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Weakly Contracting Systems and Attractors of Galerkin Approximations of Navier–Stokes Equations on the Two-Dimensional Torus*

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To my friend Alexander Chetaev

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1. Introduction

1.1. Attractors and turbulence

After the fundamental work of L. D. Landau [**24**] and E. Hopf [**16**], the heuristic explanation of turbulence from the point of view of ordinary differential equations has become widespread. The Navier–Stokes equations are considered under such an approach as a dynamical system in infinite-dimensional phase space. The points of that space are fields of velocities of the fluid—divergence-free vector fields, defined in the region of flow. The Navier–Stokes equation itself yields a vector field, now in the infinite-dimensional phase space. The trajectories of that field are then solutions of the Navier–Stokes equation. A singular point of that vector field is a stationary solution of the Navier–Stokes equation, corresponding to a laminar flow; a limit cycle corresponds to a periodic flow.

The most important characteristic of a vector field is its attractor—a set to which, with the passage of time, all or almost all of the phase curves are drawn.

The description of turbulence proposed by Landau and Hopf consists in the following. We fix the region of flow and consider the Navier–Stokes equation, depending on the Reynolds number as on a parameter. The first heuristic assumption: For every value of the Reynolds number the Navier–Stokes equation has a compact attractor. Both Landau and Hopf supposed that it was of finite dimension. For small Reynolds numbers, i.e. for large viscosities, the Navier–Stokes equation has a unique stable singular point, which attracts all the other solutions. In other words, there exists a laminar regime into which after a positive time all remaining flows are drawn. With the increase of the Reynolds number the stationary solution loses stability, and from it a limit cycle is born, which takes onto itself the function of an attractor. Now, indeed, all the solutions are drawn to it. With the further growth of the Reynolds number the attractor changes its structure, its dimension increases; the motion of the phase curves on and around it becomes ever more complicated. The second heuristic assumption consists in that near the attractor the solution behaves chaotically to a larger or lesser degree.

Landau and Hopf supposed that the attractor of the Navier–Stokes equation is for an arbitrarily large Reynolds number a multidimensional torus, and the phase curves of the Navier–Stokes equation are conditionally periodic windings on the torus. With the growth of the Reynolds number, as a result of the successive bifurcations the dimension of the attractor increases. The behavior of the phase curves on the attractor becomes more and more complicated as the number of independent frequencies defining the conditionally periodic motion increases.

In the following years a number of objections, of both experimental and theoretical character, were raised against this hypothesis, by F. A. Zhuravel′ and others [**35**], V. S. L′vov, A. A. Predtechenskii [**26**], and R. Helleman [**14**], who stated, "One of the principal objections against this picture consists in that in reality chaos in experimental observations of flows does not rise constantly with the growth of the Reynolds number **R**, but rather occurs sharply and suddenly for definite values of the Reynolds number" (R. Helleman [**14**]).

A hypothesis that appeared at the beginning of the 1960's after D. V. Anosov's discovery and investigation of U-systems, later called "Anosov systems" (see, e.g., V. I. Arnol′d [**2**], [**4**]), is free of these objections. U-systems are smooth dynamical systems which are characterized by the following remarkable property. Each solution φ of such a system has a neighborhood resembling a multidimensional saddle; there exist two smooth families of solutions whose intersection contains only the solution φ; the solutions of one of the families tend exponentially to φ as $t \to +\infty$; the solutions of the other family tend to φ as $t \to -\infty$. The solutions of Anosov systems are extremely unstable; each pair of close initial conditions may be slightly altered in such a way that the solutions under the new initial conditions will be exponentially scattered. Anosov systems exist even on three-dimensional manifolds. Their solutions behave very chaotically (D. V. Anosov [**1**], Ya. G. Sinaĭ [**32**]). The classical example is the geodesic flow on a surface of negative curvature.

V. I. Arnol′d conjectured that the attractor of a Navier–Stokes equation is a finite-dimensional smooth manifold, and that the restriction of the equation to the attractor is an Anosov system.

This conjecture makes clear, in particular, why the solution of the Navier–Stokes equation "forgets" its initial condition. Indeed, after a finite time each solution enters a small neighborhood of the attractor; two solutions, beginning at points that are close, may diverge by far; two solutions, far from one another, may become close, and then again diverge, and so forth. Attractors, near which the solutions behave so chaotically that their behavior may be adequately described in theoretical-probabilistic terms, are called stochastic attractors (Ya. G. Sinaĭ [**32a**]).

In general, attractors need not be manifolds. Such attractors are frequently called "strange", although D. Ruelle and R. Takens [**31**], the authors of the term "strange attractor", give this name to any attractor that is not a finite union of points or cycles. Ruelle and Takens conjectured that turbulence is explained by the presence of an attractor, along which the solutions coming nearby drift chaotically. The Navier–Stokes equation on the attractor itself need not be an Anosov system.

F. A. Zhuravel′, V. S. L′vov and others, analyzing experimental material for Couette flows, came to the conclusion that for comparatively small Reynolds numbers ($\mathbf{R} < 1100$) the Landau–Hopf hypothesis correctly describes the behavior of the solutions; if $\mathbf{R} > 1300$ the experimental data support the hypothesis of the "stochastic attractor".

1.2. Mathematical foundations of the heuristic picture

The first step in the explanation of the preceding hypotheses must be a theorem on the existence of solutions of the Navier–Stokes equation on the entire positive semiaxis of time and a uniqueness theorem. Such theorems have been proved only for two-dimensional flows. Below we will be dealing only with two-dimensional flows in a bounded region (with the condition of adherence on the boundary) or on closed (two-dimensional) surfaces.

The existence of a compact attractor for the Navier–Stokes equation in a bounded plane region was first proved by O. A. Ladyzhenskaya [**21**], [**22**]. C. Foias and R. Temam [**12**] proved that a subset in phase space, invariant with respect to the Navier–Stokes equation and bounded in a relatively strong norm (in fact the norm H_1), has finite dimension. The fact that the attractors discovered by O. A. Ladyzhenskaya are bounded in the norm H_1 has been proved only for Navier–Stokes equations on the two-dimensional torus (see §4 below).

The description of the behavior of the solutions of the Navier–Stokes equation for small Reynolds numbers given above has a rigorous foundation. The globally attracting singular point corresponding to the laminar flow was found by Serrin. The loss of stability of the laminar flow and the birth of a periodic regime were investigated by V. I. Yudovich. A detailed bibliography, along with a presentation of the results of Serrin and Yudovich, may be found in the book of Marsden and MacCracken [**28**].

About the behavior of the solution of the Navier–Stokes equation on the attractor, almost nothing is known. In contrast, a great deal of information has been accumulated on the behavior of flows of a nonviscous (ideal) fluid. These flows largely resemble the solutions of Anosov systems. In fact, the phase curves depicting these flows are geodesics on an infinite-dimensional manifold (on the group of diffeomorphisms of the domain of the flow, which preserve the volume). The Riemannian curvature of this group in the majority of two-dimensional directions is negative (V. I. Arnol'd [**2**]). Using that fact, Arnol'd discovered the exponential instability of the "trade-wind stream" on the two-dimensional torus. A. M. Lukatskiĭ obtained an analogous result for the two-dimensional sphere [**25**] and for the multidimensional torus. Moreover, Lukatskiĭ proved that the Ricci curvature "averaged over dimension" of the group of volume-preserving diffeomorphisms of a three-dimensional region is also negative. (The "averaged over dimension" Ricci curvature of an infinite-dimensional manifold G is defined as the limit of the Ricci curvature of a finite-dimensional manifold, "approximating" G, divided by the dimension n of the approximating manifold; the limit is taken as n tends to infinity.) These results give rise to the hope of discovering in the general case the exponential instability and chaotic behavior of the flows of an ideal fluid. It must be said that we are dealing here with the exponential instability of the fluid itself, and not of its field of velocities. That is a different kind of instability; the motions of a fluid with close fields of velocities as initial conditions may differ by a quantity which increases exponentially with time. For a flow in a three-dimensional region V. I. Arnol'd [**4**, §6] discovered a stationary flow of an ideal fluid for which the field of velocities was exponentially unstable. Recent investigations of Arnol'd, Ya. B. Zel'dovich, A. A. Rusmaĭkin and D. D. Sokolov make it possible to generalize this example to the case of a viscous fluid.

Arnol'd offered the following conjecture: The negativity of the curvature of the group of volume-preserving diffeomorphisms implies the exponential instability and chaotic behavior not only for flows of an ideal fluid, but also for the solutions of the Navier–Stokes equation.

1.3. Formulation of the results

The basic results of the present paper concerning the Navier–Stokes equation are the following.

Theorem 1. *The dimension of the attractors of the Galerkin approximations of the Navier–Stokes equation on the two-dimensional torus does not exceed* $\mathbf{C}\mathbf{R}^4$.

Here $\mathbf{R}$ *is the Reynolds number. The estimate is valid for* $C = 200$ [1] *for all* C *and, for* $C = 13$, *for large* $\mathbf{R}$.

A precise formulation of this theorem will be found in subsection 2.5. Here we note only that the dimension of the attractor is understood in the nonclassical sense. In fact, we deal with the Hausdorff dimension,[2] which is defined for an arbitrary subset of Euclidean space and which may take on not only integer but arbitrary real values. For smooth manifolds the Hausdorff dimension coincides with the classical dimension.

The following theorem shows the kind of consequences, formulated in classical terms, that one may obtain from estimates of a nonclassical—the Hausdorff or a related, the so-called entropic—dimension of the attractor.

Theorem 2. *Suppose that* $\mathbf{R} \geq 2$, $d > (2\mathbf{R})^{10}$. *Then the attractor of the Galerkin approximation to the Navier–Stokes equation on the two-dimensional torus has a one-to-one projection onto a d-dimensional plane in general position.*

Remark. The estimate given in Theorem 2 is extremely rough. It suffices to require that the number $d/2$ exceed the entropic dimension of the attractor. A possible experiment suggested by Theorem 2 is considered in subsection 1.4.

Theorems 1 and 2 are obtained as corollaries of general results on the so-called weakly contracting systems. We shall give first a definition generalizing the concept of a contracting mapping.

A smooth mapping of a domain Ω of Euclidean space is said to be *k-contracting* if it decreases the element of volume of any k-dimensional plane located in Ω.

Theorem. *An invariant set of a k-contracting mapping of a domain in Euclidean space into itself has Hausdorff dimension not larger than k.*[3]

This theorem, formulated in somewhat different terms, is proved in §3 (Theorem 5).

A differential equation in a Euclidean phase space is said to be *weakly contracting* if it has a globally absorbing domain and the divergence of the

[1] I did not try to obtain a best possible value for C.

[2] As far as is known to me, the first application of Hausdorff dimension to the theory of differential equations was carried out by J. Mallet-Paret [27].

[3] A closely related result was proved by A. Douady and J. Osterlé [11]. The theorem above was obtained independently.

corresponding vector field is negative in this domain. The phase flow transformation of this system in the absorbing domain, corresponding to the positive time, is k-contracting for some k. The value of k may be relatively easily estimated from above in terms of the right-hand side of the system. This makes it possible to find an upper estimate for the dimension of the attractor of weakly contracting systems (Theorems 3 and 4 of subsection 2.6).

There are many different examples of weakly contracting systems. One of them is a multidimensional pursuit problem arising in the theory of large biological systems (Sections 2.7, 2.8). A fundamental example of a weakly contracting system, to which this paper is devoted, is that of the Galerkin approximations to the Navier–Stokes equation on the two-dimensional torus. A third example is connected with the so-called Kuramoto–Sivashinsky equation

$$u_t + (u_x)^2 + u_{xx} + \nu u_{xxxx} = 0, \qquad \nu > 0. \tag{1.1}$$

This equation arises in chemical kinetics.

A numerical investigation of its solutions, periodic in x with period 2π, shows turbulent behavior for $\nu \ll 1$ (see for instance Y. Kuramoto and T. Yamada [**19**]). The Galerkin approximations to this equation also turn out to be weakly contracting systems. Moreover, they turn out to be k-contracting, while k is estimated from above by a quantity depending only on ν but not on the dimension of the Galerkin approximation. Another paper will be devoted to the investigation of equation (1.1).[4]

1.4. Commentary

This section contains an interpretation of Theorems 1 and 2 from the point of view of numerical experiments; the considerations are not rigorous. Suppose that the analogues of the theorems mentioned are valid for the full Navier–Stokes equation and for a three-dimensional flow. We interpret this assumption from the point of view of the experimenter observing the real flow of a fluid. The corresponding phase curve in infinite-dimensional space becomes so close to the attractor after a finite time that its points become indistinguishable from close points lying on the attractor. On the other hand, the location of points on the attractor is completely determined by a finite number of parameters (the coordinates of the projection of those points on a finite-dimensional plane). If the dimension of the attractor is small (of order 2 or 3) then the experimenter

[4] Yu. S. Il'yashenko, Global analysis of the phase portrait for the Kuramoto–Sivashinsky equation, IMA Preprint Series # 665, July 1990.

will observe a "repetition of the picture". Indeed, suppose that one photographs the fields of velocities (more precisely, two fields of directions of the flow) arising in the same flow of a fluid at different moments of time. Suppose 5 to 7 parameters characterizing these fields (for example, the directions of the flow at prescribed points) turn out to be close. Then both fields turn out to be close in the entire region of the flow.

This effect ought in principle to be observed for any Reynolds number and for any dimension of the attractor. We need only remark that with the growth of the dimension of the attractor the probability of the appearance of "close pictures" decreases exponentially.

1.5. Problems

It would be interesting to generalize Theorem 1 to the full Navier–Stokes equation on the two-dimensional torus or even to any closed surface. This generalization apparently does not present any difficulties in principle.[5] For this one would need an a priori estimate for the H_1-norm of the solutions, *uniform relative to the initial conditions* (the only thing depending on the initial conditions is the moment of time after which the estimate is valid).

For three-dimensional flows the generalization of Theorem 1 seems at present hopelessly difficult. In this direction C. Foias and R. Temam have obtained a conditional theorem on the finiteness of the dimension. It asserts that a set which is invariant relative to the Navier–Stokes equation and bounded in the norm H_1 has a finite Hausdorff dimension. The problem of boundedness of the solutions in the H_1-norm is one of the main problems of the theory. If one succeeds in proving that all the solutions of the Navier–Stokes equation with a three-dimensional region of flow are attracted to a set which is bounded in H_1, this would immediately solve the problem of existence and uniqueness of the solution on the ray $t \geq 0$. However, an approach to the solution of this "problem of global regularity" is at the present time not visible. See for example J. Marsden, M. McCracken and G. F. Oster [**28**, end of Chapter 9].

The study of weakly contracting systems was undertaken jointly with my friend Aleksandr Nikolevich Chetaev.

The impetus for the writing of this paper was the advice of V. I. Arnol'd, who read in manuscript the work of A. N. Chetaev and the author [**17**], to apply the results of that paper to the problems of hydrodynamics.

[5] This generalization for the two-torus was recently obtained by O. A. Ladyzhenskaya [**23**], the author [**37**] and A. V. Babin and M. I. VIshik [**7**], [**36**] for the full equation on the two-torus and in a bounded region in the plane. For arbitrary closed surfaces the problem remains open.

I thank V. I. Arnol′d for that advice and for many fruitful discussions. I also thank V. I. Bakhtin, A. I. Komech, E. M. Landis, A. E. Tumanov, A. L. Shnirel′man and M. A. Shubin, with whom I had very useful conversations.

2. Definitions and basic results

2.1. Maximal attractors

The word "attractor" means "attracting set". This term may be given a precise meaning in different ways.

Definition (absorbing domain). A region with piecewise smooth boundary, situated in the phase space of the differential equation

$$\dot{x} = v(x), \tag{2.1}$$

is said to be *absorbing* for that equation if the field v on the boundary of the region is directed to the interior of the region or tangent to the boundary.[6]

Remark. Suppose that g^t is the transformation of the phase current of equation (2.1) across time t, B an absorbing domain for equation (2.1). Then for any positive t, $g^t B \subset B$. Once it falls inside the absorbing domain, the phase curve never leaves it.

Definition (attractor). Suppose that B is a compact absorbing domain for equation (2.1). The set

$$M = \bigcap_{t \in \mathbf{N}} g^t B, \tag{2.2}$$

$\mathbf{N}$ being the set of natural numbers, is called a *maximal attractor* of equation (2.1).

In what follows we consider only maximal attractors, and drop the adjective "maximal".

Remark 1. The set M is nonempty. Indeed, in view of the preceding remark, for any integer t,

$$g^t B \subset B.$$

[6] The right-hand sides of the differential equations considered below are always supposed to be infinitely smooth.

Applying to both sets the transformation g^s for a natural number s, and using the group property of phase flows, we get

$$g^{t+s}B \subset g^s B.$$

Accordingly, M, as the intersection of a countable number of nested compact sets, is nonempty.

Remark 2. The set M is invariant for the time 1 transformation of the phase flow:

$$g^1 M = M. \tag{2.3}$$

Indeed, applying to both parts of equation (2.2) the transformation g^1, we get on the right-hand side the intersection of sets from the same sequence, only without the first term. Since the sequence consists of nested sets, this does not change the intersection.

2.2. The Navier–Stokes equation, elimination of the effects of dimension. The Reynolds number

The Navier–Stokes equation is considered as an ordinary differential equation in an infinite-dimensional phase space; hence we write $\dot{u}$ instead of u_t. Below, u, f are vector fields, p is a function on the torus; u, p and f are written as doubly periodic vector-functions and as functions on the plane. In the Navier–Stokes equation

$$\dot{u} = -(u, \nabla)u + \nu\Delta u - \nabla p + f, \qquad \operatorname{div} u = 0,$$

the vector-function f does not depend on time and has a zero average on the torus.

In order to eliminate the effects of scaling from the problem, we need to choose the system of units in such a way that the "characteristic size" of the domain of the flow and the "characteristic norm" of the perturbing force f are equal to unity. The choice of both quantities contains elements of arbitrariness. The characteristic dimension turns out to be the modulus of the maximal nonzero eigenvalue of the Laplace operator on the torus; the torus is chosen so that that quantity is equal to unity. Without loss of generality we may suppose that $\operatorname{div} f = 0$. The characteristic norm turns out to be the norm $\|\operatorname{rot} f\|_{L_2}$. We shall further on write $\|\operatorname{rot} f\|$ for short.

Remark. On the space of divergence-free vector fields with zero average on the torus, $\|\operatorname{rot} \cdot\|$ is indeed a norm.

Below we will denote by $m(\cdot)$ the mean value of a function or vector-function on the torus. Beginning from this point we suppose that the "de-dimensionalization" has already been accomplished;

$$T^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2, \qquad \|\operatorname{rot} f\| = 1, \qquad m(f) = 0.$$

We denote by L_2 the space of functions with summable square on the torus, equipped with the usual scalar product and norm:

$$(p, q) = \int_{T^2} pq = \int_0^{2\pi}\int_0^{2\pi} pq\,dx\,dy, \qquad \|p\|^2 = (p, p).$$

We denote by $\mathbf{L}_2$ the space of vector-functions on the torus $\mathbf{L}_2 = L_2^2$. The scalar product and the norm in $\mathbf{L}_2$ are defined as follows: If $u = (u_1, u_2)$, $v = (v_1, v_2)$, then

$$(u, v) = (u_1, v_1) + (u_2, v_2); \qquad \|u\|^2 = (u, u).$$

The scalar product and norm in the spaces $\mathbf{L}_2$ and L_2 are denoted in the same way. With which space we are dealing will be clear from the context.

We denote by V the space of divergence-free, in the generalized sense, vector fields in $\mathbf{L}_2$:

$$V = \{u \in \mathbf{L}_2 \mid (u, \nabla\varphi) = 0 \ \forall\varphi \in C^\infty(T^2)\}.$$

We denote by $\mathscr{H}$ the space of twice smooth vector fields on V. The subspace V is closed in $\mathbf{L}_2$ (O. A. Ladyzhenskaya [**21**]).

Suppose that P is the orthoprojector $\mathbf{L}_2 \to V$. The kernel of the operator P consists of gradient vector fields whose potentials have generalized first derivatives in the sense of Sobolev. It is not hard to prove that $P\Delta = \Delta P$. Therefore the Navier–Stokes equation may be written in the form

$$\dot{u} = -P((u, \nabla)u) + v\Delta u + f, \qquad u \in \mathscr{H}. \tag{N–S}$$

The torus T^2 and the perturbation f satisfy the normalizing conditions (2.4); the Reynolds number is defined by the equality $\mathbf{R} = 1/v$.

2.3. *Galerkin approximations*

Denote by $\Delta_{\mathscr{H}}$ the restriction of the Laplace operator on $\mathscr{H}$. Suppose that $\mu_1 \le \mu_2 \le \cdots$ are the eigenvalues of the operator $-\Delta_{\mathscr{H}}$, enumerated by taking account of multiplicity; in particular, $\mu_1 = \mu_2 = 0$, $\mu_3 = \cdots = \mu_6 = 1$. The eigenvectors of the operator $\Delta_{\mathscr{H}}$ have the form $\zeta^k = Ik\,e^{ikx}$, $k \in \mathbb{Z}^2$, where I is the counterclockwise rotation of the plane $\mathbb{R}^2$ through 90°: if $k = (k_1, k_2)$, then $Ik = (-k_2, k_2)$. The eigenvalue $|k|^2 = k_1^2 + k_2^2$ corresponds to the vector ζ^k. We enumerate the vectors $\{\zeta^k \mid k \in \mathbb{Z}^2\}$ in the sequence $\{\eta^j \mid j \in \mathbb{N}\}$ so that the eigenvalue μ_j corresponds to the vector η^j and so that the vectors ζ^k and ζ^{-k} stand side by side. For any even N we denote by E_N the space of all real vector fields generated by the fields $\eta^1, \ldots, \eta^N$. Denote by $P_N \colon \mathbf{L}_2 \to E_N$ the orthoprojector. Put

$$\mathfrak{B}(u) = -(u, \nabla)u + v\Delta u + f.$$

Then the Navier–Stokes equation takes the form

$$\dot{u} = P\mathfrak{B}(u), \qquad u \in \mathscr{H} \tag{N–S}$$

(the linear operator and its derivative are denoted in the same way). The Galerkin approximation to the Navier–Stokes equation has the form

$$\dot{u} = P_N\mathfrak{B}(u), \qquad u \in \mathscr{H}. \tag{G_N}$$

2.4. *Hausdorff and entropic dimension*

The idea of the definition of Hausdorff dimension is illustrated by the following example.

One is investigating a set in $\mathbb{R}^3$ about which it is known that it is either a curve, a surface, or a body. The set itself is inaccessible, but, to the question: "What is its length, area, and volume?" one can obtain an answer. Obviously, the answer: length ∞, area 1, volume 0 means that the unknown set is a surface.

To define the Hausdorff dimension, we first define the Hausdorff measure, which in turn is defined in terms of coverings, similar to the upper Lebesgue measure.

Everywhere below K is a compact subset of a Hilbert space. All the definitions retain their meaning for subsets of any metric space, but we will have no need of such generality.

Definition. A *covering* of the set K by balls (for short, a covering of the set K) is a collection of balls whose union contains K.

We denote by $\mathfrak{U}_\varepsilon(K)$ the collection of all finite coverings of K by balls whose radii do not exceed ε. $\mathfrak{B}_\varepsilon(K)$ denotes the set of all coverings of the class $\mathfrak{U}_\varepsilon(K)$ consisting of equal balls; the *d-dimensional volume* of a covering U of the set K by balls Q_j of radius R_j is the quantity

$$V_d(U) = \sum_j R_j^d.$$

Put

$$m_{\varepsilon,d}(K) = \inf_{u \in \mathfrak{U}_\varepsilon(K)} V_d(U).$$

Definition. The *d-dimensional Hausdorff measure* of the set K is the limit, finite or infinite,

$$m_d(K) = \lim_{\varepsilon \to 0} m_{\varepsilon,d}(K).$$

This limit exists, because for a fixed d the quantity $m_{\varepsilon,d}(K)$ does not decrease with the decrease of ε, the infimum in the definition being taken over an ever smaller class of coverings.

Definition. The *Hausdorff dimension* of the set K is given by the formula

$$\dim_H K = \begin{cases} \infty, & \textit{if } m_d(K) \neq 0 \quad \textit{for any } d, \\ \inf\{d \mid m_d(K) = 0\} & \textit{otherwise.} \end{cases}$$

The entropic measure and dimension are defined in the same way as the Hausdorff ones, with the only difference that everywhere, instead of coverings of the class $\mathfrak{U}_\varepsilon(K)$, one chooses coverings of the class $\mathfrak{B}_\varepsilon(K)$. The entropic d-dimensional measure and dimension are denoted by $\mathrm{Em}_d(K)$ and $\dim_E(K)$, respectively. The concept of entropic dimension was introduced by L. S. Pontryagin and L. G. Shnirel'man **[29]**, who called this quantity the "metric order of the compact set".

The properties of both dimensions are discussed in subsection 5.1.

2.5. Attractors of Galerkin approximations

Now we may precisely formulate Theorem 1 of the Introduction. Our results are divided into qualitative—we assert the existence of some estimates, and quantitative—we calculate estimates explicitly. Here we shall formulate these and other results, and prove only the qualitative ones.

Theorem 1. 1°. *The system* (G_N) *has an attractor whose entropic and Hausdorff dimensions are estimated from above by constants, depending only on the Reynolds number, and not depending on the index N of the Galerkin approximation.*

2°. *If* $\mathbf{R} \leqslant 0.69$, *then the system* (G_N) *has a stationary solution to which all the others tend as* $t \to \infty$. *The attractor consists of a single point.*

3°. *For large* $\mathbf{R}$

$$\dim_H M \leq 13\mathbf{R}^4, \qquad \dim_E M \leq 2^7\mathbf{R}^{10}.$$

Moreover,

$$\dim_H M \leq 200\mathbf{R}^4 \quad \textit{for all } \mathbf{R};$$

$$\dim_E M \leq 2^9\mathbf{R}^{10} \quad \textit{for } \mathbf{R} \geq 2.$$

2.6. *Weakly contracting systems*

Theorem 1 is obtained as a consequence of general theorems on the so-called weakly contracting systems.

Definition 1. An absorbing domain of the dynamical system (2.1) is called *globally absorbing* if each phase curve falls into that domain after some positive time.

Definition 2. The dissipative system (2.1) is called *weakly contracting* if

1°. It has a globally absorbing domain B with a compact closure.

2°. In the domain B the inequality

$$\operatorname{div} v < 0$$

is satisfied.

Obviously a weakly contracting system has an attractor $M \subset B$. One may succeed in estimating its Hausdorff and entropic dimensions if one imposes additional requirements on the system.

Beginning from this point, the space $\mathbb{R}^N$ is Euclidean. To each smooth vector field v there corresponds a quadratic form

$$F_v(x)\colon \xi \to (v_*(x)\xi, \xi), \qquad \xi \in T_x\mathbb{R}^N.$$

Suppose that $\lambda_1(x) \geq \cdots \geq \lambda_N(x)$ are the eigenvalues of the quadratic form $F_v(x)$.

Definition 3. A weakly contracting system with a globally absorbing domain B has characteristic k if for each $x \in B$,

$$\lambda_1(x) + \cdots + \lambda_k(x) \leq 0,$$

and for some $x \in B$,

$$\lambda_1(x) + \cdots + \lambda_{k-1}(x) \geqslant 0.$$

Definition 4. A system with a globally absorbing domain B is said to be *weakly contracting with constants* (λ, a, n), where λ is real, a is positive, and n is a natural number, if it is weakly contracting in the sense of Definition 2, and for any $x \in B$,

$$\lambda_1(x) \leq \lambda, \qquad \lambda_{n+1}(x) \leq -a.$$

Theorem 3. *A compact invariant set of a weakly contracting system with characteristic k has a Hausdorff dimension no larger than k.*

Theorem 4. *A compact invariant set of a weakly contracting system with constants* (λ, a, n) *has an entropic dimension not exceeding the quantity* $16n(\lambda + a)(\lambda + 5a)a^{-2}$.

If, moreover, $\lambda < 0$, *then any compact invariant set of the system in question consists of a single point.*

Theorem 4 is proved in the paper [**17**] of A. N. Chetaev and the author.

The following section is devoted to the proof of Theorem 3. A comparison of Theorems 3 and 4, and also of the estimates of Assertion 3° of Theorem 1, shows that the Hausdorff dimension is more easily and better estimated by analytical means than by entropic means. However, in numerical calculation one almost always "estimates" (on the empirical, nonrigorous level) the entropic, and not the Hausdorff, dimension. Moreover, the dimension of a plane in general position, on which one may homeomorphically project the compact set, has been successfully estimated only in terms of the entropic, and not the Hausdorff, dimension. Therefore we consider here both dimensions.

2.7. The multidimensional pursuit problem

A large number of visual examples of weakly contracting systems give rise to the so-called multidimensional pursuit problem, which in my point of view is of independent interest. It was posed by A. N. Chetaev and arises naturally in the investigation of large biological systems (A. N. Chetaev [**10**], A. A. Chestnova and A. N. Chetaev [**9**]). The following model in very rough terms describes the work of that zone of the respiratory center where the rhythmics are generated. It is a significant simplification of the model of Chestnova and Chetaev [**9**]. Suppose that $\{1, \ldots, N\}$ is a set of cells of the zone being investigated, x_i a real parameter describing the state of the cell (the mean instantaneous frequency of impulsation of that cell). An exterior signal arriving at the cell i is decoded as a requirement to "go to the state y_i". The state of the cell changes according to the law

$$\dot{x}_i = y_i - x_i.$$

This equation corresponds to an idealization of a "system with broken connections". In fact the signal y_i arrives at the i-th cell not from without, but rather is determined by the states of the remaining cells of the system, including the cell i itself:

$$y_i = f_i(x), \qquad x = (x_1, \ldots, x_N).$$

Therefore the work of the system of cells $\{1, \ldots, N\}$ is defined by a differential equation:

$$\dot{x} = f(x) - x, \qquad f = (f_1, \ldots, f_N);$$

here f is an interior signal defining the state of the system. The image of the mapping f is called the "manifold of interior signals" and is denoted by Λ. The complexity of this manifold (in particular, its dimension) characterizes the complexity of the behavior of the system of cells. Thus, if Λ consists of a single point y, the system, from any state, goes into the stable state y. In the general case one has to study the set of "stationary regimes of the work of the system of cells", depending on the properties of the mapping f.

The formal statement of the problem consists of the following.

Suppose f: $\mathbb{R}^N \to \Lambda$ is a smooth mapping of $\mathbb{R}^N$ onto a compact manifold Λ, with or without boundary. The system

$$\dot{x} = f(x) - x \tag{2.5}$$

is said to be a *multidimensional pursuit problem*. We seek to find an attractor of this system.

Proposition 1. *Suppose that B is any ball containing* Λ, $\dim \Lambda = n$, *and L is the Lipschitz constant of the restriction of f to B. Then*

1°. *The multidimensional pursuit problem* (2.5) *is a weakly contracting system with constants* $(L - 1, 1, n)$ *and absorbing domain B;*

2°. *The characteristic k of the system* (2.5) *does not exceed the integer part of* $Ln + 1$: $k \leqslant [Ln + 1]$.

Proof. The second assertion follows immediately from the first, which is proved below. The vector in the right-hand side of the system (2.5), depicted as a directed segment, has origin x and endpoint $f(x) \in \Lambda$. Therefore any ball containing Λ is a global absorbing domain for the system (2.5): any positive semitrajectory of the system (2.5) sooner or later falls into any such ball.

We will prove that the system (2.5) is weakly contracting with constants $(L - 1, 1, n)$. Fix a ball B containing Λ and suppose that $v(x) = f(x) -$

$x\, F_v(x)\colon \xi \to (v_*(x)\xi, \xi)$. Then for $x \in B$,

$$F_v(x)\xi \leq (L-1)(\xi, \xi),$$

in view of the Schwarz (Cauchy–Bunyakovskiĭ) inequality.

This proves the inequality $\lambda_1(x) \leqslant L-1$ (see Definition 4 in subsection 2.6).

Further, the rank of any mapping $f_*(x)$ at each point $x \in B$ does not exceed n. Therefore the subspace $\operatorname{Ker} f_*(x)$, as the kernel of an arbitrary mapping, cannot have codimension larger than n. On the other hand, for $\xi \in \operatorname{Ker} f_*(x)$,

$$(v_*(x)\xi, \xi) = -(\xi, \xi).$$

Accordingly, by the theorem of Rayleigh, Courant and Fischer (V. I. Arnol′d [4]) no more than n eigenvalues of the quadratic form $F_v(x)$ exceed -1: $\lambda_{n+1} \leq -1$ for each $x \in B$. Proposition 1 is proved.

Corollary. *The Hausdorff dimension of an attractor of a multidimensional pursuit problem does not exceed* $[Ln+1]$.

This assertion strengthens a result of A. N. Chetaev and the author [**17**].

2.8. *A geometrical example*

Some questions naturally arise: Can a weakly contracting system with constants (λ, a, n) have an attractor with dimension larger than n?

Can the multidimensional pursuit problem have such a property?

Two students of V. I. Arnol′d, namely D. N. Bernshteĭn and V. A. Vasil′ev, have constructed examples giving a positive answer to both questions. Vasil′ev constructed a closed curve on the plane and a field of tangent pointer-vectors on it, the endpoints of which run along a nonclosed curve (Figure 1).

Proposition 2 (V. A. Vasil′ev, D. N. Bernshteĭn). *For any n there exists a multidimensional pursuit problem for which the manifold Λ has dimension n and the attractor has dimension $2n$.*

Proof. 1°. We first construct a system (2.5) with a one-dimensional "objective manifold" Λ and a two-dimensional attractor. The crucial step is already done on Figure 1. Suppose Γ is a closed curve as depicted in the figure, and Λ a nonclosed curve running through the endpoints of the tangent vectors to Γ. The mapping $f_\Gamma\colon \Gamma \to \Lambda$ is given on Figure 1; the origin of the vector passes into its endpoint. We extend f_Γ to a smooth mapping $f\colon \mathbb{R}^2 \to \Lambda$. To this end we smoothly parametrize the curve Λ by points s of the segment $I = \{s \in [0, 1]\}$.

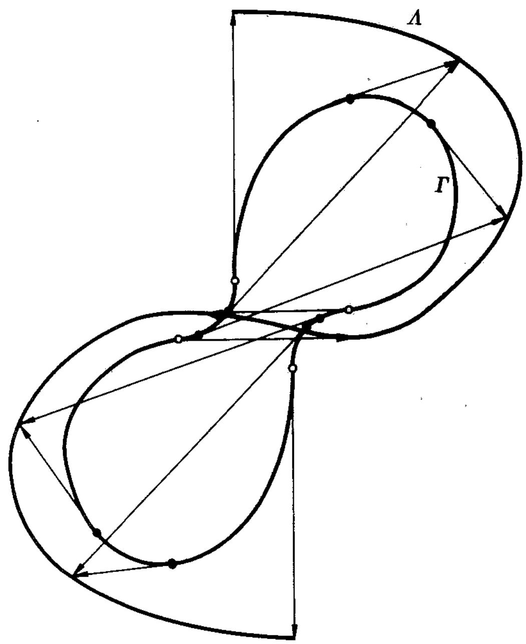

Figure 1. Λ is the objective manifold, Γ the closed phase curve of the corresponding multidimensional pursuit problem.

The mapping f_Γ defines a function $s \circ f_\Gamma\colon \Gamma \to I$ (and may be reconstructed in terms of it, because the mapping $s\colon \Lambda \to I$ is invertible). We extend the function $s \circ f_\Gamma$ to a smooth function $S\colon \mathbb{R} \to \Lambda$ (this may easily be done by an elementary construction). We define the mapping $f\colon \mathbb{R}^2 \to \Lambda$ by the formula $f = s^{-1} \circ S$ (the point s is mapped to the point on the curve Λ which corresponds to the value of the parameter $s = S(x)$).

The mapping just constructed coincides on Γ with f_Γ and yields a multidimensional pursuit problem (2.5) having a closed phase curve Γ. The interior of this curve is, obviously, invariant under the action of the phase flow, and, accordingly, lies entirely in the attractor of the system (2.5), which is thus two-dimensional.

2°. The direct product of n systems, constructed in 1° above, yields the desired multidimensional pursuit problem. Proposition 2 is proved.

Addition of the direct component $\dot{y} = -y$, $y \in \mathbb{R}^m$, for sufficiently large m, transforms the equation constructed in Proposition 2 into a weakly contracting system.

3. Attractors of weakly contracting systems

3.1. Distortion of volumes under the action of weakly contracting systems

In this subsection we prove Theorem 3. The proof of Theorem 3 is based on the following simple idea, due to A. N. Chetaev. A dynamical system for which the transformation of the phase flow for a positive time decreases the k-dimensional volume cannot have an invariant k-dimensional manifold.

Indeed, an invariant manifold under the action of a phase flow preserves its volume.

A. N. Chetaev applied this idea to the study of the attractor in the multi-dimensional pursuit problem.

V. I. Arnol′d proposed to call the transformations decreasing the k-dimensional volume "*k-contracting*".

Lemma 1. *Suppose that $\dot{x} = v(x)$, $x \in \mathbb{R}^N$, is a weakly contracting system with an absorbing domain B and characteristic k. Then the phase-flow transformation of this system for any positive time is k-contracting in the domain B.*

Proof. As above, we denote by $\lambda_j(x)$ the j-th eigenvalue of the quadratic form

$$F_v(x)\colon \xi \to (v_*(x)\xi, \xi), \qquad \lambda_1(x) \geq \cdots \geq \lambda_N(x),$$

and we suppose that

$$\Lambda_n = \max_{x \in B} \sum_1^n \lambda_j(x).$$

We will prove that for any $x \in B$ and any n-dimensional parallelepiped $\Pi^n \subset T_x B$ the inequality

$$V(g_*^t(x)\Pi^n) \leq e^{\Lambda_n t} V(\Pi^n) \tag{3.1}$$

holds, where $V(\Pi^n)$ is the nonoriented n-dimensional volume of the parallelepiped Π^n. The assertion of the lemma follows immediately from inequality (3.1). We will prove, first for small τ, the inequality

$$V(g_*^\tau(x)\Pi^n) \leq (1 + \Lambda_n \tau + o(\tau)) V(\Pi^n). \tag{3.2}$$

Suppose that Π is the n-dimensional plane containing Π^n, F_Π is the restriction of the form $F_v(x)$ on Π, $\nu_1 \geq \cdots \geq \nu_n$ the eigenvalues of the form F_Π, $\eta^1, \ldots, \eta^n$ the corresponding orthonormalized eigenvectors. Suppose that $I^n \subset \Pi$ is a unit cube with edges $\eta_1, \ldots, \eta^n$. The volumes of all n-dimensional parallelepipeds in the plane Π under the mapping $g_*^\tau(x)$ are distorted in the same way. Hence

$$\frac{V(g_*^\tau(x)\Pi^n)}{V(\Pi^n)} = V(g_*^\tau(x)I^n).$$

Further,

$$V(g_*^\tau(x)I^n) \le \prod_{j=1}^n |g_*^\tau(x)\eta^j|,$$

the volume of the parallelepiped does not exceed the product of the lengths of its edges. On the other hand,

$$g^\tau x = x + \tau v(x) + a(\tau, x), \qquad a(\tau, x) = o(\tau),$$

uniformly over $x \in B$. Accordingly,

$$g_*^\tau(x)\xi = \xi + \tau v_*(x)\xi + o(\tau), \qquad |g_*^\tau(x)\eta^j| \le 1 + \tau v_j + o(\tau).$$

By the Rayleigh–Courant–Fischer theorem (V. I. Arnol'd [**4**]), $v_j \le \lambda_j(x)$. Hence inequality (3.2) follows.

Suppose that t is arbitrary and positive. Putting $\tau = 1/\nu$ in (3.2), we get

$$V(g_*^{t/\nu}(x)\Pi^n) \le \left(1 + \frac{\Lambda_n t}{\nu} + o\left(\frac{1}{\nu}\right)\right)V(\Pi^n).$$

Taking account of the group property of the phase flow in an absorbing domain for $t > 0$ we get

$$V(g_*^t(x)\Pi^n) \le \left(1 + \frac{\Lambda_n t}{\nu} + o\left(\frac{1}{\nu}\right)\right)^\nu V(\Pi^n).$$

Passing to the limit as $\nu \to \infty$, we obtain inequality (3.2). Lemma 1 is proved.

3.2. Distortion of the "Hausdorff volume" under a diffeomorphism and an upper estimate of the dimension of attractors

3.2.1. Formulations. In this subsection we deduce Theorem 3 from Lemma 1. V. I. Arnol'd proposed to consider it as a consequence of a theorem on the distortion of Hausdorff measure under diffeomorphism. The formulations of Theorems 6 and 7 resulted from conversations between V. I. Arnol'd and the author. These theorems are formulated, proved, and used for subsets of a finite-dimensional space. Close results are valid in infinite-dimensional space.

Theorem 5. 1°. *Suppose that O is a region in Euclidean space H and $g\colon O \to H$ a twice smooth diffeomorphism with a second derivative bounded in O. Suppose* *that k is an integer and that, for any $x \in O$,*

$$V(g_*(x)\Pi^k) \le qV(\Pi^k), \tag{3.3}$$

where Π^k is a k-dimensional parallelepiped in T_xH, $V(\Pi^k)$ is its nonoriented k-dimensional volume, and q is some positive constant.

2°. *Suppose $K \subset O$ is a compactum;*

$$g(K) = K.$$

3°. *Suppose $q < 1$. Then*

$$m_k(K) = 0 \quad \textit{and} \quad \dim_H(K) \le k.$$

An analogous theorem is valid in the infinite-dimensional case. It strengthens a result of J. Mallet-Paret [**27**] and will be published elsewhere.

Theorem 6. *Suppose condition 1° of Theorem 5 is satisfied.*

2°. *Suppose that $K \subset O$ is compact, and*

$$0 < m_k(K) < \infty.$$

Then there exists a quantity $C(k) > 0$ such that

$$m_k(gK) \le C(k)qm_k(K). \tag{3.4}$$

Both of these theorems may be derived from the following one.

Theorem 7. *Suppose condition 1° of Theorem 5 is satisfied, and that $K \subset O$ is compact. Then there exists a constant $\tilde{C}(k) > 0$ such that for sufficiently small ε the inequality*

$$m_{\varepsilon,k}(g, K) \le \tilde{C}(k)qm_{\delta,k}(K) \tag{3.5}$$

holds, where $\delta = 2\sqrt{2}\,\varepsilon q^{1/k}$.

In what follows we shall constantly use concepts from subsection 3.4, where in particular the quantity $m_{\varepsilon,k}$ is defined.

3.2.2. Reductions. Suppose that Theorem 7 is already proved.

Proof of Theorem 6. Taking account of the fact that $\delta = 2\sqrt{2}\,\varepsilon q^{1/k}$, we pass to the limit in inequality (3.5) as $\varepsilon \to 0$. We obtain inequality (3.4) with $C(k) = \tilde{C}(k)$.

Proof of Theorem 5. It is sufficient to establish that $m_{\varepsilon,k}(K) \equiv 0$ for all sufficiently small ε. Suppose the contrary: There exists an arbitrarily small ε for which

$$m_{\varepsilon,k}(K) \neq 0.$$

In view of the invariance of K, $g(K) = K$ and, for any natural number t,

$$m_{\varepsilon,k}(g^t K) = m_{\varepsilon,k}(K).$$

We choose t so that

$$q^t \tilde{C}(k) < 1 \tag{3.6}$$

and apply Theorem 7 to the mapping g^t. This mapping is defined in some neighborhood $O' \subset O$ of the compactum in view of the invariance of K, and it satisfies there the inequality

$$V((g^t)_*(x)\Pi^k) \leq q^t V(\Pi^k) \quad \text{for } x \in O'.$$

Theorem 7 together with inequality (3.6) implies that

$$m_{\varepsilon,k}(g^t K) \leq m_{\varepsilon,k}(K).$$

The resulting contradiction completes the proof.[7]

3.2.3. Commentary. It would be interesting to find the smallest value for $C(k)$ in Theorem 6. Here we have proved a very rough variant of the theorem: $C(k) = k^{k/2}2^{k/2}\exp(2\sqrt{k})$. The covering theorem of Rogers [30] makes it possible to lower $C(k)$ down to the value $2^{3k/2}\rho(k)$, where $\rho(k) = k(\ln k + \ln\ln k + 5)$ is Rogers' constant. No one has yet succeeded in lowering that value of $C(k)$.

3.2.4. Plan for the proof of Theorem 7. We recall that $\mathfrak{U}_\varepsilon(K)$ is the class of coverings of the compact set K by balls of radius no larger than ε.

Lemma 2. *Suppose that g and K are the same as in Theorem 7. Then there exists a small ε such that, for each covering $U \in \mathfrak{U}_\varepsilon(K)$, there exists a covering $\tilde{U} \in \mathfrak{U}_\delta(gK)$ such that*

$$V_k(\tilde{U}) \leq \tilde{C}(k) V_k(U);$$

here $\delta = 2\sqrt{2}\,\varepsilon q^{1/k}$ and $\tilde{C}(k)$ is a positive quantity depending only on k.

Theorem 7 follows immediately from Lemma 2.

[7] This proof, as well as the formulation of Theorem 5, was inspired by the heuristic considerations in the paper of P. Frederickson, J. L. Kaplan, and J. A. Jorke [13].

Scheme of Proof of the Lemma. Suppose that the covering U consists of balls Q_j with centers x^j and radii R_j. The regions gQ_j cover gK. If ε is sufficiently small, each of these regions is very similar to the ellipsoid $g_*(x^j)Q_j$, which we denote from now on by $\mathscr{E}_j$. The ellipsoid $\mathscr{E}_j$ is flattened out, in the following sense. Let $a_1, \ldots, a_k$ be its k largest axes, $\mathscr{E}'_j$ the section of the ellipsoid spanning them, Q'_j a ball of radius $|a_k|$ in the plane orthogonal to the subspace spanning $\mathscr{E}'_j$. Then $\mathscr{E}_j \subset \mathscr{E}'_j \times Q'_j$. It is not hard to show that $|a_k| \le q^{1/k}R_j$, while the a_1 axis may be arbitrarily large.

The ellipsoid $\mathscr{E}'_j$ may be covered by balls of radius $|a_k|$ in such a way that the k-dimensional volume of the resulting covering will not be larger than $C_1(k)qR_j^k$ (Proposition 2 below). Blowing up these balls by $\sqrt{2}$ times, we obtain a covering of the whole ellipsoid $\mathscr{E}_j$; blowing them up a bit further, we obtain a covering of the region gQ_j. Taking the union of these coverings over all j, we obtain a covering of the compactum gK, the k-dimensional volume of which does not exceed $\tilde{C}(k)qV_k(U)$, as required.

3.2.5. Proof of Lemma 2. All the notation of subsection 3.2.4 remains in force.

Proposition 1. *Suppose that $A\colon H \to H$ is a linear operator such that*

$$V(A\Pi^k) \le qV(\Pi^k);$$

Q is the unit ball in K. Then there exist an $R \le \sqrt{2}q^{1/k}$ and a $C_1(k)$ such that the ellipsoid $\mathscr{E} = AQ$ can be covered by balls of radius R such that one obtains a covering (denoted from now on by U_A) *for which*

$$V_k(U_A) \le C_1(k)q.$$

Proof. Denote by $a_1, \ldots, a_k$, $|a_j| = v_j$, the k largest semiaxes of the ellipsoid, $v_1 \ge \cdots \ge v_k$, and suppose that $\mathscr{E}'$ is a section of the ellipsoid $\mathscr{E}$ by a plane spanning these axes. Obviously

$$v_1, \ldots, v_k \le q, \qquad v_k \le q^{1/k}.$$

Denote by Q' a ball of radius v_k with center 0, situated in the orthogonal complement of the plane of the ellipsoid $\mathscr{E}'$. Obviously $\mathscr{E} \subset \mathscr{E}' \times Q'$.

Proposition 2. *The ellipsoid $\mathscr{E}'$ may be covered by balls of radius v_k such that their centers lie in the plane of that ellipsoid and the k-dimensional volume of the resulting covering $\mathring{U}_A$ does not exceed $C_2(k)q$, where $C_2(k)$ is a positive quantity, depending only on k.*

Remark. The k-dimensional volume of $\mathring{U}_A$ is estimated from above by a quantity depending only on the volume but not on the form of the ellipsoid. The radius of the balls of the covering still depends on the form of the ellipsoid, and the region gQ_j may be approximated by ellipsoids of a different form. This prevents proving Theorems 5–7 for the entropic dimension in the same way as for the Hausdorff dimension.

Proof of Proposition 2. We enclose the ellipsoid $\mathscr{E}'$ in a parallelepiped Π with edges $2v_1, \dots, 2v_k$. Any segment of length v can be covered by segments of length h whose number does not exceed $1 + v/h$. Therefore the parallelepiped Π may be covered by cubes with edge h whose vertices form a cubic lattice and whose number does not exceed the quantity

$$\mathfrak{N}(k, h) = (2v_1 + h) \dots (2v_k + h)h^{-k}.$$

The parallelepiped Π may be covered by balls of radius $R = v_k$ with centers at the nodes of a cubic lattice and with step $h = 2R/\sqrt{k}$; the number of balls does not exceed $\mathfrak{N}(k, h)$. We denote the resulting covering by $\mathring{U}_A$. We have

$$\begin{aligned} V_k(\mathring{U}_A) &\le \frac{(2v_1 + h) \dots (2v_k + h)k^{k/2}}{2^k R^k} R^k \le \left(v_1 + \frac{v^k}{\sqrt{k}}\right) \cdots \left(v_k + \frac{v_k}{\sqrt{k}}\right) k^{k/2} \\ &\le v_1 \dots v_k \left(1 + \frac{1}{\sqrt{k}}\right)^k k^{k/2} \le k^{k/2}\, e^{\sqrt{k}} q, \end{aligned}$$

because $v_1 \dots v_k \le q$, $h = 2v_k/\sqrt{k}$. The proposition is proved for

$$C_2(k) = k^{k/2}/e^{\sqrt{k}}.$$

Proof of Proposition 1. We replace the k-dimensional balls of the covering $\mathring{U}_A$ (having radius $R = v_k$) by concentric balls in the space H of radius $\sqrt{2}R$. By the Pythagorean theorem, these balls form a covering of the product $\mathscr{E}' \times Q'$, and, consequently, of the ellipsoid $\mathscr{E}$. This is the desired covering U_A. Obviously

$$V_k(U_A) \le 2^{k/2} C_2(k) q.$$

Proposition 1 is proved for $C_1(k) = 2^{k/2} C_2(k) = k^{k/2} 2^{k/2}\, e^{\sqrt{k}}$.

Proof of Lemma 2 (Completion). If $M \subset H$ is some set, U a collection of balls, $a \in H$, $b \in \mathbb{R}$, then we denote by $a + bM$ and $a + bU$ the set and the collection of balls obtained from M and U, respectively, by the affine transformation $x \to a + bx$. We denote by Q the Euclidean ball with center 0 in any of the tangent spaces $T_x H$.

We choose an arbitrary $\rho \in (0, 1)$. In view of the boundedness in O of the second derivative of the mapping g, there exists an $\varepsilon(\rho)$ such that if $Q' \subset O$ is a ball with center x and radius $R < \varepsilon(\rho)$ then

$$gQ \subset gx + (1+\rho)g_*(x)Q'. \tag{3.7}$$

Suppose further that $\varepsilon(\rho)$ is less than the distance from K to ∂O. In Lemma 2 we now take $\varepsilon < \varepsilon(\rho)$. Then we may suppose that all the balls of the covering $U \subset \mathfrak{U}_\varepsilon(K)$ belong to the domain O. Suppose that the covering U consists of balls Q_j with centers x^j and radii R_j. In view of relation (3.7),

$$gQ_j \subset gx^j + (1+\rho)R_j g_*(x^j)Q.$$

Suppose that $A_j = g_*(x^j)$ and U_{A_j} is a covering of the ellipsoid A_jQ, constructed in Proposition 1. Then the collection of balls

$$U_j = gx^j + (1+\rho)R_j U_{A_j}$$

is a covering of the region gQ_j. The union of such coverings over all j forms a covering of the gK; it is the desired covering $\tilde{U}$. In view of Proposition 1,

$$V_k(U_j) = (1+\rho)^k R_j^k V_k(U_{A_j}) \leq (1+\rho)^k C_1(k) q R_j^k.$$

Accordingly,

$$V_k(\tilde{U}) \leq (1+\rho)^k C_1(k) q V_k(U),$$

which proves Lemma 2 for

$$\tilde{C}(K) = (1+\rho)^k C_1(k).$$

Remark. Since $\rho \in (0, 1)$ is arbitrary, in Lemma 2 and Theorem 6 we may take $C(k) = C_1(k) = k^{1/k} 2^{1/k} e^{\sqrt{k}}$.

4. Galerkin approximations of the Navier–Stokes equation as weakly contracting systems

In this section the first assertion of Theorem 1 is deduced from general theorems on weakly contracting systems.

We denote by E_N^C the plane $\{u \in E_N \mid m(u) = C\}$.

4.1. Main lemma

Main Lemma 1°. *The plane E_N^C is invariant for the system (G_N). The restriction of the system (G_N) to the plane E_N^C is a weakly contracting system for sufficiently large N with characteristic k and constants $(\lambda, 1, n)$. All of these parameters are estimated from above by a quantity depending on the Reynolds number and not depending on N.*

Assertion 1° of Theorem 1 follows immediately from this lemma and from Theorems 3 and 4. The remaining part of this section is devoted to the proof of the main lemma.

Proof of Assertion 1°. It is easy to prove that $m(\mathfrak{B}(u)) = 0$ for $u \in \mathscr{H}$. Further, the operators m and P_N commute (the operator P_N preserves the lower modes). Hence $m(\mathfrak{B}^n(u)) = 0$ for $u \in E_N$. Accordingly, the vector-function $m(u)$ is the first integral of the system (G_N). Assertion 1° of the main lemma is proved.

All the further proofs are carried out for $C = 0$; for $C \neq 0$ they can be obtained by simple modifications.

4.2. The absorbing domain

Lemma 1. 1°. *The set* $\mathscr{E}_N = \{u \in E_N^0 \mid \|\mathrm{rot}\, u\| \le R\}$ *is an ellipsoid.*
2°. *For* $R \le \mathbf{R}$ *this ellipsoid is absorbing for the equation* (G_N).

Remark. In view of Lemma 1 the function $\|\mathrm{rot}\, u\|^2$ decreases along any phase curve of the system (G_N) in the region $\|\mathrm{rot}\, u\| \le \mathbf{R}$. Therefore for the system (G_N) on the plane E_N^0 the ellipsoid $B = \mathscr{E}_\mathbf{R}$ is a globally absorbing domain.

Proof of Lemma 1. 1°. The form $\|\mathrm{rot}\, u\|^2$ on E_N^0 is positive definite. Indeed, in the opposite case there exists a nonzero field $h \in E_N^0$ for which $\mathrm{rot}\, h = 0$. Since the divergence and mean value of h are also zero, then h is itself zero, a contradiction.

Thus the set $\mathscr{E}_\mathbf{R}$ is indeed an ellipsoid.

In what follows the proofs of this and the following lemmas make use of the general method: First we establish a certain assertion for the full Navier–Stokes equation, and then we show that the passage to Galerkin approximations leaves the assertion in force.

2°. We denote by d/dt and d/dt_N differentiation along the vector fields $\mathfrak{B}$ and $\mathfrak{B}^N$, respectively. We recall that $\mathfrak{B}(u) = -(u, \nabla)u + \nu\Delta u + f$, $\mathfrak{B}^N = P_N \mathfrak{B}$.

Proposition 1. $d/dt\|\mathrm{rot}\, u\|^2 \le 0$ *for any* $u \in \mathscr{H}$ *for* $\|\mathrm{rot}\, u\| \ge \mathbf{R}$.

The proof is based on the two following facts.

1°. $\|\mathrm{rot}\, u\|^2$ is the first integral of Euler's equation, so that $(\mathrm{rot}[P(u, \nabla)u], \mathrm{rot}\, u) = 0$.

2°. The torus T^2 is a manifold without boundary, so that for any function $h \in C^1(T^2)$,

$$(\text{rot}\, u, h) = -(u, I\nabla h).$$

Below we apply this equation to $h = \text{rot}\, u$; it is not true for smooth functions h in a bounded region. Proposition 1 is the only place in our considerations which is not suitable for Navier–Stokes equations in a bounded plane region with the condition of adherence at the boundary.

We have

$$\frac{1}{2}\frac{d}{dt}(\text{rot}\, u, \text{rot}\, u) = (\text{rot}\, P\mathfrak{B}(u), \text{rot}\, u) = (\nu\ \text{rot}\, \Delta u + \text{rot} f, \text{rot}\, u)$$
$$= (\nu\Delta\, \text{rot}\, u + \text{rot} f, \text{rot}\, u) = -\nu(\nabla\, \text{rot}\, u, \nabla\, \text{rot}\, u) + (\text{rot} f, \text{rot}\, u).$$

Proposition 2. *For any function* $h \in C^1(T^2)$ *with a zero average,* $(\nabla h, \nabla h) \geq (h, h)$.

Proposition 1 follows immediately from Proposition 2. Indeed, $h = \text{rot}\, u$ is a function with a zero average, and since $\|h\| \geq R = 1/\nu$, $\|\text{rot} f\| = 1$, we get

$$-\nu(\nabla h, \nabla h) + (\text{rot} f, h) \leq \nu(-\|h\|^2 + R\|h\|) \leq 0.$$

Proof of Proposition 2. The minimum of $(\nabla h, \nabla h)$ over $h \in C^1(T)$, $\|h\| = 1$, is the maximal eigenvalue of the Laplace operator on T^2 on functions with zero average.

The torus T^2 is chosen so that this eigenvalue is equal to -1.

The inequality $(\nabla h, \nabla h) \geq 1$ for $\|h\| = 1$ proves the proposition.

3°. Now we pass from the full Navier–Stokes equation to Galerkin approximations in order to prove Assertion 2° of Lemma 1.

Proposition 1′. $(d/dt_N)\|\text{rot}\, u\|^2 \leq 0$ *for any* $u \in E_N$ *for* $\|\text{rot}\, u\| \geq \mathbf{R}$.

Proof.

$$\frac{1}{2}\frac{d}{dt_N}\|\text{rot}\, u\|^2 = (\text{rot}\, \mathfrak{B}^N(u), \text{rot}\, u) = -(\mathfrak{B}^N(u), \Delta u)$$
$$= -(P\mathfrak{B}(u), \Delta u) = (\text{rot}\, P\mathfrak{B}(u), \text{rot}\, u).$$

The third equation uses the inclusion $\Delta E_N \subseteq E_N$. This finishes the proof of the proposition, and hence of Lemma 1.

It remains to estimate the eigenvalues of the quadratic form, which was used to determine the characteristic of the system.

4.3. *Estimation of the quadratic form*

Lemma 2. *Suppose that*

$$F_N(u)\xi = (\mathfrak{B}_*^N(u)\xi, \xi), \qquad \xi \in T_u E_N^0.$$

Then there exists a $C(\mathbf{R})$ *such that for any* $u \in B = \{u \in E_N^0 \mid \|\mathrm{rot}\, u\| \le \mathbf{R}\}$ *the inequality*

$$F_N(u)\xi \le C(\mathbf{R})(\xi, \xi) + \frac{1}{2\mathbf{R}}(\xi, \Delta\xi)$$

holds.

We deduce the main lemma from Lemmas 1 and 2.

In view of Lemma 1, the ellipsoid B is a globally absorbing region for the system (G_N) on the plane E_N^0.

Suppose, as we did above, that $\mu_1 \le \mu_2 \le \cdots$ are the eigenvalues of the operator $-\Delta_{\mathscr{H}}$.

The quadratic form

$$\Phi\colon \xi \to C(\mathbf{R})(\xi, \xi) + \frac{1}{2\mathbf{R}}(\xi, \Delta\xi)$$

has on the plane E_N^0 the eigenvalues

$$\omega_1 \ge \omega_2 \ge \cdots, \qquad \omega_j = C(\mathbf{R}) - \frac{\mu_{j+2}}{2\mathbf{R}}, \qquad j \in (1, \ldots, N).$$

We note that $\mu_j \to \infty$ as $j \to \infty$. In view of Lemma 2 and the Rayleigh–Courant–Fischer theorem, the jth eigenvalue of the form $F_N(u)$ does not exceed ω_j. For sufficiently large N, $\sum_1^N \omega_j < 0$. By the definition of the form $F_N(u)$, the divergence of the field $\mathfrak{B}^N$ is equal to the trace of the form F_N. Therefore, for sufficiently large N, div $\mathfrak{B}^N < 0$ everywhere in B. For such N the system (G_N) is weakly contracting on E_N^0. The characteristic of this system does not exceed any k' for which $\sum_1^{k'} \omega_j < 0$. Thus the characteristic of the system (G_N) on E_N^0 is estimated from above by a quantity depending only on the Reynolds number. Finally, the system (G_N) on E_N^0 is weakly contracting with constants $(\lambda, 1, n)$, while

$$\lambda \le C(\mathbf{R}) - \frac{1}{2\mathbf{R}}$$

and the number n does not exceed a number n' for which

$$C(\mathbf{R}) - \frac{\mu_{n'}}{2\mathbf{R}} \le -1.$$

The main lemma is proved.

4.4. *Proof of Lemma* 2

The following assertion makes it possible to pass from the full Navier–Stokes equation to the Galerkin approximation. It is of a general character, and uses only the formula $\mathfrak{B}^N(u) = P_N \mathfrak{B}(u)$.

Proposition 3. 1°. *For* $u \in E_N$, $\xi \in E_N$,

$$(\mathfrak{B}_*^N(u)\xi, \xi) = (\mathfrak{B}_*(u)\xi, \xi).$$

Proof. The tangent space $T_u\mathscr{H}$ is canonically isometric to $\mathscr{H}$. Consider $\mathfrak{B}$ as a mapping $\mathscr{H} \to \mathbf{L}_2$ and $\mathfrak{B}_*$ as the derivative of this mapping. For short put $X = E_N$, $Y = E_N^\perp \cap \mathscr{H}$, $\mathfrak{B}' = \mathfrak{B}^N$, $\mathfrak{B}'' = \mathfrak{B} - \mathfrak{B}^N$. We denote by $\partial\mathfrak{B}'/\partial x$ and $\partial\mathfrak{B}'/\partial y$ the derivatives at the point u of the restrictions of $\mathfrak{B}'$ to X and Y, respectively: $(\partial\mathfrak{B}'/\partial x)\colon X \to X$, $(\partial\mathfrak{B}'/\partial y)\colon Y \to X$. Analogously one defines the action of the operators $(\partial\mathfrak{B}''/\partial x)\colon X \to E_N^\perp$, $(\partial\mathfrak{B}''/\partial y)\colon Y \to E_N^\perp$.

Then

$$\mathfrak{B}_*^N(u) = \frac{\partial\mathfrak{B}'}{\partial x}, \qquad \mathfrak{B}_*(u) = \begin{pmatrix} \dfrac{\partial\mathfrak{B}'}{\partial x} & \dfrac{\partial\mathfrak{B}'}{\partial y} \\ \dfrac{\partial\mathfrak{B}''}{\partial x} & \dfrac{\partial\mathfrak{B}''}{\partial y} \end{pmatrix}.$$

Therefore, for $\xi \in X$,

$$\mathfrak{B}_*(u)\xi = \frac{\partial\mathfrak{B}'}{\partial x}\xi + \frac{\partial\mathfrak{B}''}{\partial x}\xi,$$

$$(\mathfrak{B}_*(u)\xi, \xi) = \left(\frac{\partial\mathfrak{B}'}{\partial x}\xi, \xi\right) = (\mathfrak{B}_*^N(u)\xi, \xi),$$

as we were required to prove.

2°. *The explicit form of the formula* $(\mathfrak{B}_*(u)\xi, \xi)$. We have $(u, \nabla)u = u_* u$, where u_* is the Jacobian matrix of the vector-function u. For $u, \xi \in \mathscr{H}$,

$$\mathfrak{B}_*(u)\xi = \frac{d}{d\varepsilon}\mathfrak{B}(u + \varepsilon\xi)\Big|_{\varepsilon=0} = u_*\xi + \xi_* u + \nu\Delta\xi,$$

$$(\mathfrak{B}_*(u)\xi, \xi) = (u_*\xi, \xi) - \nu(\nabla\xi, \nabla\xi),$$

because

$$(\xi_* u, \xi) = -\tfrac{1}{2}(u, \nabla\langle\xi, \xi\rangle) = 0.$$

3°. It remains to estimate the eigenvalues of the quadratic form

$$F(u)\colon \xi \to (u_*\xi, \xi) + \nu(\xi, \Delta\xi), \qquad \xi \in \mathscr{H},$$

for all $u \in B$ with $\|\mathrm{rot}\, u\| \le \mathbf{R}$. This is equivalent to the estimation from below of the eigenvalues of the Schrödinger operator with a potential from L_2. Happily, here we find ourselves in a domain which has been studied in full detail.

Proposition 4. *If* $\|\mathrm{rot}\, u\| \le \mathbf{R}$, $\xi \in \mathscr{H}$, $m(\xi) = 0$, *then there exists a number* $C(\mathbf{R})$ *such that*

$$(u_*\xi, \xi) + \nu(\xi, \Delta\xi) \le C(\mathbf{R})(\xi, \xi) + \frac{\nu}{2}(\xi, \Delta\xi).$$

Three proofs of this proposition are known to me.

The first uses a lemma of Kato (Kato [**18**], p. 340) and gives the best estimate for $C(\mathbf{R})$. Indeed, Assertion 3° of Theorem 1 follows from this estimate.

The second makes use of an inequality of Ladyzhenskaya [**20**] and essentially uses the two-dimensionality of the torus.

The third is based on the Sobolev imbedding theorem [**33**] and works simultaneously for functions on the two-dimensional and three-dimensional torus.

The proof of Proposition 4 will be carried out by the second and third methods. Below, s is equal to 2 or 3.

For each Banach space B we will denote by $\mathbf{B}$ the space B^s with norm $\|\xi\|^2_{\mathbf{B}} = \sum_1^s \|\xi_i\|^2_{\mathbf{B}}$, where $\xi = (\xi_1, \ldots, \xi_s)$. If B is a Hilbert space, then the scalar product in $\mathbf{B}$ is defined by the formula $(\xi, \eta) = \sum_1^s (\xi_i, \eta_i)$. We will denote by $\mathring{\mathbf{H}}_1$ the Sobolev space of functions on T^s with a zero mean and norm

$\|h\|_{\mathring{\mathbf{H}}_1} = -(\nabla h, \nabla h)_{\mathbf{L}_2}$. We note that

$$\|\xi\|_{\mathring{\mathbf{H}}_1} = -(\xi, \Delta\xi)_{\mathbf{L}_2}.$$

We turn to the estimate itself. By the Schwarz inequality,

$$(u_*\xi, \xi)| \leq \left(\int_{T^s} \|u_*(x)\|^2\, dx\right)^{1/2} \|\xi\|^2_{\mathbf{L}_4};$$

the integrand is the ordinary norm of the matrix.

On the subspace $V \cap \mathring{\mathbf{H}}_1$ (the divergence-free vector fields of $\mathring{\mathbf{H}}_1$) the norms $(\int_{T^s} \|u_*(x)\|^2\, dx)^{1/2}$ and $\|\mathrm{rot}\, u\|$ are equivalent. Therefore, in view of the inequality $\|\mathrm{rot}\, u\| \leq \mathbf{R}$ there exists a constant C_1 such that

$$|(u_*\xi, \xi)| \leq C_1 \mathbf{R} \|\xi\|^2_{\mathbf{L}_4}.$$

Completion of the Proof for $s = 2$. From Ladyzhenskaya's inequality it follows that there exists a constant C_2 such that

$$\|\xi\|^2_{\mathbf{L}_4} \leq C_2 \|\xi\|_{\mathring{\mathbf{H}}_1}, \qquad \|\xi\| = C_2\sqrt{-(\xi, \Delta\xi)(\xi, \xi)}.$$

Hence it follows that for any $\varepsilon > 0$,

$$\|\xi\|^2_{\mathbf{L}_4} \leq C_2[-\varepsilon(\xi, \Delta\xi) + \varepsilon^{-1}(\xi, \xi)].$$

Finally, for some $C > 0$,

$$|(u_*\, \xi, \xi)| \leq C\mathbf{R}[-\varepsilon(\xi, \Delta\xi) + \varepsilon^{-1}(\xi, \xi)].$$

Choosing $\varepsilon = (\nu/2)(C\mathbf{R})^{-1}$, $C(\mathbf{R}) = C\mathbf{R}\varepsilon^{-1}$, we get Proposition 4.

Completion of the Proof for $s = 3$. In view of the Sobolev imbedding theorem, the imbedding $\mathring{\mathbf{H}}_1 \to \mathbf{L}_4$ is a compact operator. Hence it easily follows that for each ε there exists a number $C(\varepsilon)$ for which

$$\|\xi\|^2_{\mathbf{L}_4} \leq \varepsilon \|\xi\|^2_{\mathring{\mathbf{H}}_1} + C(\varepsilon)\|\xi\|^2.$$

The rest of the proof is carried out as for $s = 2$.

Proposition 4, and along with it the main lemma and Assertion 1° of Theorem 1, are proved.

5. Homeomorphic projections of attractors

Here we will prove the following geometrical theorem.

Theorem 2′. *A compact subset of Euclidean space, having entropic dimension d, projects homeomorphically onto a plane L in general position,*[8] *along the orthogonal complement to L, if* $\dim L > 2d$.

Theorem 2 of the Introduction follows immediately from Theorem 2′ and Theorem 1 of subsection 2.5. Theorem 2′ generalizes the following result for manifolds.

The easy Whitney Theorem (see, for example, M. Hirsch [**15**]). *A compact d-dimensional subset of Euclidean space* (here, in distinction from the preceding theory, d is an integer!) *orthogonally projects onto a plane L in general position along the orthogonal complement to L if* $\dim L > 2d$.[9]

5.1. Some properties of Hausdorff and entropic dimensions

For both dimensions one or the other of the elementary properties possessed by the usual topological dimension turns out to be false. We shall formulate, without proof, certain results, part of which are known, and part of which are recent results of the student B. I. Bakhtin, Professor E. M. Landis, and the author.

Natural properties. 1°. $\dim_H(K_1 \cup K_2) \leq \max(\dim_H K_1, \dim_H K_2)$.

2°. $\dim_H(K_1 \times K_2) \geqslant \dim_H K_1 + \dim_H K_2$ (H. Wegmann [**34**]). The same is true for the entropic dimension (V. Bakhtin [**8**]).

3°. $\dim_E(K \times K) \leq 2 \dim_E K$ (subsection 5.3 below).

It follows from Assertion 2 that this non-strict inequality is always an equality.

Relation between the dimensions. There exists a set of Hausdorff dimension 0 and arbitrarily large entropic dimension.

[8] We say that a generic k-dimensional plane in an N-dimensional linear space has a certain property if the set of planes having that property is thick in the Grassmann manifold $G(N, k)$. A thick set is a countable intersection of open everywhere dense sets.

[9] I have ventured to alter the formulation of the classical theorem somewhat in order to approximate it to the "entropic" situation.

Pathological properties. 1°. There exist two sets such that the entropic dimension of their union is larger than the dimension of each of the sets.

2°. There exist two sets such that the Hausdorff and entropic dimensions of the direct product of these sets exceed the sum of the corresponding dimensions of the factors (V. Bakhtin [**8**]).

3°. There exists a set such that the Hausdorff dimension of the Cartesian square of this set is larger than twice the dimension of the set itself. E. M. Landis reduced this assertion to Assertion 2°. The same reduction proves Assertion 1°.

I do not know whether Theorem 2′ remains true if the entropic dimension is replaced by the Hausdorff one. Assertion 3° suggests that the answer is negative.

5.2. Theorem 2′ and the easy theorem of Whitney

The considerations of this subsection relate to both theorems in the heading. Let $M \subset \mathbb{R}^N$ be the subset or submanifold mentioned in Theorem 2′ or in the "easy Whitney theorem", respectively. It suffices to prove that if $N > 2d + 1$, then the projection along a line l in general position onto the hyperplane orthogonal to l is a homeomorphism.

Suppose that S^{N-1} is the unit sphere in $\mathbb{R}^N$. To each bundle of oriented parallel lines in $\mathbb{R}^N$ there corresponds a point on the sphere S^{N-1} (the directing vector of the bundle), and conversely. In other words, the Grassmann manifold of oriented lines passing through 0 is the sphere S^{N-1}. A point $x \in S^{N-1}$ will be said to be *M-bad* if at least one of the lines in the corresponding bundle intersects M in two or more points, and *M-good* otherwise. We denote by Σ the set of all M-bad points, and by Λ the set of all M-good points of S^{N-1}. It suffices to show that the set Λ is thick.

To that end we describe the set Σ explicitly. Denote by Δ the diagonal of the direct product $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$; $\Delta = \{(x, x) \mid x \in \mathbb{R}^N\}$. Denote by δ the mapping

$$\mathbb{R}^{2N} \setminus \Delta \to \mathbb{R}^N \setminus \{0\}, \qquad (x, y) \to x - y,$$

and by π the mapping

$$\mathbb{R}^N \setminus \{0\} \to S^{N-1}, \qquad x \to x/|x|.$$

Obviously, the mapping $\pi \circ \delta$ is smooth and $\Sigma = \pi \circ \delta(M \times M \setminus \Delta)$.

The following assertions are obvious for a compact manifold and complete the proof of the "easy Whitney theorem".

Assertions. 1°. The dimension of the direct product of a compact d-dimensional manifold M onto itself is equal to $2d$.

2°. The image of a smooth (not necessarily compact) $2d$-dimensional manifold under a smooth mapping into a manifold S of dimension $2d$ has a thick complement in S.

5.3. Completion of the proof of Theorem 2′

It remains to prove the following analogues of Assertions 1° and 2° in the "entropic situation".

Analogues. 1°. *Let M be a compact subset of $\mathbb{R}^N$, with* $\dim_E M = d$. *Then*

$$\dim_E M \times M \leq 2d.$$

2°. *Suppose that K is a closed bounded subset of the space* $\mathbb{R}^N \setminus \Delta$ (K may be nonclosed in $\mathbb{R}^N$!). *Let S be a smooth $(n-1)$-dimensional manifold, and* $\varphi\colon \mathbb{R}^{2N} \setminus \Delta \to S$ *a smooth mapping. Then, if* $\dim_E K < N - 1$, *the complement* $S \setminus \varphi K$ *is thick in S.*

Proof of Assertion 1°. The definition of entropic dimension is well suited to the upper estimation of the dimension of the Cartesian square. Indeed, suppose that $k > d$ is any number. Then

$$\mathrm{Em}_k(M) = 0;$$

accordingly, for any $\varepsilon > 0$ and $\sigma > 0$ there exists a covering U of the set M by balls of the same radius $R \leq \varepsilon$, the k-dimensional volumes of which is less than σ. Suppose that the covering U consists of $\mathfrak{N}$ balls with centers $x^1, \ldots, x^{\mathfrak{N}}$. Then, first,

$$R^k \mathfrak{N} < \sigma.$$

Second, the collection of balls with centers (x^i, x^j), $i, j \in \{1, \ldots, \mathfrak{N}\}$, of radius $\sqrt{2}R$ in the space $\mathbb{R}^{2N}$ forms a covering of the set $M \times M$. We denote this covering by $\tilde{U}$, and estimate its $2k$-dimensional volume:

$$V_{2k}(\tilde{U}) = 2^d R^{2d} \mathfrak{N}^2 \leq 2^d \sigma^2.$$

Since ε and σ are arbitrary, it therefore follows that

$$\mathrm{Em}_{2k}(M \times M) = 0,$$

with which Assertion 1° is proved.

Proof of Assertion 2°. Suppose that K_s is a sequence of compact sets exhausting K. One could for instance take

$$K_s = (x, y) \in K \mid |x - y| \geq 1/s.$$

Put $\Sigma_s = \varphi K_s$. It follows easily from the definition of entropic measure that this dimension does not increase under smooth mappings. The set Σ_s is closed, as the continuous image of a compact set, and, by what was said above, $\dim_E \Sigma_s \leq N - 1$. Accordingly, the complement $\Delta_s = S \backslash \Sigma_s$ is open and everywhere dense in S (otherwise the set Σ_s would contain a subregion of the manifold S, and its entropic dimension would be equal to $N - 1$). That means that the intersection

$$\bigcap_{s=1}^{\infty} \Lambda_s = S \backslash \varphi K$$

is a thick set, as we were required to prove.

5.4. Theorem on inclusion in the graph

Suppose that K is a compact set of entropic dimension d, L a plane in general position relative to K, $\dim L > 2d$, $L^\perp$ the orthogonal complement to L. Then there exists a continuous mapping $L \to L^\perp$ whose graph contains K.

Proof. Let $\pi_0 \colon K \to L$ be the projection along $L^\perp$, $K' = \pi_0 K$. The restriction $\pi_0|_K \colon K \to K'$ is a homeomorphism (that is the reason for the requirement that L be in general position). Therefore the continuous mapping $s \colon K' \to K$ inverse to π_0 is defined. Suppose $\pi^\perp$ is the projection $K \to L^\perp$ along L. The mapping $\pi^\perp \circ s$ is continuous. The choice of a basis in $L^\perp$ transforms it into a continuous vector-function. By Urysohn's theorem, that vector-function may be continuously extended to a continuous vector-function on L, which yields the desired mapping $L \to L'$.

References

[1] D. B. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Amer. Math. Soc., Providence, RI, 1969. (Originally published in Trudy Math. Inst. Steklov. **90** (1967).) (MR **36** # 7157)

[2] V. I. Arnol'd, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier (Grenoble) **16**:1 (1966), 319–361. (MR **34** # 1956)

[3] V. I. Arnol'd, *Lectures on bifurcation in versal families*, Russian Math. Surveys **27**:5 (1972), 54–123. (Originally published in Uspekhi Mat. Nauk **27**:5 (1972), 119–184.) (MR **54** # 1310)

[4] V. I. Arnol'd, *Remarks on the behavior of flows of an ideal three-dimensional fluid under small perturbations of the initial field of velocities*, J. Appl. Math. Mech. **36**:2 (1972), 236–242. (Originally published in Prikl. Mat. Mekh. **36**:2 (1972), 255–262.)

[5] V. I. Arnol'd, Mathematical Methods of Classical Mechanics, Springer, Berlin, 1978. (Originally published by Nauka, Moscow, 1974.) (MR **57** # 14032)

[6] V. I. Arnol'd, *Some problems in differential equations*, in Unsolved Problems of Mechanics and Applied Mathematics (in Russian), Moscow State University, 1977.

[7] A. V. Babin and M. I. Vishik, *Attractors of Navier–Stokes systems and parabolic equations and estimates of their dimension* (in Russian), Zap. Nauchn. Sem. LOMI Steklov. **115** (1982), 3–15. (MR 84d:35016)

[8] V. I. Bakhtin, *Properties of entropic and Hausdorff measure* (in Russian), Vestnik Moscov. Univ. Ser. I, Mat. Mekh. **1982**:5, 25–29. (MR 84a:54066)

[9] A. A. Chestnova and A. N. Chetaev, *A circular model of the respiratory center* (in Russian), Biofizika **24**:4 (1979), 723–726.

[10] A. N. Chetaev, *Modelling the respiratory center. Organization of connections with minimal fluctuation period* (in Russian), Biofizika **24**:3 (1979), 528–532.

[11] A. Douady and J. Osterlé, *Dimension de Hausdorff des attracteurs*, C.R. Acad. Sci. Paris Sér. A-B **290**:24 (1980), A1135–A1138. (MR 82a:58033)

[12] C. Foias and R. Temam, *Some analytic and geometric properties of the evolution Navier–Stokes equations*, J. Math. Pures Appl. **58**:3 (1979), 339–368. (MR 81k:35130)

[13] P. Frederickson, J. L. Kaplan and J. A. Yorke, *The dimension of the strange attractor for a class of difference systems*, preprint, 1980.

[14] R. G. H. Helleman, *Self-generated chaotic behavior in nonlinear mechanics*, in Fundamental Problems in Statistical Mechanics (Enschede, 1980), 165–233, North-Holland, Amsterdam, 1980. (MR 83b:58033)

[15] M. Hirsch, Differential Topology, Springer, New York, 1976. (MR **56** # 6669)

[16] E. Hopf, *A mathematical example displaying features of turbulence*, Comm. Appl. Math. **1** (1948), 303–322. (MR **10**, 716)

[17] Yu. S. Il'yashenko and A. N. Chetaev, *On the dimension of attractors for a class of dissipative systems*, J. Appl. Math. Mech. **46**:3 (1982), 290–295. (Originally published in Prikl. Mat. Mekh. **46**:3 (1982), 374–381.) (MR 85i:58072)

[18] T. Kato, Perturbation Theory for Linear Operators, Springer, New York, 1966. (MR **34** # 3324)

[19] Y. Kuramoto and T. Yamada, *A reduced model showing chemical turbulence*, Prog. Theor. Phys. **56** (1976), 681–683.

[20] O. A. Ladyzhenskaya, Mathematical Questions of the Dynamics of Viscous Incompressible fluids (in Russian), Nauka, Moscow, 1970. (MR **42** # 6442)

[21] O. A. Ladyzhenskaya, *On a dynamical system generated by Navier–Stokes equations* (in Russian), Dokl. Akad. Nauk SSSR **205**:2 (1972), 318–320. (MR **46** # 4021)

[22] O. A. Ladyzhenskaya, *The dynamical system generated by the Navier–Stokes equations* (in Russian), Zap. Nauchn. Sem. LOMI Steklov. **27** (1972), 91–115. (MR **48** # 6720)

[23] O. A. Ladyzhenskaya, *Finite-dimensionality of bounded invariant sets for the Navier–Stokes system and for other dissipative systems* (in Russian), Zap. Nauchn. Sem. LOMI Steklov. **115** (1982), 137–155. (MR 84e:35020)

[24] L. D. Landau, *On the turbulence problem* (in Russian), *Dokl. Akad. Nauk SSSR* **44**:8 (1944), 339–342.

[25] A. M. Lukatskiĭ, *Curvature of a group of diffeomorphisms preserving the measure of a two-dimensional sphere*, Functional Anal. Appl. **13**:3 (1979), 174–177. (Originally published in Funktsional Anal. i Prilozhen. **13**: 3 (1979), 23–27.) (MR 80h:58017)

[26] V. S. L'vov and A. A. Predtechenskiĭ, *On Landau and stochastic attractor pictures in the problem of transition to turbulence*, Phys. D **2**:1 (1981), 38–51. (MR 82h:76027)

[27] J. Mallet-Paret, *Negatively invariant sets of compact maps and an extension of a theorem of Cartwright*, J. Differential Equations **22**:2 (1976), 331–348. (MR **54** # 11378)

[28] J. E. Marsden, M. McCracken and G. F. Oster, The Hopf Bifurcation and its Applications, Springer, New York, 1976. (MR **58** # 13209)

[29] L. S. Pontryagin and L. G. Shnirel'man, *On a metric property of dimension*, in W. Hurewicz and H. Wallman (eds.), Dimension Theory (Russian edition), W. Hurewicz and H. Wallman, Moscow, 1948.

[30] C. A. Rogers, Packing and Covering, Cambridge Univ. Press, 1964.

[31] D. Ruelle and P. Takens, *On the nature of turbulence*, Comm. Math. Phys. **20** (1971), 167–192. (MR **44** # 1297)

[32] Ya. G. Sinaĭ, *Classical dynamical systems with countably-multiple Lebesgue spectrum. II* (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. **30**:1 (1966), 15–68.

[32a] Ya. G. Sinaĭ, *The stochasticity of dynamical systems*, Selecta Math. Soviet. **1**:1 (1981), 100–119. (Originally published in Nonlinear Waves (Gor'kii, 1977), 119–212, Nauka, Moscow, 1979.) (MR 83e:58058)

[33] S. L. Sobolev, Some Applications of Functional Analysis in Mathematical Physics (in Russian), Izdat. Leningrad. Gos. Univ., Leningrad, 1950. (MR **14**, 565)

[34] H. Wegmann, *Die Hausdorff-Dimension von Kartesischen Produktmengen in metrischen Räumen*, J. Reine Angew. Math. **234** (1969), 163–171. (MR **40** # 1584)

[35] F. A. Zhuravel', Z. B. Kruglyak, E. A. Kuznetsov, L. S. L'vov, Yu. E. Nesterikhin, A. A. Predtechenskiĭ, V. S. Sobolev and E. N. Utkin, *Evolution of the spectrum in the passage to turbulence in Couette flows*, preprint 103, Inst. Automat. Electromech. Siberian Branch Acad. Sci. USSR, Novosibirsk, 1979.

[36]* A. V. Babin and M. I. Vishik, *Attractors of evolution partial differential equations and estimates of their dimension*, Russian Math. Surveys **38**:4 (1983). (Originally published in Uspekhi Mat. Nauk **38**:4 (1983), 133–187.) (MR 84k:58133)

[37]* Yu. S. Il'yashenko, *The dimension of attractors of k-contracting systems in an infinite-dimensional space*, Moscow Univ. Math. Bull. **38**:3 (1983), 61–69. (Originally published in Vestnik. Moskov. Univ. Ser. I Mat. Mekh. **1983**:3, 52–59.) (MR 84m:58080)

* Added to the translation.