Global bifurcations in generic one-parameter families with a separatrix loop on S^2

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Abstract

Global bifurcations in the generic one-parameter families that unfold a vector field with a separatrix loop on the two-sphere are described. The sequence of bifurcation that occurs is in a sense in ono-to-one correspondence with finite sets on a circle having some additional structure on them. Families under study appear to be structurally stable. The main tool is the Leontovich-Mayer-Fedorov (LMF) graph, analog of the separatrix sceleton - an invariant of the orbital topological classification of the vector fields on the two-sphere. Its properties and applications are described.

Key words: bifurcations, separatrix loops, sparkling saddle connections
Mathematics subject classification: 34C23, 37G99, 37E35

1 Introduction

Classification of various objects is the main topic of the Catastroph Theory. Various classes of singularities of maps, vector fields and bifurcations are classified up to now. Surprisingly, global bifurcations in the two sphere are not yet classified at all. In 1985 Arnold suggested that for any number of parameters there is in a sense only a finite number of different classes of weak topological equivalence of generic local families of vector fields on the two-sphere. This is not the fact even for one-parameter families. Classification of such families that unfold a vector field with a separatrix loop is the main contents of this paper.

Local and non-local one-parameter bifurcations in the plane seem to be well known. The following degeneracies may be met in an unavoidable way in typical one-parameter families:

AH: a non-hyperbolic singular point with a pair of non-zero pure imaginary eigenvalues;

SN: a saddle-node singular point;

SC: a saddle connection between two different hyperbolic saddles;

HC: a homoclinic curve of a saddle-node;

SL: a separatrix loop of a hyperbolic saddle;

PC: a parabolic cycle, that is, a non-hyperbolic limit cycle.

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For future references, this list of degeneracies will be called *a basic list.* AH stays for Andronov – Hopf. Other abbreviations correspond to the first two words in the description of a degeneracy.

The bifurcations in these classes are described in many sources [**?**] and references therein. In fact, local bifurcations are described near non-hyperbolic singular points (classes AH and SN), and semi-local ones, that is, happening in a neighborhood of a homoclinic curve or a limit cycle are described for the last four classes. No interesting bifurcation happens in these neighborhoods for the classes two, six and three. In the first two classes the non-hyperbolic singular point or cycle either splits in two on one side of the critical parameter value, or vanishes on the other side. The saddle connection simply brakes. In the first, fourth and fifth classes a limit cycle is generated under the bifurcation.

There is a prejudice that the bifurcations in the typical one-parameter families are thus completely described. Yet the discovery of sparkling saddle connections by Malta-Palis [**?**] shows that this is not the fact.

Let us give a heuristic description of bifurcations in the families of class SL. It may happen that inside the loop of the saddle L (that is, in the domain that is bounded by the loop and contains no other separatrix of the saddle L) there are several hyperbolic saddles whose separatrixes wind towards the loop, that is, have it as an α or ω limit set.

Definition 1. *Large bifurcation support of a vector field of class SL is the union of the saddle loop of this field, and the closures of all the separatrixes that wind to or from this loop.*

On a heuristic level, the large bifurcation support is a part of a phase portrait that actually bifurcates. This is a general concept whose study may be useful for the generic theory of global planar bifurcations. It is introduced in [**?**] and studied in [**?**], work in progress. We will show that all the bifurcations in the local families of class SL are completely determined by the bifurcations that occur in a germ of a neighborhood of the large bifurcation support of a vector field of class SL corresponding to the critical parameter value zero.

The term *bifurcation support* was introduced by Arnold [**?**] in 1985. Bifurcation support was addressed to be a set in whose neighborhood all the bifurcations take place. Unfortunately, the set defined by Arnold did not play this role. For the vector fields of class SL it is the separatrix loop only. Yet the bifurcations in an SL family take place near the large bifurcation support defined above. So we need another term for the set near which all the bifurcations take place.

Quite unexpectedly, the bifurcations near the large bifurcation support in an SL family are characterized by a finite set on a circle with a special equivalence relation for some pairs of points in this set. Any finite set with this equivalence relation may be realised for some local SL family. The rigorous formulation and proof of this statement is given in what follows.

2 Basic definitions

An open and dense set in $\mathrm{Vect}^k(S^n), k \geq 1$, is formed by vector fields that satisfy the following conditions:

- all the singular points and limit cycles are hyperbolic
- there are no saddle connections, that is, no mutual separatrixes of hyperbolic saddles.

Vector fields with these properties are usually called *Morse-Smale* or *generic*. Non-generic vector fields will be called *degenerate*.

Definition 2. *A vector field is called* quasigeneric *if it has exactly one degeneracy from the basic list, and satisfies the following genericity assumptions*

1. *all the singular points and limit cycles except for those mentioned in the basic list are hyperbolic;*
2. *there are no saddle connections except for those mentioned in the basic list;*
3. *for the non-hyperbolics singular point of the vector field of class AH, the third derivative at* 0 *of the corresponding normalized Poincaré map is non-zero;*
4. *the non-hyperbolic singular point of a vector field of the class SN or HC is of multiplicity two;*
5. *the homoclinic curve of a vector field of class HC enters the saddle-node singular point strictly inside the parabolic sector*
6. *the characteristic value of a saddle with a separatrix loop for a vector field of class SL is different from* 1 *(recall that a characteristic value of a hyperbolic saddle is a magnitude of the ratio of its eigenvalues, the negative one in the denominator);*
7. *for a vector field of class PC, the multiplicity of the non-hyperbolic limit cycle equals* 2.

Definition 3. *A quasigeneric vector field is of* class SL *if its separatrices of hyperbolic saddle forms a saddle-loop. The set of all these vector fields is defined by SL.*

In [**?**] it is proved that the set SL is an immersed Banach submanifold of $Vect^k(S^2)$ for any $k \geq 4$.

Definition 4. *A local one-parameter family $V = \{v_\varepsilon\}$ is* of class $\mathcal{SL}$ *if $v_0 \in SL$, and V is transversal to SL.*

Definition 5. *Two vector fields v and w on S^2 are called* orbitally topologically equivalent*, if there exists a homeomorphism $S^2 \to S^2$ that links the phase portraits of v and w, that is, sends orbits of v to orbits of w and preserves their time orientation.*

Definition 6. *Let B, B_0 be topological balls in $\mathbb{R}$. Two families of vector fields $\{v_\alpha, \alpha \in B\}$, $\{w_\beta, \beta \in B_0\}$ on $\mathbb{S}^2$ are called* weakly topologically equivalent *if there exists a map*

$$H\colon B \times \mathbb{S}^2 \to B_0 \times \mathbb{S}^2,\ H(\alpha, x) = (h(\alpha), H_\alpha(x))$$

such that h is a homeomorphism, and for each $\alpha \in B$ the map $H_\alpha : \mathbb{S}^2 \to \mathbb{S}^2$ is a homeomorphism that links the phase portraits of v_α and $w_{h(\alpha)}$.

Definition 7. *A family of vector fields is called* weakly structurally stable *if it is weakly topologically equivalent to any nearby family.*

In what follows, we deal with local families.

Definition 8. *A* local family *at $\alpha = 0$ with the base $(\mathbb{R}^1, 0)$ is a germ on $\{0\} \times S^2$ of a family given on $B \times M$, $B \ni 0$, $B \subset \mathbb{R}$ is open. Two local families are* weakly topologically equivalent *if they have locally weakly topologically equivalent representatives, and the corresponding homeomorphism of the bases maps* 0 *to* 0.

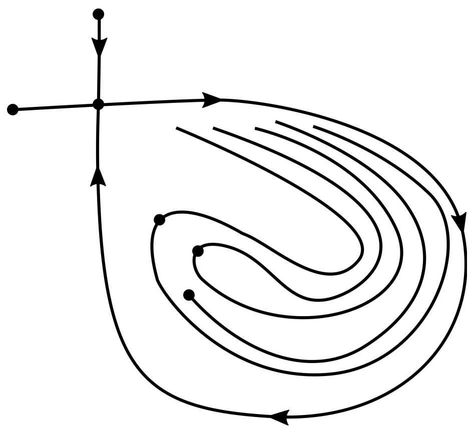

Figure 1: Complicated large bifurcation support

Definition 9. *An* unfolding *of a vector field is a local family for which this field corresponds to the critical (zero) parameter value. We say that this family* unfolds *the vector field.*

We will be mainly interested in generic unfoldings. In what follows local families (the germs) will be identified with their representatives whose base is so small that all the requirements below that are guaranteed by the smallness of the base hold true.

3 Main results

Theorem 1. *Generic one-parameter family of class SL is weakly structurally stable.*

Theorem 2. *Suppose that two quasigeneric vector fields of class SL are orbitally topologically equivalent. Then their generic unfoldings are weakly topologically equivalent.*

This theorem provides a necessary and sufficient condition for the equivalence of local $\mathcal{SL}$ families; it is called *the SL criterion.*

Large bifurcation supports defined above may have a complicated topological structure, see Figure **??**. They determine a sequence of bifurcations (the bifurcation scenario) that occur in a one-parameter family of class $\mathcal{SL}$.

This scenario, in turn may be characterized in a simple way by the so called *marked finite sets* on a circle.

Definition 10. *A* proper equivalence relation *on a finite set $A = (a_1, \dots, a_l)$ on an oriented circle is defined as follows. Equivalence classes are single points and pairs of points with the following restriction: if $a \sim b$, $c \sim d$ then the pairs (a, b), (c, d) are not intermingled on the oriented circle. This means that the arc from a to b either contains no points from the pair (c, d), or contains both. A finite set with this equivalence relation is called* marked.

Two marked sets on an oriented circle are equivalent *iff there exists an orientation-preserving homeomorphism of a circle that maps the first set into the second one and respects the order of points on the circle and the equivalence relation.*

We will prove in Section **??** that the bifurcation scenarios in the $\mathcal{SL}$-families are in one-to-one correspondence with the equivalence classes of the marked sets on a circle. The exact form of this correspondence is described below.

Theorem 3. *Any marked finite set on a circle may be realized as a set corresponding to a large bifurcation support of some vector field of class SL.*

4 LMF graphs and their applications

4.1 Generic and quasigeneric vector fields

Theorem 4. *(Andronov—Pontryagin) Generic vector fields on S^2 are structurally stable.*

Theorem 5 (see [?]). *Quasigeneric vector fields form an open and dense subset of the set Σ of structurally unstable vector fields. Quasigeneric vector fields are structurally stable inside the corresponding classes AH, SN, SC, HC, SL, PC.*

The latter statement means that for any quasigeneric vector field v there exists a neighborhood in the corresponding class such that any vector field from this neighborhood is orbitally topologically equivalent to v. A short proof of this statement is presented below.

Theorem **??** follows immediately from Theorems **??** and **??**. Indeed, let V be a local family of the class $\mathcal{SL}$, and W be a close one from the same class. Then the vector fields v_0 and w_0 are orbitally topologically equivalent by Theorem **??**. Then the families V and W are weakly equivalent by Theorem **??**.

4.2 Leontovich-Mayer-Fedorov graph for quasigeneric vector fields

A copmplete topological invariant of a vector field on a two sphere may be presented in three different ways. The first one is the scheme of the vector field obtained in [?]. The second one is the separatrix sceleton, see [?] and references therein. We define it in the next subsection just for the survey purposes; it will not be used below. The third one is a so called LMF graph, see [?] and [?]. The general definition of all these objects is rather complicated. We consider here only the case of quasigeneric vector fields.

4.2.1 Separatrix sceletons

Quasigeneric vector fields have only hyperbolic singular points, except probably for one; this one point is a slow focus or a saddle-node of multiplicity two. These vector fields have only hyperbolic limit cycles except probably for one, which is a parabolic limit cycle of multiplicity two. Separatrices of such vector fields are separatrices of hyperbolic saddles, or the phase curves that include the boundaries of the hyperbolic sectors of the saddle-node.

Definition 11. *An* extended separatrix sceleton *of a quasigeneric vector field is the union of all its singular points, limit cycles and separatrices.*

Definition 12. *A* completed separatrix sceleton *of a quasigeneric vector field is the union of its extended separatrix sceleton together with one orbit in each connected component of the complement to the sceleton.*

Theorem 6. *[see [?] Section 1.9, and references therein] Two quasigeneric vector fields are orbitally topologically equivalent iff their completed separatrix sceletons are isotopic on the two-sphere.*

In fact, a much stronger theorem is stated in [?].

A separatrix sceleton is not a graph: separatrixes that wind towards a limit cycle or a separatrix loop have infinite length. In what follows we will use graphs as invariants. This will allow us to apply the graph theory; these applications are crucial for what follows.

4.2.2 Transversal loops and modified separatrix sceletons

Consider a quasigeneric vector field v. Choose an orientation on the sphere. For any attracting or repelling hyperbolic singular point of v there exists a curve without contact (called in future transversal loop) that surrounds this point; this curve is C^1 smooth and splits the sphere in two parts. Let us call *interior* the part that contains the singular point mentioned above. The transversal loop may be so chosen that the phase portrait of v inside this curve is topologically equivalent to $\dot{x} = -x$ for an attractor or to $\dot{x} = x$ for the repeller. Let us fix such curves for all the attracting and repelling singular points of v. Let us orient the transversal loop as a boundary of the domain that contains the singular point; namely, this point lies to the left of the oriented loop.

A similar construction for any limit cycle of v provides two closed oriented transversal loops, one on each side of the cycle, such that that the vector field v in the closed annulus bounded by the cycle and the curve is orbitally topologically equivalent to one of the samples: $\dot{\varphi} = 1, \dot{r} = \pm(1 - r)$; $\dot{\varphi} = -1, \dot{r} = \pm(1 - r)$; the samples are considered in an annulus $r \in [1, 3/2]$ or $r \in [1, 1/2]$. Let us orient these transversal loops as components of the boundary of the domain that contains the limit cycle.

If v is of the class SL, then v has a separatrix loop of a hyperbolic saddle. Denote the saddle by L, and the loop by γ. Define the interior domain of γ to be a domain bounded by γ and containing no separatrixces of L. There exists a closed curve C without contact oriented as the boundary of the domain that contains L and such that v in the annulus between C (included) and γ (excluded) is topologically equivalent to one of the fields $\dot{\varphi} = 1, \dot{r} = \pm(1 - r)$, considered in an annulus $r \in (1, 1/2]$. From now on, we suppose that the infinity is outside the loop γ. Hence, the transversal loop is oriented clockwise.

Any outgoing separatrix of a quasigeneric vector field has a hyperbolic saddle or a saddlenode as an α-limit set. A separatrix loop of a vector field of class SL has a saddle as an ω-limit set too. A saddle connection of a vector field of class SC has another saddle as an ω-limit set. All the other separatrixes of a quasigeneric vector field have a hyperbolic attracting singular point or a limit cycle (not necessary hyperbolic in the class PC), or a loop γ (for the fields of class SL) as an ω-limit sets. Therefore, any such separatrix intersects the curve without contact constructed for its ω-limit set. The closure of the arc of the outgoing separatrix between its α-limit set and the intersection point with the transversal loop defined above is called the *truncated separatrix*, and its non-singular vertex is called the *truncation vertex*. In the same way truncated ingoing separatrices and their truncation verteces are defined; only α and ω-limit sets are exchanged.

Definition 13. *The modified separatrix sceleton of a quasigeneric vector field is the union of all the singular points, time oriented limit cycles and truncated separatrices, together with all the oriented transversal loops constructed above.*

There are two points of view on the spherical graphs constructed above: a set theoretical and combinatorial ones. From the first point, the graph is a subset of the two sphere. From the second one, it is a finite set of labeled verteces and labeled oriented edges that connect some of the verteces. As a subset of the sphere, the LMF graph constructed below belongs to the truncated separatrix sceleton.

4.2.3 Vertexes of the LMF graph

Vertexes of an LMF graph of a quasigeneric vector field are:

1. Singular points of the vector field

2. Truncation verteces

3. An arbitrary chosen one point of any limit cycle

4. An arbitrary chosen one point on any transversal loop, in case when this curve contains no truncation verteces; in what follows such a curve will be called *empty.*

4.2.4 Edges of the LMF graph

Edges of an LMF graph of a quasigeneric vector field are:

1. Time oriented saddle connections whose verteces are the saddles that they connect

2. Time oriented truncated separatrixes with the corresponding singular point and truncation vertex as verteces

3. Time oriented limit cycles with the unique chosen vertex

4. Arcs of the transversal loops with the orientation inherited from these curves that connect two subsequent truncation points

5. Oriented empty transversal loops with one chosen vertex.

4.2.5 Labels

Each vertex which is a singular point is labeled by its type: attractor, repeller, saddle or a saddle-node. On other verteces, we put labels showing if they are points on the cycles or points on the non-contact curves. Each edge is labeled depending on whether it is an ingoing or outgoing separatrix or both, a limit cycle, or an arc of a non-contact curve (that may coincide with the whole curve). For the arcs of transversal loops we put labels showing whether this arc is absorbing (the ω-limit set of its points is the limit set corresponding to this transversal loop) or outgoing (absorbing in the negative time).

This completes the constrsuction of the labeled oriented LMF graph for a quasigeneric vector field.

4.3 Leontovich-Mayer-Fedorov graph as an invariant

Definition 14. *Images of two embeddings of one abstract oriented labeled graph into a sphere S^2 are called* isotopic*, if there exists an orientation-preserving homeomorphism of the sphere, that maps the image of the first embedding to the image of the second one and preserves the orientation of the edges and the labels.*

We use the following result of R. Fedorov [**?**], based on the previous result of Andronov, Leontovich, Gordon, Mayer [**?**].

Theorem 7. *[see [**?**] and references therein] Two quasigeneric vector fields are orbitally topologically equivalent iff their labeled oriented LMF graphs are isotopic on the two-sphere. This means that they are the images of two isotopic embeddings of one and the same abstract oriented labeled graph.*

In [**?**], [**?**] an analogous theorem is stated in a more general setting. Our definition of the LMF graph slightly differs from the one given in [**?**], but the two kinds of graphs may be easily expressed through one another, and we skip the details.

4.4 Isotopy of spherical graphs

The theorem above allows us to reduce the problem of equivalence of vector fields to that of the embeddings of the graphs. A necessary and sufficient condition of the latter equivalence is very simple.

Theorem 8 ([**?**], [**?**])**.** *Images of two embeddings of the same connected abstract graph on a sphere S^2 are isotopic iff corresponding isomorphism of images preserves a counterclockwise order of edges at each vertex.*

If the isomorphism of oriented graphs preserves a counterclockwise order of edges at each vertex, we will say that it satisfies the *star condition* because this property is related to the star of edges at any vertex.

Definition 15. *Faces of spherical graphs are connected components of the complement to the graph.*

LMF graphs are usually not connected; their faces may be either topological discs, or annuli, see Lemma **??** below. We will use the following theorem.

Theorem 9. *Suppose that two oriented planar graphs Γ_1, Γ_2 (not necessarily connected) are embedded in S^2 by maps $\varphi_1 \colon \Gamma_1 \to S^2, \varphi_2 \colon \Gamma_2 \to S^2$, and the (open) faces of the graphs $\varphi_j \Gamma_j,\ j = 1, 2$ in S^2 are topological discs or annuli. Choose an orientation in S^2. Suppose that these graphs are isomorphic as oriented graphs, and the isomorphism, denote it by g, satisfies the star condition. Suppose that the map $\varphi_2 \circ g \circ \varphi_1^{-1}$ extends to an orientation-preserving homeomorphism of all the annuli-shaped faces of the graph Γ_1. Then the map $\varphi_2 \circ g \circ \varphi_1^{-1}$ can be extended to the orientation-preserving homeomorphism of S^2, so $\varphi_1(\Gamma_1)$ is isotopic to $\varphi_2(\Gamma_2)$.*

The idea of the proof of this theorem is to add edges through all annuli-shaped faces of our graph, so that the extended graph is connected, and then use Theorem **??**. A detailed (still easy) proof may be found in [**?**]. The latter condition of the theorem is called *the annuli faces condition.*

4.5 Applications

These results imply a very simple proof of the classical structural stability theorems for generic and quasigeneric vector fields. A sketch of this proof is presented below; it is based on the following lemma.

Lemma 1. *Consider a family of C^1 embeddings f_ε of the same finite graph M in the sphere. Suppose that the graphs $f_\varepsilon(M)$ have the discs and annuli shaped faces only. Let f_ε be the continuous in $\varepsilon \in [0, 1]$. Then the graphs $M_\varepsilon = f_\varepsilon(M)$, $\varepsilon \in [0, 1]$, are pairwise isotopic.*

Proof. Both star and annuli faces condition hold under the homotopy. □

4.5.1 Structural stability of Morse-Smale vector fields on S^2

Corrolary 1. *A Morse-Smale vector field on S^2 is structurally stable.*

Proof. We will prove that for any Morse-Smale vector field v there exists a neighborhood U of v in $\mathrm{Vect}^1(S^2)$ such that the LMF graphs of the fields from U are isotopic to that of v. For this let us take U as a ball:

$$||v - w||_{C^1(S^2)} \leq r;$$

r is so small that the following holds:

- for any singular point P of v there exists a C^1-map $\mathcal{P} : U \to S^2,\ w \mapsto \mathcal{P}(w)$, such that $\mathcal{P}(w)$ is a hyperbolic singular point of w; let $\lambda_1(w), \lambda_2(w)$ be the eigenvalues of $\mathcal{P}(w)$;
- for all $w \in U$, the points $\mathcal{P}(w)$ are hyperbolic singular points of the vector field w, and the functions $\mathrm{Re}\ \lambda_1, \mathrm{Re}\ \lambda_2$ keep sign on U;
- for any limit cycle c of v (which is hyperbolic because v is Morse - Smale) there exists a C^1 map $\Gamma : U \to S^2$ such that the orbit of the point $\Gamma(w)$ under the phase flow of the vector field w is a hyperbolic limit cycle c_w; $c_v = c$;
- for any ingoing separatrix s of a hyperbolic saddle S of v there exists a continuous family of separatrixes $s_w,\ w \in U$ such that $s_v = s$, and the α-limit set of s_w depends continuously on w. This means the following. The α-limit set of s under v may be either a repeller, denote it by P, or a limit cycle, denote it by c. The α-limit set of s_w under w is $\mathcal{P}(w)$ in the first case, and c_w in the second one. A similar statement holds for outgoing separatrixes.

Such a ball exists by the infinite dimensional implicit function theorem. Let us prove that this r may be taken so small that all the singular points and cycles of the vector fields $w \in U$ are generated by the singular points and limit cycles of the vector field v as described above. By contraposition, let w_n be a sequence of vector fields such that w_n has a singular point $P_n \neq \mathcal{P}(P)$ for any $P \in \mathrm{Sing}\ v$. By compactness of the sphere, there exists a convergent subsequence $P_{n_k} \to P$. Then, $P \in \mathrm{Sing}\ v$. Hence, $P_{n_k} = \mathcal{P}(w_{n_k})$, because the only singular points close to P of the vector fields w close to v are of the form $\mathcal{P}(w)$, a contradiction. In a similar way, let w_n be a sequence of vector fields such that w_n has a limit cycle $c_n \neq c_{w_n}$ for any limit cycle c of the vector field v. The sequence c_n has a subsequence c_{n_k} that converges in a sense of the Hausdorff distance to a limit cycle c_0 or to a polycycle γ of the vector field v. As v is Morse–Smale, it has no polycycles, hence the latter case is impossible. For the same reason, the limit cycle c_0 of the vector field v is hyperbolic. Hence, c_{n_k} emerges from c_0 as described above, a contradiction. As a byproduct, we proved that the set of Morse-Smale vector fields on the sphere is open.

Let us now check that the LMF graphs of the vector fields $w \in U$ may be chosen to be continuous in w. Take all the attracting and repelling hyperbolic fixed points of v. Let us surround them by transversal loops. These loops remain to be transversal for all the vector fields C^1-close to v. The same holds for the hyperbolic limit cycles of v and their transversal loops.

The hyperbolic saddles and their truncated separatrices depend continuously on the vector field. Hence, the LMF graphs of the vector fields C^1-close to v may be chosen to depend continuously on the field.

For any w close to v consider now a one-parameter family $V_w = \{w_\varepsilon | \varepsilon \in [0,1]\}$:

$$w_\varepsilon = v + \varepsilon(w - v),\ w_\varepsilon \in U.$$

The LMF-graph of w_ε depends continuously on ε by the choice of U. Hence, by Lemma 1, the LMF-graphs of w_ε are pairwise isotopic for all $\varepsilon \in [0,1]$. Therefore, w_ε is orbitally topologically equivalent to v. Hence, v is structurally stable. □

4.5.2 Structural stability of vector fields inside the class Σ

Corrolary 2. *A quasigeneric vector field on S^2 is structurally stable inside the corresponding class from the basic list.*

Proof. The proof is similar to that for the Morse-Smale vector fields. Let v be a quasigeneric vector field. For any hyperbolic singular point P and limit cycle c of v, let $\mathcal{P}(w)$ and c_w be the same as above.

Non-hyperbolic elements of the phase portrait of v should be considered case by case.

AH: the vector field v has a singular point Q with a pair of purely imaginary eigenvalues; the corresponding Poincaré map has a non-zero third derivative. If it is negative (positive) the point Q is attracting (repelling). For all quasigeneric vector fields w, C^3-close to v, with an AH point $Q(w)$, consider the Poincaré map corresponding to $Q(w)$. The first derivative of this map at zero is 1 because the vector field is of class AH, the third derivative keeps sign by continuity, and the same non-contact curve C may be taken around $Q(w)$ for all the vector fields w.

SN: the vector field v has a saddle-node singular point P of multiplicity 2. All the C^2-close quasigeneric vector fields w have saddle-node singular points $\mathcal{P}(w)$ with the same property: $\mathcal{P}(w)$ depends continuously on w, and $\mathcal{P}(0) = P$. For all these vector fields the point $\mathcal{P}(w)$ has a separatrix of two hyperbolic sectors. If it is outgoing, then it has an ω-limit set, which is an attracting singular point $Q(w)$, or an attracting limit cycle γ_w, both continuous in w. The case of ingoing separatrix is treated similarly.

SC: the vector field v has a saddle connection between two hyperbolic saddles, say, P and Q; the connection depends continuously on w in the C^1-topology on the space of the vector fields.

HC: the vector field v has a saddle-node singular point P, and an outgoing (or ingoing) separatrix l of P that turns back to the parabolic sector of P in the positive (respectively, negative) time. All the C^1-close quasigeneric vector fields have a singular point $\mathcal{P}(w)$ of the same type, $\mathcal{P}(v) = P$, and a separatrix of two hyperbolic sectors of $\mathcal{P}(w)$ that turns back to $\mathcal{P}(w)$ through its parabolic sector, and depends continuously on w.

SL: the vector field v has a hyperbolic saddle P with a separatrix loop γ, and the characteristic number different from 1. All the C^1-close quasigeneric vector fields have a hyperbolic saddle $\mathcal{P}(w)$ with a saddle loop γ_w and the characteristic number different from 1.

PC: the vector field v has a parabolic-limit cycle c of multiplicity 2. All the quasigeneric vector fields w C^2-close to v have a parabolic limit cycle c_w of multiplicity 2 that depends smoothly on w.

After these arguments we see that the LMF graphs of quasigeneric vector fields of the same class from the basic list close to fixed one depend continuously and even smoothly on the vector field. In [S] it is proved that the set of degenerate vector fields of the same class near a quasigeneric one form an immersed Banach submanifold. Hence, by a local diffeomorpism of the ambient space $\mathrm{Vect}^3(S^2)$, it may be transformed to a piece of a hyperplane. A linear homotopy in this hyperplane allows us to apply Lemma **??** and conclude the proof of the Corollary; Lemma **??** is applicable by Lemma **??** proved below. □

4.6 Faces of the LMF graphs

For the proof of Theorem **??**, we will need a description of the faces of the LMF graphs.

Lemma 2. *Faces of an Leontovich-Mayer-Fedorov graph of a generic or quasigeneric vector field are topological discs or annuli. The annular faces may be of the following types:*

1. *an annulus between a transversal loop and the corresponding limit cycle;*

2. *a punctured disc between a transversal loop and the corresponding attracting or repelling singular point;*

3. *an annulus between two "empty" transversal loops;*

4. *an annulus between a separatrix loop of a hyperbolic saddle and the corresponding transversal loop.*

Remark 1. *Case 4 never occurs for generic vector fields.*

Proof. Consider a face Ω of the Leontovich-Mayer-Fedorov graph of a quasigeneric vector field v, which is not of type $1, 2$ or 4 from **??**. Let p be an interior point of Ω, $\gamma(p)$ its orbit, $\alpha(p)$ and $\omega(p)$ its ω-limit sets. Any trajectory of a quasigeneric vector field has α and ω limit sets of the following four types only:

- a limit cycle,
- a sink or a source,
- a saddle,
- a separatrix loop of a hyperbolic saddle.

Neither of the sets $\alpha(p)$ and $\omega(p)$ may be a saddle, or else $\gamma(p)$ is a separatrix and belongs to the Leontovich-Mayer-Fedorov graph. Then p is a boundary point, and not the interior point of the face, a contradiction. If the set $\alpha(p)$ is a repelling singular point, or a hyperbolic limit cycle, or a separatrix loop, and the negative orbit of p does not intersect the corresponding transversal loop, then the face that contains p is of type 1, 2 or 4. This contradicts to the assumption. Hence, in this case the negative orbit of p intersects the transversal loop corresponding to $\alpha(p)$. Similarly, if $\omega(p)$ is an attracting singular point, or a hyperbolic limit cycle, or a separatrix loop, then the positive orbit of p intersects the transversal loop corresponding to $\omega(p)$.

We conclude that if the face Ω is not of the type $1, 2$ or 4 from the proposition, then the curve $\gamma(p)$ intersects the non-contact curves C^- and C^+ that surround $\alpha(p)$ and $\omega(p)$. Indeed, for every hyperbolic attractor, a point or a cycle, every orbit that tends to this attractor intersects exactly once the transversal loop corresponding to this attractor.

Let $p_\alpha = C^- \bigcap \gamma(p)$, $p_\omega = C^+ \bigcap \gamma(p)$. Then a germ of the Poincaré map is defined:

$$P\colon (C^-, p_\alpha) \to (C^+, p_\omega).$$

There are two cases:

1. the germ P may be extended to the whole curve C^-;

2. the germ P may not be extended to the whole curve C^-, see Figure **??**.

Consider Case 1. In this case $P(C^-) = C^+$, and the vector field v in the closed annulus between the loops C^-, C^+ is orbitally topologically equivalent to $\dot{r} = 1$, $\dot{\varphi} = 0$ in $\{r \in [1, 2]\}$. The annulus between these loops is a separate face of the LMF graph of v. Indeed, no separatrix enters this annulus from outside: the faces adjacent to the curves C^-, C^+ contain in their boundary the element to which the transversal loop corresponds. This element may be a singular point, a cycle, or a separtrix loop, if v is of class SL.

So in the first case Ω is of type 3.

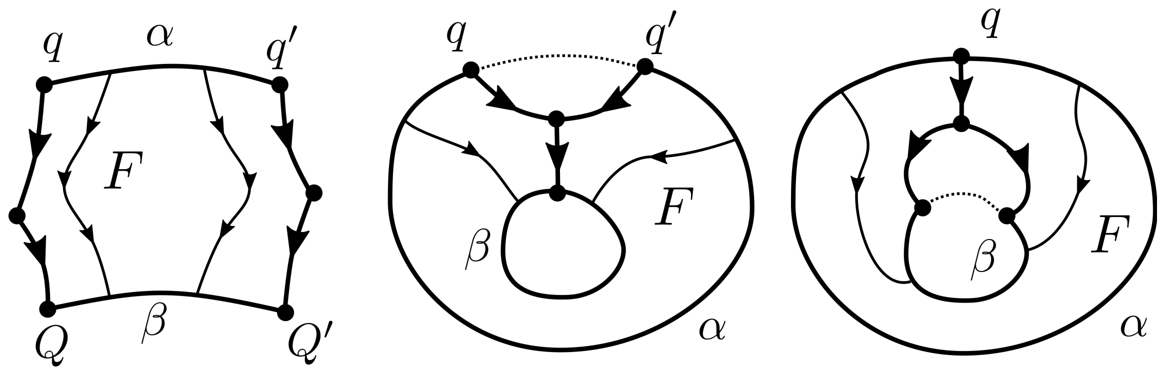

Figure 2: Faces of the LMF graph

Consider now Case 2. Suppose that P may be extended from a neighborhood of p_α to a proper subarc α of C^-. By the continuous dependence of the orbits on initial conditions, the arc α is open. Let q be its boundary point. Then the orbit $\gamma(q)$ of q does not reach C^+. Then its ω-limit set belongs to an annulus bounded by C^- and C^+. It may not be a hyperbolic sink or a limit cycle, because the basins of attraction of these sets is open, and the orbits from a whole neighborhood of q would not reach C^+. But $q \in \partial\alpha$, a contradiction. So, $\omega(q)$ is a saddle.

The arc α may have one endpoint, say q, or two endpoints, say q and q', see Figure **??**. In the first case, $\alpha = C^- \setminus q$. Consider the second case. Take a sequence $p_n \in \alpha$, $p_n \to q$. Let γ_n be the arcs of the phase curves of v between p_n and $P(p_n) \in C^+$. Then there exists an arc $\gamma = \lim \gamma_n$ in sense of the Hausdorff distance; γ is a union of the (arcs of) separatrices. For quasigeneric vector fields γ contains no more than one saddle connection, hence no more than two saddles. Denote the truncation vertex $\gamma \cap C^+$ by Q.

In the same way take a sequence $p'_n \in \alpha$, $p_n \to q'$, and construct an analogous curve γ' combined by arcs of separatrices of v with the endpoints $q' \in C^-$, $Q' \in C^+$. Let β be the arc of C^+ between Q and Q' such that $P(\alpha) = \beta$. Then the face Ω of the LMF graph that contains p is a topological disc bounded by the union

$$\partial\Omega = \alpha \cup \gamma \cup \beta \cup \gamma';$$

we neglect the orientation.

The case of the unique endpoint q is treated in the same way, only the sequence p_n converges to q from one side, and the sequence p'_n from another one. $\square$

5 Sparkling saddle connections in $\mathcal{SL}$ families

5.1 Marked finite sets

Consider a vector field v of class SL. Let L be a saddle of v with the separatrix loop γ. The interior domain of γ is the one that contains no other separatrix of L. By assumption, the characteristic number λ of L is different from 1. Without loss of generality we may assume that $\lambda < 1$; otherwise we reverse the time. Let C be a transversal loop close to γ and oriented as a component of the boundary of the topological annulus bounded by C and γ, hence, clockwise.

As $\lambda < 1$, the separatrix loop repels the orbits: nearby phase curves of v inside γ wind to γ in the negative time. All the saddles located inside γ are hyperbolic and form a finite set. Consider all the separatrices of this saddle that wind to γ in the negative time. Each one intersects C at exactly one point. Let us enumerate these points in the order as they are located on the oriented curve C: $A = \{a_1, ..., a_K\}$. Say that two points in this set are equivalent iff they belong to two separatrices of the same saddle. A saddle whose separatrix passes through a_k is denoted by I_k, and the corresponding separatrix is denoted by l_k. If another separatrix of the same saddle passes through a point a_l, this saddle is also denoted by I_l:

$$I_k = I_l \Leftrightarrow a_k, a_l \text{ are equivalent.}$$

Thus the set A becomes marked. This completes the construction of the marked finite set that corresponds to a vector field of class SL.

Let w be any vector field C^1- close to v. A similar construction provides the following objects that play for w the same role as L, C, A, I_k play for v: a saddle $L(w)$ close to L; the same non-contact curve C, separatrixes $l_k(w)$ of the saddles $I_k(w)$, the marked set $A(w)$ of their intersections with C; when $w = v$, these objects coincide with L, C, A, I_k. The elements of the set $A(w)$ are ordered in the same way as the corresponding elements of A.

5.2 Parameter depending monodromy map of the loop γ

Let Γ be a cross section transversal to γ and oriented inside γ; $O = \gamma \cap \Gamma$. Let $a = \Gamma \cap C$; suppose that Γ is so chosen that a is located between a_K and a_1 on the oriented curve C. Let $V = \{v_\varepsilon\}$ be a one-parameter unfolding of v transversal to the Banach manifold SL. It is well known that for ε on one side of zero the vector field v_ε has a limit cycle close to γ. Suppose that this happens for $\varepsilon < 0$; otherwise we reverse the parameter. Let U_ε (respectively, S_ε) be the unstable, same as "outgoing" (respectively, stable, same as "ingoing") separatrix of $L(\varepsilon) := L(v_\varepsilon)$, continuously depending on ε and such that for $\varepsilon = 0$ they coincide with γ. Recall that Γ is oriented inside the loop γ; "up" and "down" on Γ is understood in sense of this orientation. In our assumptions, for $\varepsilon > 0$, U_ε crosses Γ below S_ε; in the opposite case, a limit cycle is generated from γ. Let $u(\varepsilon)$ be the first intersection point of $U_\varepsilon \cap \Gamma$. In a similar way $s_\varepsilon = S_\varepsilon \cap \Gamma$ is defined. Note that $u(0) = s(0) = O$. Let x be a coordinate on Γ that induces the same orientation on Γ as already chosen; $x_\varepsilon = x - x(s_\varepsilon)$. Let us call this ε a *separatrix splitting parameter.* For any small $\varepsilon > 0$ and small $x_0 > 0$ a parameter depending monodromy map of segments of Γ corresponding to the loop γ is well defined:

$$\Delta_\varepsilon : [s(\varepsilon), x_0] \to [u(\varepsilon), x_0'], \ x_0' = \Delta_\varepsilon(x_0).$$

Let $d_k(\varepsilon)$ be the x_ε-coordinate of the point of the countable intersection $l_k(\varepsilon) \cap \Gamma$, which is the last one on the time oriented separatrix $l_k(\varepsilon)$ before $a_k(\varepsilon)$. Let us change the parameter in such a way that $x_\varepsilon(u(\varepsilon)) = \varepsilon$. We will call this ε *the separatrix splitting parameter.* Then the equation

$$\Delta_\varepsilon^n(\varepsilon) = d_k(\varepsilon) \tag{1}$$

is equivalent to the following geometric statement: *the separatrix U_ε of the saddle $L(\varepsilon)$ after n winds hits the point $d_k(\varepsilon) \in l_k(\varepsilon)$ and thus forms a saddle connection between $L(\varepsilon)$ and $I_k(\varepsilon)$.* Equation (**??**) is called a *connection equation.*

The following properties of the monodromy map are proved in [**?**]:

For any smooth function d_k in $(\mathbb{R}, 0)$, any $k \in \{1, ..., K\}$ and any $n \in \mathbb{N}$, the solution ε_{kn} of equation (**??**) *exists. It is monotonically decreasing in n and tends to 0 as $n \to \infty$. It is also monotonically decreasing in $k \in \{1, ..., K\}$ for n fixed. Moreover, for any fixed n, the function $\Delta_\varepsilon^n(\varepsilon)$ monotonically decreases with ε small, and tends to 0 as $\varepsilon \to 0$. Moreover,*

$$D_\varepsilon \Delta_\varepsilon^n(\varepsilon) \to \infty \text{ as } \varepsilon \to 0. \tag{2}$$

5.3 The bifurcation scenario

Recall that $a = \Gamma \cap C$. Let $d(\varepsilon) = x_\varepsilon(a)$. Consider an equation

$$\Delta_\varepsilon^n(\varepsilon) = d(\varepsilon).$$

By the statement from [?] quoted above, this equation has a solution ε_n; $\varepsilon_n \to 0$ as $n \to \infty$. The solution ε_n corresponds to the value of ε for which the separatrix U_ε makes n winds along γ and hits the point $a = \Gamma \cap C$.

When ε decreases in $[\varepsilon_{n-1}, \varepsilon_n)$, the intersection point $d_k(\varepsilon)$ starts from $d = x_{\varepsilon_{n-1}}(a)$ and tends to $d' = x_{\varepsilon_n}(\Delta_{\varepsilon_n}^{-1}(a))$. Note that $d_k(\varepsilon_n) = x_{\varepsilon_n}(a)$. Denote the (unique!) intersection point of U_ε with C by $a(\varepsilon)$. When $d(\varepsilon)$ changes from d to d', the point $a(\varepsilon)$ makes one turn around C starting from a, and comes back to a. By the way, it meets the points $a_k(\varepsilon)$. As $D_\varepsilon a_k$ is bounded in a neighborhood of 0, and $D_\varepsilon a(\varepsilon) \to \infty$ by (**??**), we conclude that the equation

$$a(\varepsilon) = a_k(\varepsilon),\ \varepsilon \in [\varepsilon_{n-1}, \varepsilon_n),$$

has a unique solution ε_{kn} for any k and n large enough. This implies that when ε changes from ε_{n-1} to ε_n, the saddle connections

$$U_\varepsilon = l_k(\varepsilon),\ \varepsilon = \varepsilon_{kn},$$

occur. Moreover, on the interval $[\varepsilon_{n-1}, \varepsilon_n)$ the values ε_{kn} decrease monotonically as k increases.

This completes the description of the bifurcation scenario in an $\mathcal{SL}$ family for $\varepsilon > 0$.

5.4 Realization theorem

Here we give a sketch of the proof of Theorem **??**. It relies upon Lemma **??** below. The lemma is proved in [?], and we do not reproduce the proof here.

Consider a Morse–Smale vector field in a disc D with the boundary C transversal to v. Let A be the set of the intersection points of the separatrices of v with C; as above, two points of A are equivalent iff they belong to the separatrices of the same saddle. Thus A is a marked set. Let us call it *the characteristic set* of v.

Lemma 3. *Consider a marked set A on a circle C that is a boundary of a disc D. Then there exists a C^∞ Morse-Smale vector field v in D such that A is a characteristic set of v.*

Let us take now an arbitrary smooth quasigeneric vector field w of class SL; suppose that L is a saddle of w with a separatrix loop γ and with a characteristic number $\lambda < 1$. Let C be a corresponding transversal loop inside γ. Consider a disc D bounded by C and not containing γ. As $\lambda < 1$, the disc is absorbing. Let $D' = S^2 \setminus D \cup C$, and w_0 be the restriction of w to D'.

Let now A be the prescribed marked set. Let v_0 be a vector field in D provided by Lemma **??** having an absorbing disc D and the characteristic set A. Let us glue together the vector fields w_0 and v_0 to obtain one smooth vector field v on S^2. This field is of class SL and has a prescribed characteristic set A. This proves Theorem **??**.

6 Proof of the SL-criterium

Here we prove Theorem **??**.

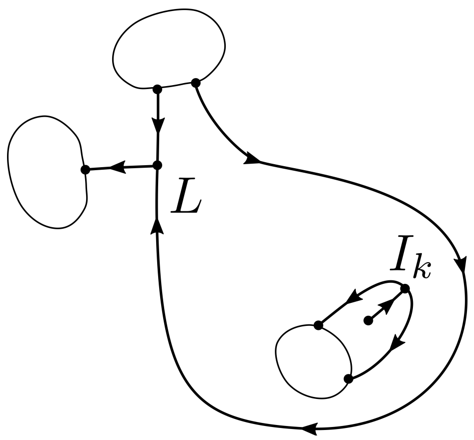

Figure 3: Part of the graph X_ε that contains saddles L and I_k

6.1 Plan of the proof

Consider two quasigeneric vector fields of class SL, v and w, and suppose that they are orbitally topologically equivalent. Let $\hat{H} : S^2 \to S^2$ be the corresponding homeomorphism.

Let V and W be two local one-parameter families that unfold v and w respectively:

$$V = \{v_\varepsilon | \varepsilon \in (\mathbb{R}, 0)\}, \; v_0 = v; \; W = \{w_\delta | \delta \in (\mathbb{R}, 0)\}, \; w_0 = w.$$

Let $M_\varepsilon (\tilde{M}_\delta)$ be the LMF-graph of v_ε (respectively, w_δ). We will find a germ of a homeomorphism $h : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that the LMF-graphs M_ε and $\tilde{M}_{h(\varepsilon)}$ will be isotopic on S^2. This will imply the weak equivalence of the families V and W.

6.2 The non-bifurcating subgraph

We will prove first weak equivalence of families V^+ and W^+ that correspond to $\varepsilon > 0$ and $\delta > 0$. Let us first study the family of graphs M_ε for the vector fields v_ε. We will determine a subgraph $X_\varepsilon \subset M_\varepsilon$ that depends continuously on ε and thus does not bifurcate at all.

Roughly speaking, X_ε is M_ε with all the truncated separatrices that eventually form sparkling saddle connections deleted.

Let us turn to a more detailed construction. Let $U(\varepsilon)$ be the unstable separatrix of the saddle $L(\varepsilon)$ whose germ at $L(\varepsilon)$ depends continuously on ε and belongs to the loop γ for $\varepsilon = 0$.

Let $\hat{U}(\varepsilon)$ be: the entire separatrix $U(\varepsilon)$ if the latter forms a saddle connection; the truncated separatrix $U(\varepsilon)$ elsewhere. Let $\hat{l}_k(\varepsilon)$ be the entire separatrix $l_k(\varepsilon) = U(\varepsilon)$ if the latter forms a saddle connection; the truncated separatrix $l_k(\varepsilon)$ elsewhere. For $\varepsilon > 0$, let:

$$X_\varepsilon = M_\varepsilon \setminus \hat{U}(\varepsilon) \setminus \cup_1^K \hat{l}_k(\varepsilon),$$

see Figure **??**.

Proposition 1. *The subgraph X_ε depends continuously on ε for $\varepsilon > 0$ small.*

Proof. We will treat separately different types of verteces and edges of the graph X_ε.

Singular points of the vector fields v_ε are all hyperbolic and persist under small perturbation. These are verteces of type 1. They depend continuously on ε. Limit cycles of the vector fields v_ε have the same property. They are edges of type 3, together with verteces of type 3. They depend continuously on ε.

Transversal loops are the same for the vector fields $v_\varepsilon,\ \varepsilon > 0$. Indeed, they correspond to hyperbolic objects that persist for all small ε. These loops may be chosen non-depending on ε. Empty transversal loops are edges of type 5 with the verteces of type 4 on them. They do not change with ε.

Arcs on the transversal loops and truncated separatrices of the hyperbolic saddles and their truncation verteces are the only elements of X_ε remained. Any separatrix of v_ε except for $U(\varepsilon)$ and $l_k(\varepsilon)$ has a hyperbolic limit set; another one is the corresponding saddle. The first limit set persists as ε changes; its transversal loop remains the same. The truncation vertex of the separatrix considered varies continuously on ε. The same holds for the arcs of the transversal loops between the subsequent truncation vertexes. These are edges of type 2 and 4, and verteces of type 2. They depend continuously on ε.

Edges of type 1 are not mentioned in the proof above. Indeed, they are saddle connections, and do not belong to X_ε.

This proves the proposition. $\square$

6.3 Parabolic arcs

Let us now describe the arcs on the transversal loops where the truncation verteces of M_ε occur. Let C be the transversal loop corresponding to the separatrix loop γ. A variety of singular points and limit cycles of the vector field v_0 may occur inside the loop γ; they are accompanied by corresponding transversal loops. Non-empty loops are divided to arcs by the truncation verteces of the separatrices of v_0. The saddles of v_0 are hyperbolic; they persist under small perturbations; the transversal loops persist as well. Hence, the truncation verteces of v_0 generate continuous families of such verteces of v_ε.

Definition 16. *An arc of a transversal loop of v_0 between two subsequent truncation verteces of v_0 is called* parabolic *for v_0 provided that there exists an orbit of v_0 that connects this arc with the transversal loop C.*

The endpoints of the parabolic arc for v_0 are truncations verteces of the separatrices of v_0; they generate two families of truncation verteces continuous in ε for the vector fields v_ε.

Definition 17. *A parabolic arc for v_ε is an arc whose endpoints are truncation verteces described above; it varies continuously with ε, and for $\varepsilon = 0$ is a parabolic arc of v_0 in sense of Definition* **??**.

Proposition 2. *For any vector field $v \in SL$, the truncation arcs on the transversal loop C corresponding to the separatrix loop γ are in one-to-one correspondence with the parabolic arcs located inside γ.*

Proof. Let a_k, a_{k+1} be the endpoints of the truncation arc on C; denote this arc by α_k. By definition of the point a_k, it belongs to a separatrix l_k of the saddle I_k. The same holds for $a_{k+1}, l_{k+1}, I_{k+1}$. Let us prove that any point $a \in \alpha_k$ different from the endpoints has the same ω-limit set. Indeed, the ω-limit set of a is not a saddle, or else a should belong to A which is not the case. Hence, $\omega(a)$ is a hyperbolic attracting singular point or limit cycle. This attractor has an open basin. Hence, the set of points of the arc α_k with the same ω-limit set is open. By the connectedness of the interval, all the points of α_k except for the endpoints have the same hyperbolic ω-limit set. Let C' be the transversal loop

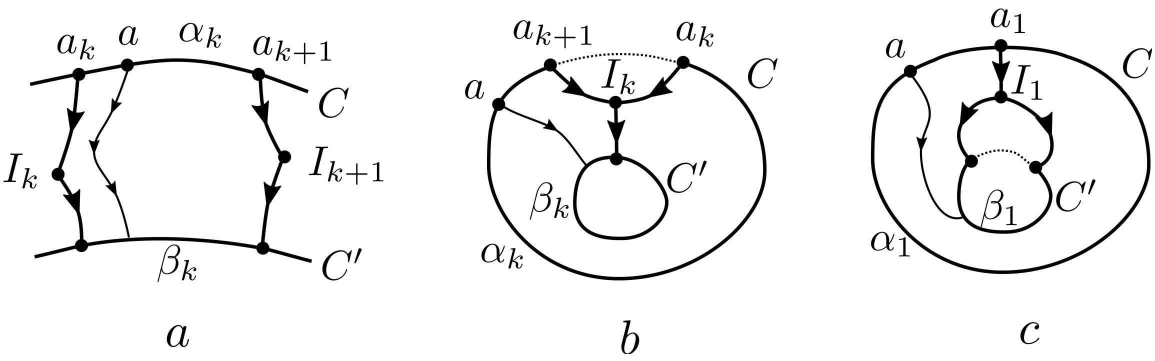

Figure 4: The Poincaré map on the parabolic arcs

corresponding to this limit set. Then any orbit emerging from int α_k (the arc α_k with the endpoints deleted) crosses this loop. Hence, the Poincaré map

$$P_k : \text{ int } \alpha_k \to C'$$

is well defined. Let

$$\beta_k = Cl \; P_k(\text{ int } \alpha_k). \tag{3}$$

Then the endpoints of β_k are the truncation verteces of certain unstable separatrices of I_k, I_{k+1} on C', see Figure **??**. The arc β_k is parabolic by definition.

The union of arcs α_k equals to C. Hence, all the parabolic arcs have the form (**??**). This proves the proposition. □

Remark 2. *The arc β_k may coincide with the whole curve C', see Figure* **??** *b. As well, the arc α_k may coincide with the whole curve C. In this case $K = 1, a_k = a_{k+1} = a_1, I_k = I_{k+1} = I_1$. Both these examples are particular cases in the proof of Proposition* **??**.

6.4 Equivalence of graphs X_ε

Let $\pi_\varepsilon : X_0 \to X_\varepsilon, \pi_0 = \text{id}$ be the map of the spherical graphs continuous in ε with the following two properties:

1. π_ε is a homeomorphism as a map of subsets on the sphere;
2. π_ε is an equivalence of graphs, that is, maps verteces to verteces and edges to edges, preserving the incidence, orientation and labels, see 6.9 for an explanation.

Let $\tilde{X}_0, \tilde{X}_\delta, \tilde{\pi}_\delta$ be the same objects for the family W. Let us modify the homeomorphism $\hat{H}$ so that it maps transversal loops of v to smooth transversal loops of w that belong to $\tilde{X}_0$, and respects the verteces of the graphs X_0 and $\tilde{X}_0$ chosen on the limit cycles and empty transversal loops. Then $\hat{H}$ maps X_0 to $\tilde{X}_0$, and has the same two properties as π_ε. Then the map:

$$G_{\varepsilon\delta} : X_\varepsilon \to \tilde{X}_\delta, \; G_{\varepsilon\delta} = \tilde{\pi}_\delta \circ \hat{H} \circ \pi_\varepsilon^{-1} \tag{4}$$

has the same two properties as π_ε for any small $\varepsilon > 0, \; \delta > 0$.

Remark 3. *Note that the map $G_{\varepsilon\delta}$ satisfies the star condition at any vertex of X_ε. Indeed, the map $\hat{H}$ meets this condition as a homeomorphism. The family π_ε is a homotopy of X_ε identical for $\varepsilon = 0$, same for $\tilde{\pi}_\delta$. Hence, $G_{\varepsilon\delta}$ meets the star condition too.*

6.5 Homeomorphism of bases

Let now $V = \{v_\varepsilon\}$ and $W = \{w_\delta\}$ be two local $\mathcal{SL}$ families from Theorem **??**. Let ε be the separatrix splitting parameter for the family V, and δ be the analogous parameter for W. We will define the map h of the bases of the families V and W for $\varepsilon > 0$, that maps bifurcation diagram of V to that of W. This map is far from being unique.

Let $v = v_0,\ w = w_0$ be vector fields of class SL, and $\hat{H} : S^2 \to S^2$ be the homeomorphism that links their phase portraits. Let $\gamma, C, \Gamma, O, a, a_1 \ldots, a_K, I_1, \ldots, I_K$ be the objects for the vector field v defined in Section **??**. We will define the same objects for the vector field w. Let $\tilde{\gamma} = \hat{H}(\gamma)$ be the separatrix loop of w, and $\tilde{O} = \hat{H}(O)$. Let $\tilde{I}_k = \hat{H}(I_k)$ be the saddles of w inside $\tilde{\gamma}$; their separatrices wind to $\tilde{\gamma}$ in the negative time. These separatrices cross the clockwise oriented curve $\tilde{C} = \hat{H}(C)$ at the points $\tilde{a}_k = \hat{H}(a_k),\ k = 1, \ldots K$. Recall that Γ is so chosen that a is located between a_K and a_1 on the oriented curve C. The point $\hat{H}(a),\ a = \Gamma \cap C$, lies on $\hat{H}(C)$ between $\hat{H}(a_K)$ and $\hat{H}(a_1)$. Let us slightly change $\hat{H}$ in a neighborhood of $\hat{H}(\Gamma) \cap \hat{H}(C)$ so that the new map, still denoted by $\hat{H}$, links v and w as before, but the curves $\tilde{\Gamma} = \hat{H}(\Gamma)$ and $\tilde{C} = \hat{H}(C)$ are smooth, and $\tilde{\Gamma}$ is transversal to $\tilde{C}$. The points $\tilde{a}$, and $\tilde{a}_k,\ k = 1, ..., K$ follow on $\tilde{C}$ in the same order as the points a, a_k follow on C.

Let, as before, Δ_ε be the Poincaré map for the separatrix loop of the vector field v_ε of the family V. Let $\tilde{\Delta}_\delta$ be the Poincaré map for the separatrix loop of the vector field w_δ of the family W. Consider now the solutions ε_n of the equation $\Delta_\varepsilon^n(\varepsilon) = a$, and solutions δ_n of the following equation: $\tilde{\Delta}_\delta^n(\delta) = \tilde{a}$. The first requirement for the map $h : (\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$ is: $h(\varepsilon_n) = \delta_n$. As explained in 5.2, solutions ε_{kn} of equation (**??**) follow on the semi-interval $[\varepsilon_{n-1}, \varepsilon_n)$ in the increasing order: $k > k' \Rightarrow \varepsilon_{kn} > \varepsilon_{k'n}$. The same holds for the solutions δ_{kn} of the connection equation for the family W located on the interval on $[\delta_{n-1}, \delta_n)$. We now set a homeomorphism h of semi intervals $[0, \varepsilon_{n_0}] \to [0, \delta_{n_0}]$ for some large n_0 in such a way that

$$h(\varepsilon_{kn}) = \delta_{kn}. \tag{5}$$

This completes the choice of the homeomorphism h of the bases of the families V and W.

6.6 Correspondence of the graphs

We will now define the map $G_\varepsilon : M_\varepsilon \to \tilde{M}_{h(\varepsilon)}$. Define first the restiction $G_\varepsilon : X_\varepsilon \to \tilde{X}_{h(\varepsilon)}$ as

$$G_\varepsilon = G_{\varepsilon\delta},\ \delta = h(\varepsilon),$$

where $G_{\varepsilon\delta}$ is the same as in (**??**).

The graph M_ε differs from X_ε by the separatrices $U(\varepsilon)$ and $l_k(\varepsilon)$. We will now define the map G_ε on these separatrices. This definition is illustrated by Figure **??**.

Define $G_\varepsilon(U(\varepsilon)) = \tilde{U}(h(\varepsilon))$. Let us prove that this definition agrees with the construction of G_ε on the other parts of the graph M_ε.

Consider first the case: $\varepsilon = \varepsilon_{kn}$ for some k, n. Then $U(\varepsilon)$ has two verteces: $L(\varepsilon)$ (as ever); and $I_k(\varepsilon)$. At the same time, $h(\varepsilon) = \delta_{kn}$. Hence, $\tilde{U}(h(\varepsilon)) = \tilde{U}(\delta_{kn}) = G_\varepsilon(l_k(\varepsilon_{kn}))$. This agrees with the definition of G_ε on X_ε.

Consider now the case when no ε_{kn} coincides with ε. Let k, n be such that $\varepsilon \in (\varepsilon_{kn},\ \varepsilon_{k+1n})$. Then $U(\varepsilon)$ intersects C by the interior point of the arc $\alpha_k(\varepsilon)$. By Proposition **??**, the truncation vertex of $U(\varepsilon)$ belongs to the arc $\beta_k(\varepsilon)$. The arcs $\beta_k(\varepsilon)$ depend continuously on ε, as well as $\tilde{\beta}_k(\delta)$. We have:

$$\hat{H}(\beta_k(0)) = \tilde{\beta}_k(0).$$

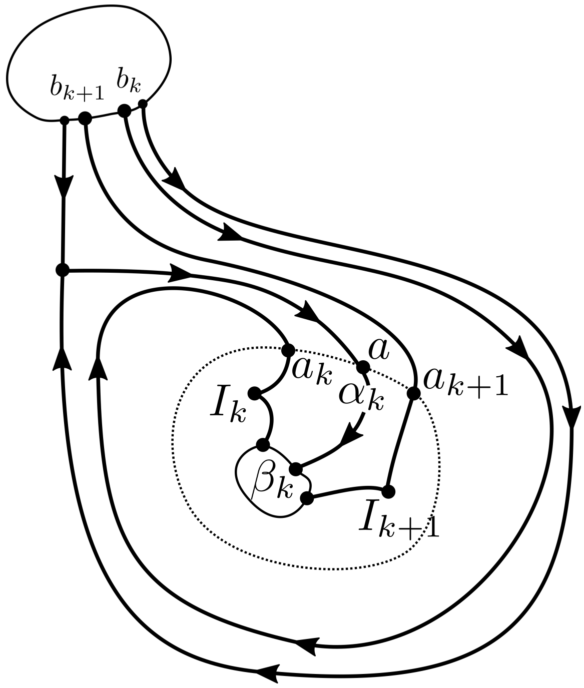

Figure 5: Bifurcations of the sparkling saddle connections

The arcs $\beta_k(\varepsilon),\ \tilde{\beta}_k(\delta)$ are the edges of the graphs $X_\varepsilon, \tilde{X}_\delta$. Hence,

$$G_\varepsilon(\beta_k(\varepsilon)) = \tilde{\beta}_k(h(\varepsilon)).$$

As $h(\varepsilon) \in (\delta_{kn}, \delta_{k+1n})$, we conclude that $\tilde{U}(h(\varepsilon))$ intersects $\tilde{C}$ by an interior point of the arc $\tilde{\alpha}_k$. By Proposition **??**, the truncation vertex of $\tilde{U}(h(\varepsilon))$ belongs to $\tilde{\beta}_k(h(\varepsilon))$. This vertex splits $\tilde{\beta}_k(h(\varepsilon))$ into two edges of $\tilde{M}_{h(\varepsilon)}$; the other verteces of these edges are the endpoints of $\tilde{\beta}_k(h(\varepsilon))$. Similarly, the truncation vertex $C(\varepsilon)$ of $U(\varepsilon)$ splits $\beta_k(\varepsilon)$ in two edges of M_ε; the other verteces of these edges are the endpoints of $\beta_k(\varepsilon)$. Hence, G_ε is well defined on $U(\varepsilon)$ and two truncation arcs that contain its vertex.

Let us now extend G_ε to the separatrix $l_k(\varepsilon)$. Let $G_\varepsilon(l_k(\varepsilon)) = \tilde{l}_k(h(\varepsilon))$.

Consider first the case $\varepsilon = \varepsilon_{kn}$. Then $l_k(\varepsilon)$ is a saddle connection between $L(\varepsilon)$ and $I_k(\varepsilon)$. At the same time, $G_\varepsilon(l_k(\varepsilon)) = \tilde{l}_k(\delta_{kn})$. Hence, $G_\varepsilon(l_k(\varepsilon))$ connects $\tilde{L}(\delta_{kn})$ and $\tilde{I}_k(\delta_{kn})$. This agrees with the definition of G_ε on X_ε.

Suppose now that $\varepsilon \neq \varepsilon_{kn}$ for all k, n. Take m and n so that $\varepsilon_{mn} < \varepsilon < \varepsilon_{m+1n}$. Then, as before, $a(\varepsilon) \in \alpha_m$. Let $S'(\varepsilon)$ be an ingoing separatrix of L different from $S(\varepsilon)$. Let $\alpha(S'(\varepsilon))$ be its α-limit set, and C' be the corresponding truncation loop. For $\varepsilon > 0$, the separatrix $S(\varepsilon)$ has the same α-limit set as $S'(\varepsilon)$, see Figure **??**. Let $b'(\varepsilon)$ and $b_0(\varepsilon)$ be the truncation verteces of $S'(\varepsilon)$ and $S(\varepsilon)$ on C'. Let C' be oriented as a boundary of the domain that contains $\alpha(S'(\varepsilon))$. Let $\beta(\varepsilon)$ be the arc of the oriented curve C' from b' to b_0, and $int\beta(\varepsilon) = \beta(\varepsilon) \setminus b'(\varepsilon) \setminus b_0(\varepsilon)$. For $\varepsilon > 0$, a Poincaré map $P_\varepsilon : C \setminus a(\varepsilon) \to int\beta(\varepsilon)$ along the orbits of v_ε is well defined. The images $b_j(\varepsilon) = P_\varepsilon(a_j(\varepsilon))$ follow in the order $b_{m+1}(\varepsilon), \ldots, b_k\varepsilon),\ b_1(\varepsilon), \ldots, b_m(\varepsilon)$ from b' to b_0.

At the same time, by construction of h, $\delta_{mn} < h(\varepsilon) < \delta_{m+1n}$. Then $\tilde{U}(\varepsilon)$ intersects $\tilde{\alpha}_m$, and the truncation verteces of the separatrices $\tilde{l}_j(h(\varepsilon))$ follow in the same order on $\tilde{\beta} : \tilde{b}_{m+1}(\delta), \ldots, \tilde{b}_k(\delta), \tilde{b}_1(\delta), \ldots, \tilde{b}_m(\delta),\ \delta = h(\varepsilon)$. This shows that the extension of G_ε to all the separatrices of v_ε is well defined.

6.7 Checking the star condition

Let us now check the star condition for the map G_ε. In all the verteces of X_ε it is already checked. All the singular points of the vector fields v_ε belong to X_ε. It remains to check the star condition in the truncation verteces of the separatrices $U(\varepsilon)$ and $l_k(\varepsilon)$. These verteces have index 3. Recall that the transversal loops are so oriented that the corresponding singular points or limit cycles or the separatrix loop lie to the left of them. Hence, the separatrices if any, come to these transversal loops from the right. So for any truncation vertex the order of the edges in the counterclockwise direction is: incoming truncation arc, outcoming truncation arc, the separatrix. This order is preserved by G_ε.

6.8 Annuli faces lemma

Lemma 4. *The homeomorphism G_ε may be extended from the boundary of any annular face of M_ε to a homeomorphism of all this face to some annular face of $\tilde{M}_{h(\varepsilon)}$.*

Proof. For $\varepsilon > 0$, the vector field v_ε has no separatrix loops; hence, the annular faces of its LMF graph are of types 1, 2, 3 only, by Lemma **??**. In case 3, both boundary components of the annular face are transversal loops and they do not depend on ε. The map G_ε on them does not depend on ε as well, and coincides with $\hat{H}$ that is a homeomorphism of the whole sphere. This proves the lemma in Case 3.

In cases 1 or 2, one boundary component is a transversal loop that does not depend on ε, and the other one is the corresponding singular point or cycle $\pi_\varepsilon(c)$ where c is a singular point or cycle of the vector field v_0. The image of this boundary under G_ε is

$$\tilde{\pi}_{h(\varepsilon)}(\hat{H}(c)) \cup \tilde{C},\ \tilde{C} = \hat{H}(C).$$

The map that is π_ε^{-1} on the singular point or cycle and identity on the transversal loop mentioned above may be easily extended to a homeomorphism F_ε of the annular face of M_0 to that of M_ε. In the same way a homeomorphism of the corresponding annular faces of the graphs $\tilde{M}_\delta$ is defined. The homeomorphism $\tilde{F}_{h(\varepsilon)} \circ \hat{H} \circ F_\varepsilon^{-1}$ is the desired one. This proves the lemma in Cases 1 and 2. □

6.9 Preserving the labels

The map G_ε for $\varepsilon > 0$ preserves the types of the verteces and edges of the LMF graph. Hence, it preserves the labels.

The construction of the map G_ε for $\varepsilon > 0$ is over. The star and annuli faces conditions for this map hold. Hence, vector fields v_ε and $w_{h(\varepsilon)}$ are orbitally topologically equivalent for $\varepsilon > 0$. The main part of the equivalence criterion is proved. Only the case $\varepsilon \leq 0$ remains.

The fields v_0 and w_0 are orbitally topologically equivalent by assumption. Let us check the same property for $\varepsilon < 0$.

6.10 Case $\varepsilon < 0$

Let now $X_0 = M_0 \setminus \gamma$. This subgraph changes continuously with ε for $\varepsilon \leq 0$, that is, generates a family of subgraphs $X_\varepsilon \subset M_\varepsilon$ continuous in ε. Let π_ε be the natural homotopy. Let $\tilde{X}_0, \tilde{X}_\varepsilon, \tilde{\pi}_\varepsilon$ be the similar objects for the family $W, \varepsilon \leq 0$. Let G_ε on X_ε be given by the formula (**??**) again, with $h(\varepsilon) = \varepsilon$.

The difference $M_\varepsilon \setminus X_\varepsilon$ for $\varepsilon < 0$ constitutes of the following elements, see Figure **??**:

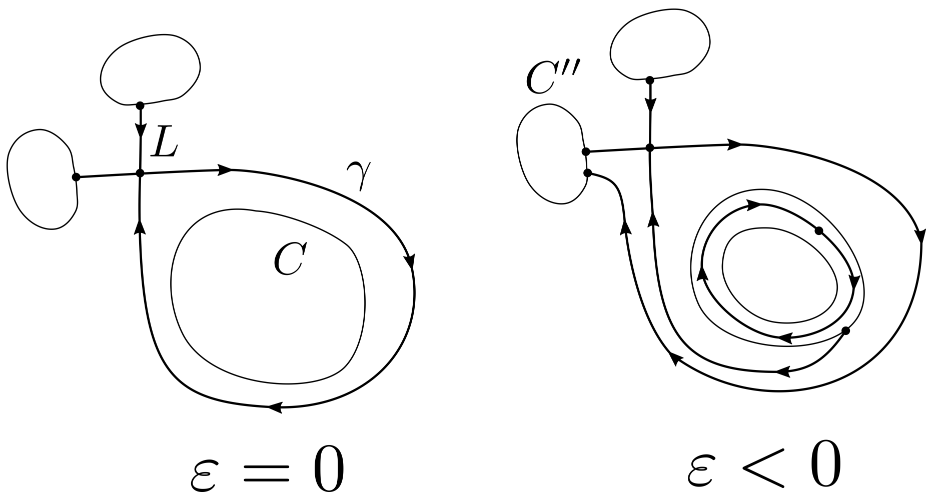

Figure 6: The bifurcating part of the LMF graph for the non-positive parameter

- the truncated separatrix $U(\varepsilon)$;
- the limit cycle $c(\varepsilon)$ with the vertex on it born from the separatrix loop γ;
- a new transversal loop $C(\varepsilon)$ outside this cycle, with the truncation vertex of the separatrix $S(\varepsilon)$ on it;
- the truncated separatrix $S(\varepsilon)$.

The map G_ε on M_ε for $\varepsilon < 0$ is defined as follows. On X_ε it is defined by (**??**) with $h(\varepsilon) = \varepsilon$. Let $\gamma, c(\varepsilon), S(\varepsilon), U(\varepsilon)$ be the same as above, see Figure **??**, $\tilde{\gamma}, \tilde{c}(\varepsilon), \tilde{S}(\varepsilon), \tilde{U}(\varepsilon)$ be the similar objects for the family W. Let

$$G_\varepsilon(c(\varepsilon)) = \tilde{c}(\varepsilon), G_\varepsilon(S(\varepsilon) = \tilde{S}(\varepsilon); G_\varepsilon(U(\varepsilon) = \tilde{U}(\varepsilon).$$

At the verteces of these three edges, the star condition is fulfilled. For all the annular faces of the graph X_ε, the annuli faces condition holds because it holds for the homotopies $\pi_\varepsilon, \tilde{\pi}_\delta$ and because $\hat{H}$ is a homeomorphism. For two annular faces whose boundary contains the new born limit cycle $c(\varepsilon)$, this condition is obvious, see Figure **??**.

The equivalence of the graphs M_ε and $\tilde{M}_{h(e)}$ is proved, and star and annuli faces conditions for it are checked. Hence, these graphs are isotopic on the sphere. Thus, the families V and W are equivalent. This proves Theorem **??**.

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