be used in the proof of Lemma SL $3_{n,\mathrm{reg}}$ for $f \succ\!\succ g$ or $f\circ g^{-1} \in G^{m-1}_{\mathrm{rap}}$.

We prove (9.5) with the use of Lemma 1.

In the next two subsections, condition 1^0 of this lemma is verified separately for $m = n$ and $m = n-1$ in Propositions 1 and 2 respectively. Then the other assumptions are justified.

$\boxed{\texttt{sub:grn1}}$

C. Group properties of special admissible germs: level n. In order to apply Lemma 1 to the cochain (9.5), we have to check the assumptions of this lemma. The first assumption is

$\boxed{\texttt{eqn:d12}}$ (9.6) $$\sigma\circ D^k = D^m \cup (\text{nonessential germs}).$$

Assumption (9.6) is checked separately for $m = n$ (level n), and $m = n-1$ (level $n-1$). In this subsection we consider the case $m = n$.

$\boxed{\texttt{prop:levn}}$ PROPOSITION 1. *Let ρ be the same as in (9.3) for $m = n$:*

$$\rho = \exp^{[k]}\circ h\circ\ln^{[n]},\ k \le n,$$

$$h \in G^{n-1},\ \text{and for } k = n,\ h \in .$$

Let $\sigma = \rho^{-1}$. Then

$\boxed{\texttt{eqn:d12k}}$ (9.7) $$\sigma\circ D^k_1 \subset D^n_0 \cup (\textit{nonessential germs})$$

$\boxed{\texttt{eqn:d12n}}$ (9.8) $$\sigma\circ D^n_0 \subset D^n_0 \cup (\textit{nonessential germs})$$

PROOF. The proof is purely combinatorial.

Case 1: $k \leq n-2$. In this case, the germs from $\sigma \circ D_1^k$ are nonessential, by Assertion c of Lemma 9 and the definitions of the sets of special admissible germs recalled at the beginning of the previous section.

Case 2: $k = n-1$. In this case,

$$\sigma = \exp \circ A^{1-n} h, \;\; h \in G^{n-1}.$$

Let $\sigma_0 \in D_1^{n-1}$.

If $\sigma_0 \in D_0^{n-1}$, then $\sigma \circ \sigma_0$ is nonessential, by Assertion c of Lemma 9.

If $\sigma_0 \in D_*^{n-1} \circ \mathcal{L}^{n-1}$, then

$$\sigma_0 = A^{1-n} g_0, \;\; g_0 \in G^{n-1},$$

by definition of $\mathcal{L}^{n-1}$. Indeed, in this case

$$\sigma_0 = A^{1-n} g_1 \circ l, \;\; g_1 \in G^{n-2}, \;\; l \in \mathcal{L}^{n-1}; \;\; l = A^{1-n} g_2, \;\; g_2 \in G^{n-1}.$$

Hence,

$$\sigma_0 = A^{1-n} g_0, \;\; g_0 = g_1 \circ l, \;\; g_0 \in G^{n-1}$$

Therefore,

eqn:ssig0 (9.9) $$\sigma \circ \sigma_0 = \exp \circ A^{1-n} g, \;\; g = h \circ g_0 \in G^{n-1}.$$

Hence, the germ $\sigma \circ \sigma_0$ is admissible of class D_0^n or nonessential depending on g by Lemmas 7 and 9. If $g \in G_{\text{slow}}^{n-1}$, then $\sigma \circ \sigma_0 \in D_0^n$ by definition **??**QQ If $g \in G^{n-1} \setminus G_{\text{slow}}^{n-1^-}$, then $\sigma \circ \sigma_0$ is nonessential by Lemma 9a. This proves (9.7).

Case 3: $k = n$. In this case

$$\sigma = A^{-n} h^{-1}, h^{-1} \in$$

We have to prove (9.8). For an arbitrary $\sigma_0 \in D_0^n$,

$$\sigma_0 = A^{-n} g_0 \circ \exp, \;\; g_0 \in .$$

Then

$$\sigma \circ \sigma_0 = A^{-n} (h^{-1} \circ g_0) \exp \circ.$$

Depending on the germ $h^{-1} \circ g_0 \in G^{n-1}$, the germ $\sigma \circ \sigma_0$ either is nonessential, or belongs to D_0^n. This implies (9.8). $\square$

sub:grn2

D. Group properties of special admissible germs: level $n-1$. Here we prove relation (9.6) for $m = n-1$.

prop:levn1 PROPOSITION 2. *Let ρ be the same as in* (9.3) *for $m = n-1$:*

$$\rho = \exp^{[k]} \circ h \circ \ln^{[n-1]}, \;\; k \leq n-1,$$

$$h \in G^{n-2}, \;\; \textit{and for } k = n-1, \;\; h \prec\prec id \;\; \textit{in } G^{n-2}.$$

Let $\sigma = \rho^{-1}$. Then

$$\sigma \circ D_1^k \subset D_1^{n-1} \cup (\; \textit{the set of nonessential germs})$$

PROOF. **Case** 1: $k \leq n-3$. This case is treated as Case 1 in Proposition 1: the germs from $\sigma \circ D_1^k$ have "at least two extra exponents", and hence, are nonessential.

Case 2: $k = n-2$. This case is treated as Case 2 in Proposition 1. Namely, in this case

$$\sigma = \exp \circ A^{2-n} h, \;\; h \in G^{n-2}.$$

Let $\sigma_0 \in D_1^k = D_1^{n-2}$. If $\sigma_0 \in D_0^{n-2}$, then the germ $\sigma \circ \sigma_0$ is nonessential. If $\sigma_0 \in D_*^{n-2} \circ \mathcal{L}^{n-2}$, then

$$\sigma_0 = A^{2-n} g_0,\ g_0 \in G^{n-2}.$$

Hence,

$$\sigma \circ \sigma_0 = \exp \circ A^{2-n} g,\ g = h \circ g_0 \in G^{n-2}.$$

This germ either belongs to D_0^{n-1} or is nonessential by the same arguments as in case 2 of Proposition 1.

Case 3: $k = n-1$. Then

eqn:sk3 (9.10)
$$\sigma = A^{1-n} h^{-1},\ h^{-1} \in G_{\text{slow}}^{n-2^+}.$$

We have to prove that

eqn:inclun1 (9.11)
$$\sigma \circ D_1^{n-1} \subset D_1^{n-1}.$$

Let $\sigma_0 \in D_1^{n-1}$. If $\sigma_0 \in D_0^{n-1}$, then the arguments of case 3, Proposition 1, imply: $\sigma \circ \sigma_0 \in D_0^{n-1}$ or is nonessential.

Now, let $\sigma_0 \in D_*^{n-1} \circ \mathcal{L}^{n-1}$. Then

$$\sigma_0 = A^{1-n} g_0 \circ l,\ g_0 \in G_*^{n-2^+}.$$

Then, by (9.10),

$$\sigma \circ \sigma_0 = A^{1-n} g \circ l,\ g = h^{-1} \circ g_0 \in G_{\text{slow}}^{n-2^+},$$

because $G_{\text{slow}}^{n-2^+} \circ G_*^{n-2^+} = G_{\text{slow}}^{n-2^+}$. Hence,

$$\sigma \circ \sigma_0 \in D_*^{n-1} \circ \mathcal{L}^{n-1} \subset D_1^{n-1}.$$

□

sub:ass2

E. Checking assumption 2^0 of Lemma 1. Let us first prove the second part of this assumption: if $\sigma_1 \rhd \sigma_2 \in D^k$ and σ is as in (9.2), then "$\sigma \circ \sigma_2$ is nonessential" implies that $\sigma \circ \sigma_1$ is nonessential of class $\mathbf{\Omega_m}$.

By definition of a germ, there exists a Ω of class m such that

$$(\sigma \circ \sigma_1)^{-1}(\Omega) \subset \Pi_*.$$

Note that $\sigma_1 \rhd \sigma_2$ implies that for any $\varepsilon > 0$,

$$\sigma_2(\Pi_*, \infty) \subset \sigma_1(\Pi_*^{(\varepsilon)}, \infty).$$

Hence,

$$\sigma_1^{-1} \circ \sigma_2(\Pi_*, \infty) \subset (\Pi_*^{(\varepsilon)}, \infty).$$

Therefore,

$$(\sigma \circ \sigma_1)^{-1} = (\sigma_1^{-1} \circ \sigma_2) \circ (\sigma \circ \sigma_2)^{-1} : (\Omega, \infty) \to (\Pi_*, \infty) \to (\Pi_*^{(\varepsilon)}, \infty).$$

This implies that $\sigma \circ \sigma_1$ is of class $\mathbf{\Omega_m}$.

Now let us prove that if $\sigma_1 \rhd \sigma_2 \in D^k, \sigma$ is as in (9.2), QQQ and $\sigma \circ \sigma_1$ is $\mathbf{\Omega}$-admissible, then $\sigma \circ \sigma_2$ is also $\mathbf{\Omega}$-admissible. The proof is combinatorial with many cases to consider.

In the assumption of Lemma 2.6, QQQ $m = n-1$ or $m = n$. Suppose that $k < m-1$. Then the germ $\sigma \circ \sigma_1$ is by Lemma 9, a contradiction. Hence, $k = m-1$ or $k = m$.

Case 1**:** Let $k = m-1$; the arguments in this case work for both $m = n-1$ and $m = n$. Then

$$\sigma = \exp \circ A^{1-m} h,\ h \in G^{m-1}.$$

As $\sigma \circ \sigma_1$ is admissible, the composition $\sigma \circ \sigma_1$ contains but one exponent. Hence,

$$\sigma_1 \in D_1^{m-1},\ \sigma_1 = A^{1-m} g,\ g \in G^{m-1}.$$

Then

$$\sigma \circ \sigma_1 = \exp \circ A^{1-m} g_1,\ g_1 = h \circ g \in G^{m-1}.$$

As $\sigma \circ \sigma_1$ of this form is admissible, it belongs to D_0^m. Now Lemma 9 a implies that $g_1 \in G_{\text{slow}}^{m-1^-}$. As $\sigma_1 \rhd \sigma_2$, we have:

$$\sigma_2 = A^{1-m} \tilde{g},\ \tilde{g} \in G^{m-1},\ \sigma \circ \sigma_2 = \exp \circ A^{1-m} g_2,\ g_2 = h \circ \tilde{g} \in G^{m-1}.$$

Case 1a**:** σ_1 is not weakly equivalent to σ_2. Then $\sigma_1 \succ \sigma_2,\ g \succ \tilde{g},\ g_1 \succ g_2$. Therefore, $g_1 \in G_{\text{slow}}^{m-1^-}$ implies $g_2 \in G_{\text{slow}}^{m-1^-}$, and $\sigma \circ \sigma_2$ is $\mathbf{\Omega}$-admissible.

Case 1b**:** σ_1 is weakly equivalent to σ_2. Then

$$\sigma_1^{-1} \circ \sigma_2 = id + \Phi,$$

Φ is bounded on $(\mathbb{R}^+, \infty)$. Let g_1, g_2 be the same as before. As $g_1 \in G_{\text{slow}}^{n-1^-}$,

$$\frac{(A^{-m} g_1)}{\xi} \to 0 \text{ on } (\mathbb{R}^+, \infty).$$

On the other hand,

$$id + \Phi = (\sigma \circ \sigma_1)^{-1} \circ (\sigma \circ \sigma_2) = A^{1-m}(g_1^{-1} \circ g_2).$$

Hence,

$$A^{-m} g_2 = (A^{-m} g_1) \circ A^{-1}(id + \Phi).$$

Note that

$$A^{-1}(id + \Phi) = \zeta \cdot \exp \circ \Phi \circ \ln.$$

Hence, the ratio $A^{-1}(id + \Phi)/\xi$ is bounded on $(\mathbb{R}^+, \infty)$. Therefore,

$$\frac{A^{-m} g_2}{\xi} = \frac{(A^{-m} g_1) \circ A^{-1}(id + \Phi)}{A^{-1}(id + \Phi)} \cdot \frac{A^{-1}(id + \Phi)}{\xi} \to 0 \text{ on } (\mathbb{R}^+, \infty).$$

Hence, $g_2 \in G_{\text{slow}}^{m-1^-}$, and the germ $\sigma \circ \sigma_2 \in D_0^m$ is $\mathbf{\Omega}$-admissible.

Case 2**:** $k = m = n$. In this case, $D^k = d^m = D_0^n$, and, by (9.2),

$$\sigma = A^{-n} h,\ h \in G_{\text{slow}}^{n-1^+}.$$

Indeed, $h = g \circ f^{-1}$, and $(n, f) \prec (n, g)$ by assumption of Lemma 2.6. Now, take $\sigma_1 \in D_0^n$:

$$\sigma_1 = A^{-n} g_1 \circ \exp,\ g \in G_{\text{slow}}^{n-1^-}.$$

We have:

$$\sigma \circ \sigma_1 = A^{-n}(h \circ g_1) \circ \exp.$$

As $\sigma_2 \in D_0^n$, we get:

$$\sigma \circ \sigma_2 = A^{-n}(h \circ g_2) \circ \exp$$

for some $g_2 \in G_{\text{slow}}^{n-1^-}$. As $\sigma \circ \sigma_1$ is admissible, we conclude that $h \circ g_1 \in G_{\text{slow}}^{n-1^-}$. As $\sigma_1 \rhd \sigma_2$, we conclude, exactly as in the previous case, that $h \circ g_2 \in G_{\text{slow}}^{n-1^-}$. Hence, the germ $\sigma \circ \sigma_2$ is admissible (of class D_0^n).

Case 3: $k = m = n-1$. In this case

eqn:sssigma (9.12) $$\sigma = A^{1-n}h,\ h \in G^{n-2^+}_{\text{slow}},$$

and $\mathrm{D}^m = D_1^{n-1}$. Recall that

$$D_1^{n-1} = D_0^{n-1} \cup D_*^{n-1^+} \circ \mathcal{L}^{n-1}.$$

Case 3*a*: Both σ_1, σ_2 belong to D_0^{n-1}. Then $\sigma \circ \sigma_2$ is admissible by the same arguments as above.

Case 3*b*: $\sigma_1 \in D_0^{n-1},\ \sigma_2 \in D_*^{n-1} \circ^{n-1}$. Recall that in this case

$$\sigma_2 = (AS^{1-n}g) \circ l,\ g \in G^{n-2^+}_{\text{slow}},\ l \in \mathcal{L}^{n-1}.$$

By (9.12),

$$\sigma \circ \sigma_2 = A^{1-n}(h \circ g) \circ l,\ h \circ g \in G^{n-2^+}_{\text{slow}}.$$

sub:ass3

F. Checking assumption 3^0 of Lemma 1. The germ $\sigma = \exp^{[m]} \circ h \circ \ln^{[k]}$ is either nonessential or admissible. Indeed, for $k < m-1$ it is nonessential by Lemma 9 *c*. For $k = m-1$, $\sigma = \exp \circ A^{1-m} \circ h,\ h \in G^{m-1}$. The germ σ is either nonessential or belongs to D_0^{m-1} by Lemma 9 *a*. For $k = m$, $\sigma = A^{-m}h,\ h \in G_*^{m-1^+}$. In this case σ is admissible.

Suppose first that σ is nonessential. Then there exists a standard domain $\tilde{\Omega}$ of class n such that

$$\rho(\tilde{\Omega}, \infty) \subset (\Pi_*, \infty).$$

For such $\tilde{\Omega}$, requirement 3^0: $\rho\tilde{\Omega} \subset \Omega$, holds.

Suppose now that σ is admissible. Then 3^0 holds by the requirement 1^0 of the definition of admissible germs.

sub:ass4

G. Checking assumption 4^0 of Lemma 1. This assumption is treated the same way both for $m = n-1$ and $m = n$. Let us check it for $m = n$.

Suppose first that $k < n$. Then

$$\rho = \ln^{[n-k]} A^{-n}h^{-1}.$$

Let $k = n-1$. Then, by Lemma 2, ρ decreases faster than any power. Indeed, by (7.2), in ,

$$|A^{-n}h^{-1}| \prec \exp|\zeta|^{\varepsilon}$$

for any $\varepsilon > 0$. Then, for $k = n-1$,

$$|\rho| \prec |\zeta|^{\varepsilon},$$

But in the of class n for any $n > 0$, $|\zeta| \prec |\mathrm{Re}\ \zeta^3|$. Hence, for any $\varepsilon > 0, \alpha > 0$

$$\mathrm{Re}\ \rho \prec \xi^{\varepsilon} \prec \alpha\xi$$

in this domains. For $k < n-1$, the upper estimate of Re ρ is even stronger.

Suppose now that $k = n$. Then

$$\rho = A^{-n}h^{-1},\ h^{-1} \in D^{n-1^-}_{\text{slow}}.$$

Assumption 4^0 now follows from Lemma 7 b.

Thus, all the assumptions of Lemma 1 are checked, and the lemma may be applied. This proves implication A, see (9.1).

H. Implication B. By the ADT_k, which appears in the induction hypothesis for $k \leq n-1$, there exists for each $f \in G^k$ an expansion

$$f^{-1} = a + \sum \varphi_j, \quad \varphi_j \in \mathcal{F}^{k_j^+}_{\mathrm{reg}, g_j},\ g_j \in G^{k_j - 1}.$$

Then

$$\begin{aligned}\mathrm{Ad}(f)(\mathrm{id} + \mathcal{F}^m_{\mathrm{reg},g}) &= [a \circ (\mathrm{id} + \mathcal{F}^{m^+}_{\mathrm{reg},g}) + \sum \varphi_j \circ (\mathrm{id} + \mathcal{F}^{m^+}_{\mathrm{reg},g})] \circ f \\ &\overset{1}{\subset} \left(a + \sum \varphi_j + \mathcal{F}^{m^+}_{\mathrm{reg},g}\right) \circ f = \mathrm{id} + \mathcal{F}^{m^+}_{\mathrm{reg},g\circ f}.\end{aligned}$$

The inclusion 1 QQQ ne ponial follows from the affineness of a and Lemma $\mathrm{SL}\,2_{k,n,\mathrm{reg}}$ proved above with implication A; here the latter appears in the induction hypothesis. To use this lemma we must set $\varphi = \varphi_j$ in its formulation. The implication B is proved, and with it $\mathrm{SL}\,2_{n,\mathrm{reg}}$.

S 2.10. The third shift lemma, function-theoretic variant, SL $\mathbf{3}_{n,\mathrm{reg}}$

A. Formulations. Recall that, by the convention in S1.10

$$\mathcal{D}^m = \begin{cases} \mathcal{D}^m_1 & \text{for} m \leq n-1, \\ \mathcal{D}^m_0 & \text{for} m = n. \end{cases}$$

Let, for brevity, $\mathcal{F}\mathcal{C}^{m(+)}_{\mathrm{reg}} = \mathcal{F}\mathcal{C}^{(+)}_{\mathrm{reg}}(\mathcal{D}^m)$.

LEMMA $\mathrm{SL3}_{n,\mathrm{reg}}$. a. *Suppose that $m \leq n$, and either $f \succ\succ g$ in G^{m-1} or $f \circ g^{-1} \in G^{m-1}_{\mathrm{reg}}$. Then*

$$\mathcal{F}^{m(+)}_{\mathrm{reg},f} \circ (\mathrm{id} + \mathcal{F}^{m^+}_{\mathrm{reg},g}) \subset \mathcal{F}^{m(+)}_{\mathrm{reg},f}.$$

b. For any $g \in G^{m-1}$, the set of germs

$$id + \mathcal{F}^{m^+}_g$$

forms a group with the operation "composition".

B. Proof of the shift property: SL $\mathbf{3}_{n,\mathrm{reg}}$ a.

PROOF. It is required to prove that for arbitrary $F_1 \in \mathcal{F}\mathcal{C}^{(+)}_{\mathrm{reg}}(\mathcal{D}^m)$, $F_2 \in \mathcal{F}\mathcal{C}^{+}_{\mathrm{reg}}(\mathcal{D}^m)$, and f, g satisfying the condition of the lemma there exists a germ $F \in \mathcal{F}\mathcal{C}^{(+)}_{\mathrm{reg}}(\mathcal{D}^m)$ such that

$$F_1 \circ \exp^{[m]} \circ f \circ (\mathrm{id} + F_2 \circ \exp^{[m]} \circ g) = F \circ \exp^{[m]} \circ f.$$

We investigate the germ F given by this equality. We have:

$$F := F_1 \circ Ad(\exp^{[m]} \circ f)(id + F_2 \circ \exp^{[m]} \circ g).$$

Let

$$h = g \circ f^{-1},\ \rho = A^{-m} h,\ h \in G^{m-1^-}_*.$$

By the Remark 15 at the end of Subsection B, we have the there exists QQQ a germ $\tilde{F}_2 \in^+ (D^m)$ such that

$$F = F_1 \circ (id + \tilde{F}_2 \circ \rho.)$$

The scheme of the further proof of assertion a is the following. By Lemma 1 we will prove that $\tilde{F}_2 \circ \rho = F_3 \in \mathcal{F}\mathcal{C}_{\mathrm{reg}}(D^m)$. Then we will give an upper bound for F_3 and prove that $F_3 \in (D^m)$. Application of Lemma 2 will complete the proof of assertion a.

In order to apply Lemma 1, we have to check its assumptions. Recall that $G_*^{n-1^-} = G_{\text{slow}}^{n-1^-} \cup G_{\text{rap}}^{n-1}$.

Case 1: $h \in G_{\text{slow}}^{n-1^-}$. In this case, assumption 1^0 of Lemma 1 is checked in Subsections C and D of Section 2.9; assumptions $2^0, 3^0$ and 4^0 (strengthened) are checked in Subsections E, F, G of Section 2.9 respectively.

Case 2: $h \in G_{\text{rap}}^{n-1}$. In this case, assumptions of Lemma 1 are checked in Subsections B and C of Section 2.8.

Hence, Lemma 1 is applicable. It implies that

$$\tilde F_2 \circ \rho = F_3 \in \mathcal{FC}_{\text{reg}}(D^m).$$

Let us now prove that $F_3 \in (D^m)$. Namely, we will prove a stronger estimate:

eqn:weaked

$$|\tilde F_2 \circ \rho| \prec |\zeta|^{-\mu} \tag{10.1}$$

for any $\mu > o$. Indeed, $\rho = A^{-m}h,\ h \in G^{m-1}$. Such a germ satisfies the logarithmic-exponential estimate by Lemma 2. In particular, $|\rho| \succ |\ln \zeta|^C$ for any $C > 0$ in some $\tilde\Omega$ of class m. This domain may be so chosen that $\rho\tilde\Omega \subset \Omega$ for some $\Omega \subset \boldsymbol{\Omega}_m$. But in Ω, $|\zeta| \prec |\text{Re } \zeta|^3$. Hence, in $\tilde\Omega$, $|\text{Re } \rho| \succ c|\ln\zeta|$ for any c.

On the other hand, $\tilde F_2$ is rapidly decreasing, that is, $|\tilde F_2| \prec \exp(-\nu\xi)$ in some for some $\nu > 0$. Hence,

$$|\tilde F_2 \circ \rho| \prec \exp(-\nu c|\ln \zeta|) = |\zeta|^{-\nu c}.$$

Since c is arbitrary, this implies (10.1) for any $\mu > 0$, and proves Lemma SL $3_{n,\text{reg}}$ a.

C. Proof of the group property of the set $id+^{m^+}$**.** To prove the group property in the title, we have to check that the set $id+^{m^+}$ for any $g \in G^{m-1}$ is closed under multiplication and taking the inverse (multiplication and inversion properties).

The multiplication property follows from assertion a just proved. Indeed, let $F_1, F_2 \in^{m^+}$. Then:

$$(id + F_1 \circ \exp^{[m]} \circ g) \circ (id + F_2 \circ \exp^{[m]} \circ g) = id + F_2 \circ \exp^{[m]} \circ g + F \circ \exp^{[m]} \circ g.$$

Such an $F \in^{m^+}$ exists by assertion a.

sub:inver

D. Reduction to the inversion property of the set $id + \mathcal{FC}_{wr}^{m^+}$**.** Let us prove the inversion property for $m = n$ or $m = n-1$

eqn:inver

$$(\text{id} + \mathcal{F}_{\text{reg},g}^{m^+})^{-1} = \text{id} + \mathcal{F}_{\text{reg},g}^{m^+}. \tag{10.2}$$

According to Lemma $\text{CL}_{m-1,m,\text{reg}}$, we have that for $g \in G^{m-1}$

$$\text{id} + \mathcal{F}_{\text{reg},g}^{m^+} = \text{Ad}(g)(\text{id} + \mathcal{F}_{\text{reg,id}}^{m^+}).$$

Therefore, it suffices to prove assertion b for $g = \text{id}$. Further, we have the following equality, proved below:

$$\text{id} + \mathcal{F}_{\text{reg,id}}^{m^+} = A^m(\text{id} + \mathcal{F}\,\mathcal{C}_{\text{reg}}^{m^+}). \qquad \text{eqn:(am}*)\,(10.3)$$

An equivalent assertion:

$$A^{-m}(\text{id} + \mathcal{F}_{\text{reg,id}}^{m^+}) = \text{id} + \mathcal{F}\,\mathcal{C}_{\text{reg}}^{m^+}.$$

We prove by induction on k the equality

$$A^{-k}(\mathrm{id}+\mathcal{F}^{m^+}_{\mathrm{reg,id}}) = \mathrm{id}+\mathcal{F}\mathcal{C}^{m^+}_{\mathrm{reg}} \circ \exp^{[m-k]}$$

$(0 \le k \le m)$. The needed assertion is obtained for $k = m$.

THE INDUCTION BASE: $k = 0$. In this case the equality to be proved becomes an obvious identity.

INDUCTION STEP. Suppose that the assertion has been proved for some $k \le m-1$; we prove it for $k+1$. By the induction hypothesis,

$$A^{-(k+1)}(\mathrm{id}+\mathcal{F}^{m^+}_{\mathrm{reg,id}}) \subset A^{-1}(\mathrm{id}+\mathcal{F}\mathcal{C}^{m^+}_{\mathrm{reg}} \circ \exp^{[m-k]})$$
$$= \exp\circ(\ln+\mathcal{F}\mathcal{C}^{m^+}_{\mathrm{reg}} \circ \exp^{[m-k+1]}) = \zeta\cdot\exp\mathcal{F}\mathcal{C}^{m^+}_{\mathrm{reg}} \circ \exp^{[m-k+1]}\,.$$

By Corollary 1 in S2.2,

$$\exp\mathcal{F}\mathcal{C}^{m^+}_{\mathrm{reg}} = 1+\mathcal{F}\mathcal{C}^{m^+}_{\mathrm{reg}}\,.$$

Then

$$\exp\mathcal{F}\mathcal{C}^{m^+}_{\mathrm{reg}} \circ \exp^{[m-k-1]} = 1+\mathcal{F}\mathcal{C}^{m^+}_{\mathrm{reg}} \circ \exp^{[m-k-1]}\,.$$

We next get by Lemma 1 in S2.1 that

$$\zeta\cdot\mathcal{F}\mathcal{C}^{m^+}_{\mathrm{reg}} \circ \exp^{[m-k-1]} = (\ln^{[m-k-1]}\mathcal{F}\mathcal{C}^{m^+}_{l,\mathrm{reg}})\circ\exp^{[m-k-1]}$$
$$= \mathcal{F}\mathcal{C}^{m+1}_{\mathrm{reg}} \circ \exp^{[m-k-1]}\,.$$

From this,

$$\zeta\cdot\exp\mathcal{F}\mathcal{C}^{m^+}_{\mathrm{reg}} \circ \exp^{[m-k-1]} = \mathrm{id}+\mathcal{F}\mathcal{C}^{m^+}_{\mathrm{reg}} \circ \exp^{[m-k-1]}\,.$$

The equality (10.3) is proved.

Accordingly, assertion b of Lemma SL$\dot{3}_{m,\mathrm{reg}}$ is equivalent to the following:

$$[\mathrm{Ad}(\exp^{[m]}\circ g)(\mathrm{id}+\mathcal{F}\mathcal{C}^{m^+}_{l,\mathrm{reg}})]^{-1} = \mathrm{Ad}(\exp^{[m]}\circ g)(\mathrm{id}+\mathcal{F}\mathcal{C}^{m^+}_{l,\mathrm{reg}}),$$

or

$$(\mathrm{id}+\mathcal{F}\mathcal{C}^{m^+}_{\mathrm{reg}})^{-1} = \mathrm{id}+\mathcal{F}\mathcal{C}^{m^+}_{\mathrm{reg}}\,.$$

This is exactly the inversion property of the set $\mathrm{id}+\mathcal{F}\mathcal{C}^{m^+}_{\mathrm{reg}}$ that follows from Lemma 1 of Section 2.3. □

S 2.11. The fourth shift lemma b, function-theoretic version, level $n-1$

sec:blev

In this section we give a survey of the further proof, and prove that for any $g \in G^{n-2}$,

eqn:ls4lev1

$$\mathcal{F}^{n-1^{(+)}}_{\mathrm{reg},g} \circ J^{n-2} \subset \mathcal{F}^{n-1^{(+)}}_{\mathrm{reg},g} \tag{11.1}$$

sub:blev

A. Strategy of the proof of the fourth shift lemma. Now, the regular parts of the first three shift lemmas are proved, and we turn to the fourth one, see Lemma **??**. The proof of this lemma is more involved than the proofs of the previous shift lemmas, and we present here the strategy of this proof. Assertion “a” of the lemma for n replaced by $n-1$ enters the induction assumption. With the use of this assertion we prove assertion “b” for $m = n-1$. This is done in the present section. From this, in the next section, we deduce assertion “a” of the lemma as it is. At last, from assertion “a” we deduce assertion “b” for $m = n$. This is done in Section 2.13.

Like in the previous arguments, the main part is the study of a cochain F defined by

$$F = F_1(\rho),\ F_1 \in \mathcal{FC}^m, \rho = A^{-m} j,\ j \in J^{m-1}.$$

The composition ρ in this expression requires a special study; we do that in the next subsection. After that we reduce the proof of (11.1) to the simple shift lemmas for cochains, namely to Lemma 1. Again a new obstacle arises. To check the assumption 1 of this lemma, we have to investigate the classes $L^m \circ D^m$. These classes did not occur in the definition of special admissible germs. We have to prove that the germs from these compositions are *equivalent* to special admissible germs of class m. This gives rise to an extra technical part of the proof.

sub:prel

B. Preliminaries: properties of the groups $A^{-m}J^{m-1}$ and $A^{1-m}J^{m-1}$. Recall the definition of the group J^{m-1}. Namely, it is a group generated by the elements

eqn:genj

$$j = Ad(g)A^{m-1}f,\ f \in \mathcal{A}^0,\ g \in G^{m-1} \tag{11.2}$$

Recall that

$$\lambda_{m-1}(g) = \lim_{(\mathbb{R}^+,\infty)} \frac{A^{1-m}g}{\xi}$$

Let us introduce a set L^m which is in a sense close to $\mathcal{L}^m$, see Proposition 2 below.

Definition 1. Let $m < n$. Denote by L^m the set

eqn:lm

$$L^m = \{A^{-m}(Ad(g)A^{m-1}f)|\ f \in \mathcal{A}^0,\ g \in G^{m-1}, \lambda_{m-1}(g) = 0\} \tag{11.3}$$

lem:genj1

Lemma 1. *For any element j from* (11.2),

eqn:inf

$$A^{-m}j \in id + \mathcal{FC}_{wr}(\mathcal{D}_0^m) \text{ for } \lambda_{m-1}(g) = \infty. \tag{11.4}$$

eqn:rp

$$A^{-m}j \in \mathcal{D}^m_{rap} \circ (id + \mathcal{FC}_{wr}(\mathcal{D}_0^m)) \text{ for } \lambda_{m-1}(g) \in (0,\infty), \tag{11.5}$$

eqn:zero

$$A^{-m}j \in L^m \text{ for } \lambda_{m-1}(g) = 0. \tag{11.6}$$

We suppose that this lemma is proved for $m \le n-1$ (the induction assumption). We will prove it below with the references to Chapter 5 for $m = n-1$.

Note that (11.6) is a direct consequence of definitions.

sub:red1

C. Reduction to the simple shift lemma. Let us now turn to the proof of the lemma $LS4_{n-1}b$, namely, to the proof of (11.1), with g replaced by g_1. In order to justify (11.1), we have to prove that for any $F_1 \in \mathcal{FC}_{\text{reg}}(\mathcal{D}_1^{n-1}),\ g_1 \in G^{n-2},\ j \in J^{n-2}$:

eqn:genj1

$$j = Ad(g)A^{n-2}f,\ g \in G^{n-2},\ f \in \mathcal{A}^0 \tag{11.7}$$

there exists $F \in \mathcal{FC}_{\text{reg}}(\mathcal{D}_1^{n-1})$ such that

$$F_1 \circ \exp^{[n-1]} \circ g_1 \circ j = F \circ \exp^{[n-1]} \circ g_1.$$

This equality implies

$$F = F_1 \circ \rho,\ \rho = A^{1-n}(Ad(g_1)j),$$

where j is from (11.7). The germ

$$\tilde{j} = Ad(g_1)j$$

still belongs to J^{n-1} and has the form:

$$\tilde{j} = Ad(\tilde{g})A^{n-1}f,\ \tilde{g} = g \circ g_1^{-1},\ f \in \mathcal{A}^0$$

where $g, g_1 \in G^{n-2}$. Obviously, $\tilde{g} \in G^{n-2}$. Finally,

$$\rho = A^{1-n}\tilde{j}.$$

Consider three cases.

Case 1: $\lambda_{n-2}(\tilde{g}) = \infty$. Then, by (11.4), with $m = n-1$,

$$\rho \in id + \mathcal{FC}_{\mathrm{wr}}(\mathcal{D}_0^{n-1}).$$

By Lemma **??**, $F_1 \circ \rho \in \mathcal{FC}^{(+)}_{\mathrm{reg},g}(D_1^{n-1})$.

Case 2: $\lambda_{n-2}(\tilde{g}) \in (0, \infty)$. Then, by (11.5), with $m = n-1$,

$$\rho \in \sigma_2 \circ (id + \mathcal{FC}_{\mathrm{wr}}(\mathcal{D}_0^{n-1})),\ \sigma_2 \in \mathcal{D}^{n-1}_{\mathrm{rap}}.$$

Hence,

$$F_1 \circ \rho \in F_1 \circ \sigma_2 \circ (id + \mathcal{FC}_{\mathrm{wr}}(\mathcal{D}_0^{n-1})),\ \sigma_2 \in \mathcal{D}^{n-1}_{\mathrm{rap}}.$$

By the regular version of the first shift lemma 2.6,

$$F_1 \circ \sigma_2 = F_2 \in \mathcal{FC}^{(+)}_{\mathrm{reg}}(\mathcal{D}_1^{n-1}).$$

By Lemma **??**,

$$F_2 \circ (id + \mathcal{FC}_{\mathrm{wr}}(\mathcal{D}_0^{n-1})) \in \mathcal{FC}^{(+)}_{\mathrm{reg}}(\mathcal{D}_1^{n-1}).$$

Case 3: $\lambda_{n-2}(\tilde{g}) = 0$. Then $\rho \in L^{n-1}$.

We have to prove that

$$F_1 \circ \rho \in \mathcal{FC}_{\mathrm{reg}}(\mathcal{D}_1^{n-1}).$$

For this we have to prove that ρ satisfies the properties required in Lemma 1 The main one is proposition 1 below.

sub:grp1

D. Group properties of the sets L^{n-1} and $\mathcal{D}_1^{n-1}$. Assumption 1^0 of Lemma 1 is checked by the following proposition:

prop:ld1

PROPOSITION 1.

eqn:eq1

$$L^{n-1} \circ \mathcal{D}_1^{n-1} \approx \mathcal{D}_1^{n-1}. \tag{11.8}$$

Relation (11.8) means that for any $l \in L^{n-1}$, $\sigma_0 \in \mathcal{D}_1^{n-1}$, there exists $\sigma_1 \in \mathcal{D}_1^{n-1}$ such that

eqn:eq2

$$l \circ \sigma_0 \text{ is equivalent to } \sigma_1 \tag{11.9}$$

in sense of Definition **??**.

This proposition is proved at the end of this section.

Let us check other assumptions of Lemma 1. Assumption 2^0 holds in a trivial way: there are no nonessential germs in the set $L^{n-1} \circ \mathcal{D}^{n-1}$.

Assumptions 3^0 and 4^0 follow from the two statements: Lemma 7, which claims, in particular, that the germs of class $\mathcal{L}^{n-1}$ are admissible, and the following Proposition:

prop:lm

PROPOSITION 2. *There exists a bijection between the sets L^m and $\mathcal{L}^m$ such that the corresponding germs are equivalent in sense of Definition* **??**.

This proposition is proved in Chapter 5 as an addendum to Lemma 8

Application of Lemma 1 completes the proof of $LS4_{n-1}b$.

PROOF. of Proposition 1. Recall that

$$D_1^{n-1} = D_0^{n-1} \cup D_*^{n-1} \circ \mathcal{L}^{n-1}$$

where $D_*^{n-1} = (D_{\text{slow}}^{n-1^+} \cup D_{\text{rap}}^{n-1}) = A^{n-1}(G_{\text{slow}}^{n-2^+} \cup G_{\text{rap}}^{n-2})$.

We will prove (11.9) separately for $\sigma_0 \in D_0^{n-1}$ and $\sigma_0 \in D_*^{n-1} \circ \mathcal{L}^{n-1}$.

prop:ld0 PROPOSITION 3. *For $m \leq n$,*

$$L^m \circ D_0^m \approx D_0^m.$$

By induction in n, we may assume that this proposition is already proved for $m < n$. We need it here for $m = n-1$. Proposition 3 for $m = n$ is proved in Subsection B.

prop:ld2 PROPOSITION 4. *For any $l \in L^{n-1}$, $\sigma_0 \in D_*^{n-1} \circ \mathcal{L}^{n-1}$ there exists $\sigma_1 \in D_*^{n-1} \circ \mathcal{L}^{n-1}$ such that*

eqn:eq3
$$l \circ \sigma_0 \approx \sigma_1. \tag{11.10}$$

Together, Propositions 3, 4 imply Proposition 1. □

PROOF. of Proposition 4. Let

$$\sigma_0 = A^{1-n} g_0 \circ l_0, \; g_0 \in G_{\text{slow}}^{n-2^+}, \; l_0 \in \mathcal{L}^{n-1}.$$

Without loss of generality, we may assume that l is a generating element of the group L^{n-1}:

$$l = A^{1-n}(Ad(g)A^{n-2}f), \; g \in G^{n-2}, \; f \in \mathcal{A}^0.$$

Let

$$\tilde{l} = Ad(A^{1-n}g_0)l.$$

Then

eqn:tll
$$\tilde{l} = A^{1-n}(Ad(\tilde{g})A^{n-2}f), \; \tilde{g} = g \circ g_0 \in G^{n-2}. \tag{11.11}$$

Hence, $\tilde{l} \in A^{1-n}J^{n-2}$, and Lemma 1 for $m = n-1$ is applicable. Consider 3 corresponding cases for $\tilde{g}$ from (11.11).

Case 1: $\lambda_{n-2}(\tilde{g}) = \infty$. Then, by Lemma 1,

eqn:ltil
$$\tilde{l} \in id + \mathcal{FC}_{\text{wr}}(D_0^{n-1}). \tag{11.12}$$

Let us prove that the germ $\tilde{l}$ is negligible. Let $\rho_1 = \sigma_1^{-1} = A^{1-n}g_0^{-1}$ be in a domain Ω of class n. Recall that $\tilde{l} = Ad(\sigma_1)l$. First we prove that $\tilde{l}$ is in some domain $\rho_1\tilde{\Omega}$, where $\tilde{\Omega}$ is a standard domain of class n. By Proposition 2, l is equivalent to some germ $\tilde{l} \in \mathcal{L}^{n-1}$. By Lemma 7, the germ $\tilde{l}$ is admissible. Hence, there exists a domain $\tilde{\Omega} \in \mathbf{\Omega_n}$ such that $\tilde{l}\tilde{\Omega} \subset \Omega$. This domain may be so chosen that $\tilde{\Omega} \subset \Omega$. Then $\tilde{l}$ is well defined and holomorphic in the domain $\rho_1\tilde{\Omega}$. Its correction is estimated by Lemma 1. We proved that $\tilde{l} -$ id restricted to $\rho_1\tilde{\Omega}$ is a particular case of a weakly decreasing functional cochain, namely, a holomorphic function.

prop:case1 PROPOSITION 5. *In assumption (11.12), for any $l_0 \in \mathcal{L}^{n-1}$, and the holomorphic germ $\tilde{l}$ from (11.12)*

$$\tilde{l} \circ l_0 \approx l_0 : \tilde{l} \circ l_0 = l_0 \circ N,$$

where N is negligible in $l_0^{-1}\rho_1\tilde{\Omega}$.

PROOF. This follows from the two statements: $\tilde{l}$ is negligible in $l_0^{-1}\rho_1\tilde{\Omega}$, and l_0 is admissible with a derivative that tends to 1. $\square$

Let us prove (11.10) with $\sigma_1 = \sigma_0$. We have:

$$l \circ \sigma_0 = A^{1-n} g_0 \circ \tilde{l} \circ l_0 = A^{1-n} g_0 \circ l_0 \circ N \approx A^{1-n} g_0 \circ l_0 = \sigma_0.$$

This completes the proof of Proposition 4 in Case 1.

Case 2: $\lambda_{n-1}(\tilde{g}) \in (0, \infty)$. Then, by Lemma 1, equation (11.5),

$$\tilde{l} = (A^{1-n}\hat{g}) \circ N, \ \hat{g} \in G^{n-2}_{\text{rap}}, \ N \text{ is neglidgible.}$$

Negligibility of N is proved like in Case 1. Take

$$\sigma_1 = (A^{1-n}(g_0 \circ \hat{g})) \circ l_0.$$

The germ σ_1 belongs to D_1^{n-1} for the following reason: if $g_0 \in G^{n-2^+}_{\text{slow}} \cup G^{n-2}_{\text{rap}}, \ g \in G^{n-2}_{\text{rap}}$, then

$$g_0 \circ g \in G^{n-2^+}_{\text{slow}} \cup G^{n-2}_{\text{rap}}$$

by definitions of $G^{n-2^+}_{\text{slow}}, \ G^{n-2}_{\text{rap}}$. Once again,

$$l \circ \sigma_0 = A^{1-n} g_0 \circ \tilde{l} \circ l_0 = A^{1-n}(g_0 \circ \hat{g}) \circ N \circ l_0 = A^{1-n}(g_0 \circ \hat{g}) \circ l_0 \circ N_1 \approx A^{1-n}(g_0 \circ \hat{g}) \circ l_0 = \sigma_1,$$

because the germ N_1 is negligible by Proposition 5.

Case 3: $\lambda_{n-1}(\tilde{g}) = 0$. In this case,

$$l \circ \sigma_0 = (A^{1-n} g_0) \circ \tilde{l} \circ l_0,$$

where $\tilde{l} \in L^{n-1}, \ l_0 \in \mathcal{L}^{n-1}$. By Proposition 2, $\tilde{l} = l_1 N, \ l_1 \in \mathcal{L}^{n-1}$, N is negligible. By Proposition 5, $N \circ l_0 = l_0 \circ N_1$, N_1 is negligible again. Then $l \circ \sigma_0 = (A^{1-n} g_0) \circ l_1 \circ l_0 \circ N_1$, where $l_1 \circ l_0 \in \mathcal{L}^{n-1}$.We may take $\sigma_1 = (A^{1-n} g_0) l_1 \circ l_0 \in D_1^{n-1}$. Then $l \circ \sigma_0 = \sigma_1 N_1 \approx \sigma_1$.

This completes the proof of Proposition 4. $\square$

S 2.12. The fourth shift lemma, assertion a

Here we prove the lemma named in the title. Namely,

eqn:jn1 (12.1)
$$J^{n-1} \subset \text{ Gr } (id + \mathcal{F}^{n-1^+}_{\text{reg},g} | g \in G^{n-1}).$$

PROOF. The group J^{n-1} is generated by the set

$$\tilde{J}^{n-1} = \{Ad(g) A^{n-1} f | g \in G^{n-1}, \ f \in \mathcal{A}^0\}.$$

It is sufficient to prove the lemma for the generators of J^{n-1}, i.e. to justify (12.3) for J^{n-1} replaced by $\tilde{J}^{n-1}$. That is, for any $g \in G^{n-1}, \ f \in \mathcal{A}^0$ we have to prove:

eqn:jn2 (12.2)
$$j = Ad(g) A^{n-1}(f) \in Gr(id + \mathcal{F}^{n-1^+}_{\text{reg},g}).$$

sub:gid

A. Case $g = id$**.** Let us first consider the simplest elements of the group and prove (12.2) for $g = id$. Thus, we prove first that

$$A^{n-1}f \in \text{ id } + \mathcal{F}^{n-1^+}_{\text{reg, id}}$$

for $f = \text{ id } + F$, $F \in \mathcal{F}\mathcal{C}^0_+$. By the definitions of the operation A and the set $\mathcal{F}^{n-1^+}_{\text{reg, id}}$, this is equivalent to the condition

$$\ln^{[n-1]} \circ (\text{ id } + F) \circ \exp^{[n-1]} - \text{ id } \in \mathcal{F}\mathcal{C}^{n-1^+}_{\text{reg}} \circ \exp^{[n-1]},$$

or

$$\tilde{F} = \ln^{[n-1]} \circ (\text{ id } + F) - \ln^{[n-1]} \in \mathcal{F}\mathcal{C}^{n-1^+}_{\text{reg}}.$$

In fact, a stronger assertion is valid: $\tilde{F} \in \mathcal{F}\mathcal{C}^{0^+}_{\text{reg}}$ for $F \in \mathcal{F}\mathcal{C}^{0^+}_{\text{reg}}$. We prove this. All the requirements of the definition of simple functional cochains (Example 2 after Definition 13 in S1.6) are satisfied, because the derivative $(\ln^{[n-1]})'$ is bounded. Indeed the functional cochain $\tilde{F}$ is defined in the same standard domain as F, and corresponds to the same partition. Let σ be the type of this partition, and suppose that the cochain F is (σ, ε)-extendible. Then the cochain $\tilde{F}$ has this property, too, since the functions in the tuples F and $\tilde{F}$ are defined in the same domains. The estimates in requirements 3 and 4 of Definition 10 in S1.6 follow for $\tilde{F}$ and $\delta\tilde{F}$ from the inequalities

$$|\tilde{F}| < C|F|, \qquad |\delta\tilde{F}| < C|\delta F|,$$

where C is the supremum of the modulus of the derivative of $\ln^{[n-1]}$ in the domain of the cochain $\tilde{F}$.

This proves assertion a of the lemma for $g = \text{ id }$.

sub:gec

B. The general case. Let us proceed to the case of an arbitrary $g \in G^{n-1}$ in the expression for j:

eqn:jn1 (12.3) $$j = Ad(g)A^{n-1}f, \ f \in \mathcal{A}^0, \ g \in G^{n-1}.$$

We will make use of the statement b of the Fourth Shift Lemma for $m = n-1$

eqn:sl4bn1 (12.4) $$\mathcal{F}^{n-1^{(+)}}_{\text{reg},g} \circ J^{n-2} \subset id + \mathcal{F}^{n-1^{(+)}}_{\text{reg},g}$$

proved in the previous section. In the previous subsection we proved that $A^{n-1}(id + F) = id + \varphi$, $\varphi \in \mathcal{F}^{n-1^+}_{\text{reg,id}}$. Let now $j = Ad(g)(id + \varphi)$. According to the multiplicative decomposition theorem MDT_{n-1}, which appears in the induction hypothesis, we get that

eqn:gn1 (12.5) $$g = g_1 \circ j_1 \circ u, \qquad g_1 \in G^{n-2}, \ j_1 \in J^{n-2}, \ u \in H^{n-1}.$$

By the conjugation lemma $\text{CL}_{m,\text{reg}}$ for $m = n-1$, we get that

$$Ad(g_1)(\text{ id } + \varphi) = \text{ id } + \psi \in \text{ id } + \mathcal{F}^{n-1^+}_{\text{reg},g_1}.$$

Arguing as in the proof of the conjugation lemma in S2.7 and using Lemma $\text{SL}\,4_{n-1,\text{reg}}a$ appearing in the induction hypothesis and $\text{SL}\,2_{n,\text{reg}}$ (proved before) we get that

$$j_1^{-1} \circ (\text{ id } + \psi) = j_1^{-1} + \tilde{\psi}, \qquad \tilde{\psi} \in \mathcal{F}^{n-1^+}_{\text{reg},g_1}.$$

From this,

$$j_1^{-1} \circ (\text{ id } + \psi) \circ j_1 = \text{ id } + \tilde{\psi} \circ j_1, \qquad \tilde{\psi} \circ j_1 \in \mathcal{F}^{n-1^+}_{\text{reg},g_1}.$$

by Lemma $\mathrm{SL}4_{n-1,\mathrm{reg}}b$. Accordingly,

$$Ad(g_1 \circ j_1)(\text{ id } + \varphi) = id + \tilde{\varphi},\ \tilde{\varphi} \in \mathcal{F}^{n-1^+}_{\mathrm{reg},g_1}.$$

Finally, $u \in Gr(\text{ id } + \mathcal{F}^{n-1^+}_{0,\mathrm{reg}})$, and

eqn:gn2

$$j = Ad(u)(id + \tilde{\varphi}),\ \tilde{\varphi} \in \mathcal{F}^{n-1^+}_{\mathrm{reg},g_1}. \tag{12.6}$$

Together with Lemma SL 2_n this implies assertion a of the lemma. □

sub:struc

C. Structure of the group J^{n-1}. The following result is a free byproduct of the previous arguments.

lem:jnn1

LEMMA 1. *Let j be a generating element of J^{n-1} from* (12.3). *Then*

eqn:jnn1

$$j = id + \Sigma\varphi_l,\ \varphi_l \in \mathcal{F}^{n-1^+}_{reg,g_m},\ f_l \prec\prec f_{l+1},\ f_1 = g_1, \tag{12.7}$$

g_1 is the same as in (12.5).

The lemma claims that the principal term in the decomposition for $j - \mathrm{id}$ is from $\mathcal{F}^{n-1^+}_{\mathrm{reg},g_1}$ and the other terms are from the faster decreasing classes.

PROOF. For j from (12.3) and g_1, u from (12.5) we have proved that (12.2) holds. By definition,

$$u = \prod_2^N (id + \psi_l),\ \psi_l \in \mathcal{F}^{n-1^+}_{\mathrm{reg},f_l}.$$

By the $MDT_{n-1}II$, we may assume that in this product $f_l \prec\prec f_{l+1}$. Consider first a composition

eqn:conj2

$$j_\psi = Ad(id + \psi) \circ (id + \tilde{\varphi}), \tag{12.8}$$

$\tilde{\varphi} \in \mathcal{F}^{n-1^+}_{\mathrm{reg},g_1},\ \psi \in \mathcal{F}^{n-1^+}_{\mathrm{reg},f_0},\ f_0 \prec g_1,\ f_0 \in G^{n-2}$.

Let $(id + \psi)^{-1} = id - \tilde{\psi}$. Then, by $SL2_{n-1,\mathrm{reg}}$ and $SL3_{n-1,\mathrm{reg}}$ there exists $\tilde{\varphi}_1, \hat{\varphi} \in \mathcal{F}^{n-1^+}_{\mathrm{reg},g_1}$ such that

$$j_\psi = (id + \tilde{\varphi} - \tilde{\psi} - \tilde{\varphi}_1) \circ (id + \psi) = id + (\tilde{\varphi} - \tilde{\varphi}_1) \circ (id + \psi) = id + \hat{\varphi}.$$

Here $\tilde{\varphi}_1 = \tilde{\psi} \circ (id + \tilde{\varphi}) - \tilde{\psi} = \tilde{\varphi} \cdot o(1)$, and

$$\tilde{\varphi}_1 \circ (id + \psi) - \tilde{\varphi}_1 = \tilde{\varphi}_1 \circ o(1).$$

Hence,

$$\tilde{\varphi} \circ (id + \psi) - \tilde{\varphi} = \tilde{\varphi} \circ o(1).$$

$\hat{\varphi} = \tilde{\varphi}(1 + o(1)) \neq 0$.

The same arguments prove that the composition (12.8) for $f \approx g_1$, that is, $g_1 \circ f^{-1} \in G^{n-2}_{\mathrm{rap}}$, has the form $id + \hat{\varphi},\ \hat{\varphi} \neq 0,\ \hat{\varphi} \in \mathcal{F}^{n-1^+}_{g_1}$.

Consider now the composition (12.8) for $f \succ g_1$. Then, by the Conjugacy Lemma $CL_{n-1},$reg,

$$(id + \psi)^{-1} \circ (id + \tilde{\varphi}) = (id + \tilde{\varphi}) \circ (id + \tilde{\psi}),\ \tilde{\psi} \in \mathcal{F}^{n-1^+}_{f,\mathrm{reg}},\ f \succ g_1.$$

Summarizing, one may say that

$$j = (id + \varphi)\Pi(id + \psi_j),$$

$\varphi \in \mathcal{F}^{n-1^+}_{g_1} \setminus \{0\},\ \psi_j \in \mathcal{F}^{n-1^+}_{f_j},\ f_j \succ g_1$. Reference to Lemma $SL2_{n-1,\mathrm{reg}}$ (the same argument as in the proof of the ADT_n) completes the proof of Lemma 1.

This lemma will be used in the next two sections.

□

sec:blevn

S 2.13. The fourth shift lemma assertion b, level n

In this section we prove the lemma named in the title. That is, we prove

lem:ls4nb

LEMMA 1. *For any $g \in G^{n-1}$,*

eqn:ls4n

$$\mathcal{F}^{n^{(+)}}_{reg,g} \circ J^{n-1} = \mathcal{F}^{n^{(+)}}_{reg,g}. \tag{13.1}$$

We stress that at this point Lemma $L4_n a$ is already proved.

sub:red0

A. Reduction to the simple shift lemma revisited. The reduction below is parallel to that of C. We have to prove that for any $F_1 \in \mathcal{FC}_{\text{reg}}(D_0^n)$, $g_1 \in G^{n-1}$, $j \in J^{n-1}$, there exists $F \in \mathcal{FC}_{\text{reg}}(D_o^n)$ such that

$$F_1 \circ \exp^{[n]} \circ g_1 \circ j = F \circ \exp^{[n]} \circ g_1.$$

This equality implies

eqn:rho5n

$$F = F_1 \circ \rho, \ \rho = A^{-n}(Ad(g_1)j). \tag{13.2}$$

We want to prove the statement

eqn:efn0

$$F = F_1 \circ \rho \in \mathcal{FC}_{\text{reg}}(D_0^n) \tag{13.3}$$

Let us investigate the germ ρ. It is sufficient to consider only the case when j in (13.2) is a generator of the group J^{n-1}:

eqn:jenjn

$$j = Ad(g)A^{n-1}f, \ g \in G^{n-1}, \ f \in \mathcal{A}^0. \tag{13.4}$$

Then

$$\tilde{j} := Ad(\tilde{g})A^{n-1}f \in J^{n-1}, \ \tilde{g} = g \circ g_1, \ \rho = A^{-n}\tilde{j}.$$

Case 1: $\lambda_{n-1}(\tilde{g}) = \infty$. Then

$$\rho \in id + \mathcal{FC}_{\text{wr}}(D_0^n)$$

and (13.3) holds by Lemma **??**.

Case 2: $\lambda_{n-1}(\tilde{g}) \in (0, \infty)$ Then

$$\rho = \rho_0 \circ \mathcal{M}, \ \rho_0 \in D^n_{\text{rap}}, \ \mathcal{M} \in id + \mathcal{FC}_{\text{wr}}(\mathcal{D}_0^n)$$

In this case,

$$F = F_1 \circ \rho_0 \circ \mathcal{M}.$$

By Lemma 1, $F_1 \circ \rho_0 = F_2 \in \mathcal{FC}^n_{\text{reg}}(\mathcal{D}_0^n)$. By Lemma **??**, $F_2 \circ \mathcal{M} \in \mathcal{FC}^n_{\text{reg}}(\mathcal{D}_0^n)$. This proves (13.3) in Case 2.

Case 3: $\lambda_{n-1}(\tilde{g}) = 0$. In this case, $\rho \in L^n$, and we want to deduce (13.3) from Lemma 1. For this we need to check the assumptions of this lemma. The main one is considered in the next subsection.

sub:grn0

B. Group properties of the sets L^n and $\mathcal{D}_0^n$.

prop:grn0 PROPOSITION 1.

$$L^n \circ D_0^n \approx D_0^n.$$

PROOF. Let $\sigma_0 \in D_0^n$. By Definition **??** it is sufficient to prove that the germ N determined by the equality

$$l \circ \sigma_0 = \sigma_0 \circ N$$

is negligible.

Let l be a generating element of the group L^n of the form:

$$l = A^{-n}(Ad(g)A^{n-1}f),\ g \in G^{n-1},\ f \in \mathcal{A}^0,\ \lambda_{n-1}(g) = 0.$$

By definition of $\mathcal{D}_0^n$,

$$\sigma_0 = A^{-n}g_0 \circ \exp,\ g_0 \in G_{\text{slow}}^{n-1^-}$$

Then

$$N = Ad(\sigma_0)l = A^{1-n}(Ad(\tilde{g})A^{n-1}f),\ \tilde{g} = g \circ g_0.$$

Note that $\lambda_{n-1}(\tilde{g}) = \lambda_{n-1}(g) = 0$, because $g_0 \in G_{\text{slow}}^{n-1^-}$. CCC Hence,

eqn:enn

$$N \in AL^n \subset A^{1-n}J^{n-1}. \tag{13.5}$$

At the same time,

eqn:N

$$N = \rho_0 \circ l \circ \sigma_0,\ \rho_0 = \sigma_0^{-1}. \tag{13.6}$$

Let us prove that the germ N is negligible. This is done word by word as in Subsection D. First we prove that N is in some domain $\rho_0\tilde{\Omega}$, where $\tilde{\Omega}$ is a standard domain of class n. Let ρ_0 be in a domain Ω of class n. By Proposition 2, l is equivalent to some germ $\tilde{l} \in \mathcal{L}^{n-1}$. By Lemma 7, the germ $\tilde{l}$ is admissible. Hence, there exists a domain $\tilde{\Omega} \in \mathbf{\Omega_n}$ such that $\tilde{l}\tilde{\Omega} \subset \Omega$. This domain may be so chosen that $\tilde{\Omega} \subset \Omega$. Then N is well defined and holomorphic in the domain $\rho_0\tilde{\Omega}$. Its correction is estimated by the following lemma.

lem:genj LEMMA 2. *For any element $j \in J^{n-1}$ and any ρ such that $\rho^{-1} \in D_0^{n-1}$, the correction*

$$A^{1-n}j - id = \varphi,$$

is well defined in a domain $U = \rho\Omega$, where Ω is a standard domain of class n, and decreases there faster than any power. The germ $A^{1-n}j$ is also defined in $(\Pi^{\forall}, \infty)$.

Lemma 2 is proved in the next subsection. It implies Proposition 1. Indeed, by (13.5) we have: $N = A^{1-n}j$ for some J^{n-1}. By Lemma 2 $\varphi = N -$ id decreases faster than any power in the domain $\rho_0\tilde{\Omega}$ considered above. □

C. Applications of Lemma 1. In order to apply Lemma 1 to the germ $F_1 \circ \rho$, see (13.3), (13.2), it is sufficient to check the assumptions of the lemma. The first assumption is

$$\sigma \circ D_0^n \approx D_n \cup (\text{ nonessential germs}),\ \sigma = \rho^{-1}.$$

A stronger version of this requirement, Proposition 1, is proved: there are no nonessential germs in the above relation.

Assumptions $2^0, 4^0, 4^0$ of this lemma are checked in the very same way as in subsection D. Now, Lemma 1 may be applied to the composition $F_1 \circ \rho,\ \rho \in L^{n-1}$. This application proves $SL4_{\text{reg},n}b$, modulo Lemma 2.

D. Properties of the group $A^{1-n}J^{n-1}$: proof of Lemma 2.

PROOF. By LS4$_n$a,

$$J^{n-1} \subset Gr(id + \mathcal{F}^{n+1^+}_{\text{reg }g} | g \in G^{n-2})$$

Therefore, any element of this group has the form:

$$j = \prod (id + \tilde{\varphi}_j), \ \tilde{\varphi}_j \in \mathcal{F}^{n-1^+}_{g_j}, \ g_j \prec\prec g_{j+1}.$$

Note that

$$\exp^{[n-1]} = \exp \circ \exp^{[n-2]} \in \mathcal{F}^{n-2}_{\text{reg }id}.$$

This implies that, by LS2$_{\text{reg}}$,

$$\exp^{[n-1]} \circ j = \exp^{[n-1]} + \sum \varphi_j, \ \varphi_j \in \mathcal{F}^{n-1^+}_{\text{reg }g_j}.$$

Therefore,

$$A^{1-n}j = id + \sum \varphi_j \circ \ln^{[n-1]} .$$

Note that

$$\varphi_j = F_j \circ \exp^{[n-1]} \circ g_j.$$

Then

$$\varphi_j \circ \ln^{[n-1]} = F_j \circ A^{1-n} g_j.$$

Let us prove that the germ $A^{1-n}j$ is defined in a domain $U = \rho\Omega$ from Lemma 2. For this we have to prove that any term of the form $F \circ A^{1-n}g_0$, $g_0 \in G^{n-1}$, $F \in \mathcal{FC}^{n-1}_{\text{reg}}$ may be extended to U. Let $\rho = \sigma^{-1}$, $\sigma \in D^{n-1}_0$. Then for some $g \in G^{n-1}$, $\rho = A^{1-n}g \circ \ln$. The domain of F has the form $\rho_0\tilde{\Omega}$ for some $\tilde{\Omega}$, where $\rho_0 = A^{1-n}g_0^{-1}$. We want to prove that there exists Ω such that

$$\rho_0\tilde{\Omega} \supset \rho\Omega,$$

or, equivalently,

$$\tilde{\Omega} \supset A^{1-n}(g_0 \circ g) \circ \ln \Omega.$$

The domain $\ln \Omega$ belongs to $\Pi = \{|\operatorname{Im} \zeta| < \frac{\pi}{2}\}$. By assertion 1^0 of Lemma 2, the image of any horizontal strip under a map $A^{1-n}(g_0 \circ g)$ belongs to a germ of any sector $|\arg \zeta| < \alpha$, $\alpha > 0$. For $\alpha < \frac{\pi}{2}$, the germ of such a sector belongs to the germ (Ω, ∞).

These arguments also prove that the germ $A^{1-n}j$ is well defined in $(\Pi^\forall, \infty)$.

To prove the second statement we will use Lemma 2. By this lemma, the germ $A^{n-1}g_j$ satisfies the log-exp estimate. Hence, for any $C > 0$,

$$C|\ln \zeta| \prec |A^{n-1}g_j|$$

in $(\Pi^\forall, \infty)$. Together with (**??**), this implies that $|A^{n-1}g_j| = (1 + o(1))ReA^{n-1}g_j$ in $(\Pi^\forall, \infty)$. Moreover, cochains F_j decrease exponentially in $(\Pi^\forall, \infty)$. Hence, in $(\Pi^\forall, \infty)$,

$$|\varphi_j \circ \ln^{[n-1]}| = |F_j \circ A^{1-n}g_j| \prec \exp \circ(-\varepsilon C \ln |\zeta|) = |\zeta|^{-\varepsilon C}.$$

As $C > 0$ is arbitrary, the left hand side decreases faster than any power. This proves the second assertion of the lemma. □

S 2.14. Properties of the group $A^{-n}J^{n-1}$

A. Proof of the first statement of Lemma 1. The first statement of Lemma 1 will be completely proved in this subsection, and the second statement will be partly proved; the proof is completed in the next subsection.

Lemma 1 is stated in subsection B. Only statements (11.4) and (11.5) for $m = n$ must be proved. In order to prove (11.5) we need the following proposition.

prop:18

Proposition 1. *Suppose that j has the form* (11.7)*:*

$$j = Ad(g)A^{n-1}f,\ g \in G^{n-1},\ f \in \mathcal{A}^0.$$

and

eqn:munu

$$\lambda_{n-1}(g) > \mu > 0,\ f = id + F,\ |F| \prec \exp(-\nu\xi),\ \mu\nu \geq 7. \tag{14.1}$$

Then

eqn:infnu

$$A^{-n}j \in id + \mathcal{FC}_{wr}(D_0^n). \tag{14.2}$$

Proof. Statements (11.4) and (14.2) are proved in parallel.

Let $g = g_1 \circ j_1 \circ u$ as in (12.5). Then $\lambda_{n-1}(g) = \lambda_{n-1}(g_1)$.

Case 1: $\lambda_{n-1}(g) = \infty$. In this case we will prove (11.4).

Case 2: $\lambda_{n-1}(g) = \lambda_{n-1}(g_1) > \mu > 0$. In this case we will prove Proposition 1.

By Lemma 1, j has the form (12.7). By Lemma 1,

$$A^{1-n}j = id + \sum_1^N F_m \circ A^{1-n}g_m,\ g_m \in G^{n-2},\ g_m \prec g_{m+1},\ F_m \in \mathcal{FC}^{n-1^+}_{\text{reg},g_m},$$

g_1 is the same as in the previous paragraph.

Let us now consider $A^{-n}j$:

$$A^{-n}j = \exp\circ(\ln + \Sigma\varphi_m \circ \ln) = \zeta \exp \Sigma\varphi_m \circ \ln = \zeta \Pi \exp\circ(\varphi_m \circ \ln),\ \varphi_m = F_m \circ A^{1-n}g_m.$$

By the Completeness lemma,

$$\exp\circ\varphi_m \circ \ln = 1 + \tilde\varphi_m \circ \ln,\ \tilde\varphi_m = \tilde F_m \circ A^{1-n}g_m,\ \tilde F_m \in \mathcal{FC}^{n-1^+}_{\text{reg},g_m}.$$

prop:mult

Proposition 2. *Let $F, G \in \mathcal{FC}^{n-1^+}_{reg}$, $f, g \in G^{n-2}$, $f \prec\prec g$ in G^{n-2}. Then*

$$(F \circ \exp^{[n-1]} \circ f) \cdot (G \circ \exp^{[n-1]} \circ g) = H \circ \exp^{[n-1]} \circ g,\ H \in \mathcal{FC}^{n-1^+}_{reg}.$$

This proposition is proved below. It implies:

$$\Pi(1 + \tilde\varphi_l \circ \ln) = 1 + \Sigma\psi_l \circ \ln,\ \psi_l = G_l \circ A^{1-n}g_m,\ G_l \in \mathcal{FC}^{n-1^+}_{\text{reg.}}$$

Finally, we get:

$$A^{-n}j = id + \Sigma\psi_m \circ \ln,\ \psi_m = G_m \circ A^{1-n}g_l,\ G_m \in \mathcal{FC}^{n-1^+}_{\text{reg.}}$$

Note that g_1 is the same as above: $\lambda_{n-1}(g_1) = \mu > 0$ or ∞.

Let us first prove that the terms $\psi_l \circ \ln$ are regular cochains of class D_0^n. Indeed, they have the form

$$\psi_m \circ \ln = G_m \circ \rho_m,$$

where

$$\rho_m = A^{1-n}g_m \circ \ln.$$

Let us skip the subscript m and consider a germ

$$F = G \circ \rho,\ G \in \mathcal{FC}^{n-1^+}_{\text{reg}},\ \rho = A^{1-n}g \circ \ln,\ g \in G^{n-1}.$$

We want to prove that $F \in \mathcal{FC}_{\text{reg}}^{n-1^+}(\mathcal{D}_0^{n-1})$. This was already done in the proof of $SL2_{n,\text{reg}}$.

This implies that

$$A^{-n}j = id + \Sigma F_m,\ F_m \in \mathcal{FC}_{\text{reg}}^{n-1^+}(\mathcal{D}_0^{n-1}).$$

In Case 1, for any rapidly decreasing cochain G_m, the germ ψ_m decreases faster than any exponent, and the germ $\psi_l \circ \ln$ decreases faster than any power. This proves Lemma 1 in Case 1; in other words, relation (11.4) is proved.

Consider now Case 2 : $\mu < \lambda_{n-1}(g) \in (0, \infty)$. We have already proved that

$$A^{-n}j = id + \tilde{F},\ \tilde{F} \in \mathcal{FC}_{\text{reg}}^{n-1}(\mathcal{D}_0^{n-1}).$$

We want to prove that

$$\tilde{F} \in \mathcal{FC}_{\text{wr}}(\mathcal{D}_0^{n-1}),$$

that is,

eqn:tf (14.3) $$|\tilde{F}| \prec |\zeta|^{-5}.$$

We have:

$$A^{1-n}j = Ad(A^{1-n}g)(id + F) := id + \tilde{F} = A^{-1}(id + \tilde{G}).$$

By Lemma 2, $A^{1-n}g$ is conformal in $(\Pi^\forall, \infty)$, and $\lim_{(\Pi^\forall,\infty)}(A^{1-n}g)' = \lambda_{n-1}(g) > \mu$ exists. Then

$$\text{Re } (A^{1-n}g) \succ \mu\xi \text{ in } (\Pi^\forall, \infty),$$

Moreover,

$$|\tilde{G}| = |A^{1-n}g^{-1} \circ (A^{1-n}g + F \circ A^{1-n}g) - \text{ id } | \prec C|F \circ A^{1-n}g|,\ \forall C > \lambda_{n-1}(g)$$

in $(\Pi^\forall, \infty)$. But

$$|F(\zeta)| \prec (-\nu \text{Re } \zeta)$$

in $(\Pi^\forall, \infty)$ by assumption. Then

$$|\tilde{G}| \prec C \exp(-\nu \text{Re } A^{1-n}) \prec C \exp(-\mu\nu\xi).$$

Now

$$|\tilde{F}| \prec |A^{-1}(id + \tilde{G}) - id| = |\zeta \exp \tilde{G} \circ \ln - \zeta| \prec 2|\zeta| \exp(-\mu\nu |\ln \zeta|) = 2C|\zeta|^{-6} \prec |\zeta|^{-5}.$$

This proves (14.3), and thus completes the proof of Proposition 1.

B. Proof of the second statement of Lemma 1. We can now complete the proof of the second statement of Lemma 1. Let, as before,

$$j = Ad(g)A^{n-1}f,\ g \in G^{n-1},\ f \in \mathcal{A}^0,$$

and suppose that $\lambda_{n-1}(g) = \mu$. It must be proved that there exist an $h_1 \in A^{-n}G_{\text{rap}}^{n-1}$ and an $h_2 \in$ id $+ \mathcal{F}\mathcal{C}_{\text{wr}}(\mathcal{D}_0^n)$ such that $h = h_1 \circ h_2$. Take the constant $\nu > 0$ such that $\nu\mu \geq 7$. By Definition 1 in S1.3, CCC the germ $f \in \mathcal{A}^0$ can be expanded in a Dulac series with linear part the identity. Let f_1 be a partial sum of this series approximating the germ f to within $o(1)\exp(-\nu\xi)$ in some standard domain of class n. Then

$$f = f_1 \circ f_2, \quad |f_2 - \text{ id } | = o(1)\exp(-\nu\xi).$$

Let us prove that $h_1 \in A^{-n}G_{\text{rap}}^{n-1}$. For this we verify the requirements of Definition 10 in S1.6. It will first be proved that $h_1 \in A^{-n}G^{n-l}$. We have that

$$A^n h_1 = Ad(g)A^{n-1}f_1.$$

But $f_1 \in \mathcal{R}^0$. Consequently, $A^{n-1}f_1$ belongs to the group G^{n-1} and even to the group G_{n-1}; see Definition 1 in S1.3. Further, $g \in G^{n-1}$. Since G^{n-1} is a group, we get that $A^n h_1 \in G^{n-1}$.

We prove that the germ h_1 increases no more rapidly and no more slowly than a linear germ on $(\mathcal{R}^+, \infty)$. This is equivalent to the assertion that the correction of the germ Ah_1 is bounded on $(\mathcal{R}^+, \infty)$. Let us prove this. We have

$$Ah_1 = \mathrm{Ad}(A^{1-n}g)f_1.$$

In this case the germ $A^{1-n}g$ has a bounded derivative together with the inverse on $(\mathbb{R}^+, \infty)$, since $\mu_n(g) \in (0, \infty)$. The germ f_1 has an exponentially small correction. Therefore, the germ Ah_1 has not only a bounded but even an exponentially small correction on $(\mathbb{R}^+, \infty)$. Proposition 1 gives us immediately that $h_2 \in \mathrm{id} + \mathcal{F}\mathcal{C}^+_{\mathrm{wr}}(\mathcal{D}^n_0)$. This finishes the proof of Lemma 1. □

C. Multiplication Proposition. Here we prove Proposition 2.

Proof. By definition of H in Proposition 2,

$$H = F \circ \rho G,\ \rho = A^{1-n}h,\ h = f \circ g^{-1} \in G^{n-2^-}_{\mathrm{slow}}.$$

The germ ρ satisfies all the assumptions of Lemma 1. This is checked in Subsections D-G of Section 2.9. In particular,

Assumption 1^0:

$$\sigma \circ D_1^{n-1} = D_1^{n-1}$$

is checked in Subsection D, Case 3. So, Lemma 1 is applicable to the composition $F \circ \rho$. By this lemma, $F \circ \rho \in \mathcal{F}\mathcal{C}^{n-1^+}_{\mathrm{reg}}$. By assumption, $G \in \mathcal{F}\mathcal{C}^{n-1^+}_{\mathrm{reg}}$.

By Lemma 1, $H = F \circ \rho \cdot G \in \mathcal{F}\mathcal{C}^{n-1^+}_{\mathrm{reg}}$ as well.

□

S 2.15. The Regularity Lemma

We end this chapter with the following natural lemma.

Regularity Lemma RL_n. *A partial sum of a STAR-m series with $m = n$ or $m = n-1$ belongs to the set $\mathcal{F}^m_{\mathrm{reg,\ id}}$.*

Proof. Suppose that $F \in \mathcal{F}\mathcal{C}^m$, Σ_∞ is the corresponding STAR-m for the composition $F \circ \exp^{[m]}$, and Σ is a partial sum of it. It is required to prove that

$$\Sigma \circ \ln^{[m]} \in \mathcal{F}\mathcal{C}^m.$$

The proof is by induction on n. For $n = 0$ a STAR-0 is a Dulac series; a partial sum of it is holomorphic in the whole plane and increases in $(\mathbb{C}^+, \infty)$ no more rapidly than an exponential. For $n = -1$ a STAR-n is a first-degree polynomial. Thus, the lemma is trivial for $n = 0$ (induction base).

Induction step. Suppose that Lemma RL_{n-1} has been proved. We prove Lemma RL_n. Consider the case $m = n$; the proof is analogous in the case $m = n-1$. It suffices to consider one term of the series: $a \exp \mathbf{e}$, $a \in \mathcal{K}^n$, $\mathbf{e} \in E^n$. We have that $a \circ \ln^{[n]} \in \mathcal{F}\mathcal{C}_{\mathrm{reg}(\mathcal{D}^n)}$. This follows from Propositions 1 and 2 in S2.7 and Lemma 3 in S2.3.

By the induction hypothesis, $\mathbf{e} \circ \ln^{[n-1]} \in \mathcal{F}\mathcal{C}_{\mathrm{reg}(\mathcal{D}^{n-1})}$, since $\mathbf{e}$ is a partial sum of a STAR-$(n-1)$; see Definition 2^n in S1.7. By requirement 3° of this definition,

$$|\,\mathrm{Re} \circ \mathbf{e} \circ \ln^{[n]}\,| < \mu\xi$$

in some standard domain Ω of class n.

Consequently,

$$|\exp\circ\mathbf{e}\circ\ln^{[n]}| < \exp\mu\xi \text{ in } \Omega.$$

Finally, we estimate the coboundary of the cochain $\exp\circ\mathbf{e}\circ\ln^{[n]}$. By the induction hypothesis,

$$|\delta\mathbf{e}\circ\ln^{[n]}| < m,$$

where m is some rigging cochain of the partition corresponding to the cochain $\mathbf{e}\circ\ln^{[n]}$. Then

$$\begin{aligned}|\delta(\exp\circ\mathbf{e}\circ\ln^{[n]})| &< |\exp\circ\mathbf{e}\circ\ln^{[n]}||\delta\mathbf{e}\circ\ln^{[n]}|\cdot O(1)\\ &< Cm\exp\mu\xi.\end{aligned}$$

But it was already proved above that the product of a rigging cochain of a regular partition by an exponential is majorized by another rigging cochain of the same partition.

This verifies the last requirement of Definition 10 in S1.6 and finishes the proof of Lemma RL_n. $\square$