

Global bifurcations in generic one-parameter families on \mathbb{S}^2 .

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Abstract

In this paper we prove that generic one-parameter families of vector fields on \mathbb{S}^2 in the neighbourhood of the fields of classes AH, SN, HC, SC (Andronov-Hopf, saddle-node, homoclinic curve, saddle connection) are structurally stable. We provide the classification of bifurcations in these families.

Key words: bifurcations, equivalence, structural stability

Mathematics subject classification: 34C23, 37G99, 37E35

1 Introduction

This paper is a part of a large program suggested in [I] and [IKS]. Classical bifurcation theory in the plane may be split in two parts: local bifurcations (that occur near a degenerate singular point) and semilocal ones (oftenly called nonlocal) that occur near polycycles and non-hyperbolic cycles. Recently a third part of the theory was developed: global bifurcations on the sphere. Its striking difference with the two previous parts occurs because of sparkling saddle connections discovered in [MP].

Semilocal bifurcation theory in the plane has important applications to the Hilbert 16th problem, see the founding paper [DRR] and future publications on the subject. The global theory is too young as to have such applications; yet they may be expected in the future.

Two of the most important problems of the new theory are:

- Classify the global bifurcations that occur in few (one, two, three) parameter families.
- Study the structural stability of these families.

This paper completes the solution of both problems for generic one-parameter families of vector fields in the two-sphere (families for brevity in what follows). Two previous papers on the subject are [IS] and [GIS]. The first result of the three papers is:

Theorem 1. *Generic one parameter families are structurally stable.*

The second one is the classification of the bifurcations in such families.

Note that despite the conjecture of Arnold [AAIS], not all generic finite parameter families are structurally stable: counterexamples for three parameter families were discovered [IKS].

The investigation of global bifurcations on the two-sphere was initialed by Sotomayor [S]. He described all the degeneracies that may occur in generic one-parameter families; the corresponding vector fields are called *quasigeneric*. Sotomayor also proved that quasigeneric vector fields are structurally stable in the class of all quasigeneric vector fields. But he did not investigate their bifurcations.

There are six classes of quasigeneric vector fields, see the basic list below. One of them consists of vector fields with a separatrix loop, another one of fields with a parabolic cycle. The unfoldings of the fields of the first class are investigated in [IS], of the second in [GIS]. Here we investigate bifurcations in the remaining four classes. These bifurcations are called “tame” because the bifurcation diagram for these families consists of one point, and no sparkling saddle connections occur.

The second main result of this paper is the following (see Theorem 6 below for the precise statement):

Unfolding of two quasigeneric vector fields of the four classes considered are equivalent if and only if two quasigeneric fields perturbed are orbitally topologically equivalent.

This is sort of a classification result.

Let us pass to the detailed presentation.

1.1 Preliminaries.

Below we provide some definitions that are necessary for our work. They are classical; we borrow them from [IS].

All the vector fields and families of vector fields that we consider are of the class C^∞ .

Definition 1. Two vector fields v and w on \mathbb{S}^2 are called orbitally topologically equivalent, if there exists a homeomorphism $\mathbb{S}^2 \rightarrow \mathbb{S}^2$ that links the phase portraits of v and w , that is, sends orbits of v to orbits of w and preserves their time orientation.

Definition 2. Let B, B_0 be topological balls in \mathbb{R} . Two families of vector fields $\{v_\alpha, \alpha \in B\}, \{w_\beta, \beta \in B_0\}$ on \mathbb{S}^2 are called weakly topologically equivalent if there exists a map

$$H: B \times \mathbb{S}^2 \rightarrow B_0 \times \mathbb{S}^2, \quad H(\alpha, x) = (h(\alpha), H_\alpha(x))$$

such that h is a homeomorphism, and for each $\alpha \in B$ the map $H_\alpha: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is a homeomorphism that links the phase portraits of v_α and $w_{h(\alpha)}$.

Definition 3. A vector field is called structurally stable if it is orbitally topologically equivalent to any nearby vector field. A family of vector fields is called weakly structurally stable if it is weakly topologically equivalent to any nearby family.

In what follows, we mainly deal with local families.

Definition 4. A local family at $\alpha = 0$ with the base $(\mathbb{R}^1, 0)$ is a germ on $\{0\} \times \mathbb{S}^2$ of a family given on $B \times \mathbb{S}^2$, $B \ni 0$, $B \subset \mathbb{R}$ is open. Two local families are weakly topologically equivalent if they have locally weakly topologically equivalent representatives, and the corresponding homeomorphism of the bases maps 0 to 0. A local family of vector fields is called weakly structurally stable if it is weakly topologically equivalent to any nearby local family.

1.2 Structural stability of vector fields

Before considering families of vector fields let us provide the classical results related to the structural stability of vector fields.

Definition 5. A vector field on the sphere is called a Morse-Smale system if it satisfies two following conditions:

1. all its singular points and limit cycles are hyperbolic;

2. it has no saddle separatrices.

In 1937 A. Andronov and L. Pontryagin derived the following criterion of structural stability of vector fields on the sphere:

Theorem 2. *A C^1 vector field on the sphere is structurally stable if and only if it is a Morse-Smale system.*

More general result is true if we define Morse-Smale systems on arbitrary C^1 manifolds. A vector field v on a C^1 manifold is a Morse-Smale system if it satisfies the condition 1 from the definition 5 and two additional conditions:

- 2' the set of non-wandering points of v consists of a finite number of singular points and limit cycles;
- 3' stable and unstable manifolds of singular points and limit cycles of v intersect each other transversally.

In 1971 M. Peixoto proved that the statement of the theorem 2 is true for vector fields on compact two-dimensional C^1 manifolds.

Remark 1. *Vector fields on the sphere that satisfy conditions 1 and 2 also satisfy conditions 1, 2' and 3'. However, it is already not true for vector fields on the torus: an irrational flow is not a Morse-Smale system.*

1.3 Basic list

Let us remind the list of degeneracies that can be met in an unavoidable way in **generic** one-parameter family:

1. *AH*: a non-hyperbolic singular point with a pair of non-zero pure imaginary eigenvalues;
2. *SN*: a saddle-node singular point;
3. *HC*: a homoclinic curve of a saddle-node;
4. *SC*: a saddle connection between two different hyperbolic saddles;
5. *SL*: a loop of a hyperbolic saddle;
6. *PC*: a parabolic cycle, that is, a non-hyperbolic limit cycle.

For future references, this list of degeneracies will be called a *basic list*.

Definition 6. *A vector field is called quasi-generic if it has exactly one degeneracy of the type *AH, SN, HC, SC, SL* or *PC*, and satisfies the following genericity assumptions:*

1. *all the other singular points and limit cycles are hyperbolic;*
2. *for the non-hyperbolic singular point of the vector field with a degeneracy of the type *AH*, the third derivative at 0 of the corresponding normalized Poincaré map is non-zero;*

3. the non-hyperbolic singular point of a vector field with a degeneracy of the type *SN* or *HC* is of the multiplicity two;
4. the homoclinic curve of a vector with a degeneracy of the type *HC* is not a separatrix loop;
5. the characteristic value of a saddle with a separatrix loop for a vector with a degeneracy of the type *SL* is not equal 1;
6. for a vector field with a degeneracy of the type *PC*, the second derivative of the Poincaré map of the parabolic limit cycle is nonzero.

In our work we will consider the first four degeneracies from the list.

We say a vector field v is of class *AH* (*SN*, *HC*, *SC* correspondingly) if v is a quasi-generic vector field with a degeneracy 1 (2, 3, 4 respectively) from the basic list.

In [S] it is proved that C^k vector fields of class *AH* (*SN*, *HC*, *SC*) form a Banach submanifold of codimension one in the space $\text{Vect}^k(\mathbb{S}^2)$ of C^k -vector fields on \mathbb{S}^2 for $k \geq 3$, in particular it is true for C^∞ vector fields. These submanifolds will be denoted by *AH*, *SN*, *HC*, *SC* respectively. In what follows, we will consider local families that cross this submanifolds and are transversal to them. The transversality will be used in the description of the corresponding local and semilocal bifurcations.

Local bifurcations are those that occur in a neighbourhood of a singular point; *semilocal* ones occur in a neighbourhood of saddle connections or separatrix polygons. Local and semilocal bifurcations in the classes *AH*, *SN*, *HC*, *SC* are studied long ago and described in many sources; see [AAIS] for the description and references. Global bifurcations in the classes *AH* and *SC* are reduced to the local and semilocal ones; those in the classes *SN* and *HC* are not. The difference is due to separatrices of hyperbolic saddles that enter the parabolic sector of the saddle-node singular point, see Fig.1 below.

1.4 Structural stability

“Global” structural stability theorem follows (we skip the details) from the following “local” structural stability theorem.

Theorem 3. *Generic local one-parameter family is structurally stable.*

This theorem is non-trivial only in case when the family is an unfolding of one of the degeneracies from the basic list; in other words when the vector field is of one of the classes *AH*, *SN*, *HC*, *SC*, *SL*, *PC*. For the last two classes Theorem 3 is proved.

Theorem 4. ([IS], [GIS]) *Generic unfoldings of vector fields of classes *SL* and *PC* are structurally stable.*

In this paper we prove

Theorem 5. *Generic unfoldings of vector fields of classes *AH*, *SN*, *HC*, *SC* are structurally stable.*

Families of these classes are called *tame*, because their bifurcation diagrams consist of one point in contrast to those of the families *SL* and *PC*.

1.5 Equivalence criterion

Theorem 6. *Two generic local families of classes AH, SN, HC, SC are weakly equivalent iff the corresponding quasigeneric vector fields are orbitally topologically equivalent.*

Analogous theorem is proved for local families of class SL [IS]. For local families of class PC analogous theorem is wrong. In this case a class of a phase portrait modulo topological orbital equivalence does not determine uniquely a bifurcation scenario, the latter depends on the "order" in which sparkling saddle connections appear after separation of the parabolic cycle, see details in [GIS].

The main part of the paper contains the proof of Theorem 6. Theorem 5 easily follows from this one, see Section 5 below.

2 LMF graphs

Here we reproduce some definitions and results from [IS]; this is necessary to make the paper independent.

2.1 The construction

The LMF (Leontovich-Mayer-Fedorov) graph is a modified union of singular points, separatrices and limit cycles of a vector field. We will define it for the vector fields that are perturbations of quasigeneric vector fields of classes AH, SN, HC, SC only. These vector fields have all the singular points hyperbolic, except for at most one; all the limit cycles hyperbolic; no polycycles at all. All the separatrices of such fields are saddle separatrices or phase curves that include the boundaries of hyperbolic sectors of the saddle-node. For any vector field v of this set, we will construct a spherical graph that is a complete topological invariant for this vector field. Let us surround any attracting or repelling singular point of v by small closed curve transversal to v ; it will be called a transversal loop. Let us do the same for any limit cycle of v (two closed curves from each side). Recall that these limit cycles are hyperbolic. Every unstable separatrix that enters the transversal loop in the positive time is truncated, that is, replaced by its arc between the singular point and its intersection with the loop; this intersection point is called a *truncation vertex*. An analogous truncation is done for stable separatrices. A transversal loop that contains no truncation vertices is called *empty*.

We suppose that the sphere is oriented. We orient all the transversal loops in such a way that the corresponding singular point or cycle is to the left of them.

Definition 7. [IS] *LMF-graph of a vector field v , denoted by $LMF(v)$, is an oriented graph embedded in \mathbb{S}^2 consisting of the following elements:*

- *Vertices:*
 1. All the singular points of v ;
 2. All the truncation vertices of the separatrices of v ;
 3. An arbitrary chosen point on each limit cycle;
 4. An arbitrary chosen point on each "empty" transversal loop.
- *Edges:*

1. *Unstable (stable) separatrices of singular points, if such a separatrix doesn't intersect any curve without contact. The orientation on all the orbits if v is induced by the time parametrization;*
2. *Truncated separatrices;*
3. *Limit cycles with the chosen vertices on them;*
4. *Truncation arcs, that is, arcs of the oriented transversal loops between subsequent truncation vertices;*
5. *Empty transversal loops with the chosen vertices on them.*

Labelling: Each vertex which is a singular point is labelled by the type of this singular point - attractor, repeller, saddle or saddle-node. Other vertices are labelled to show if they are points on limit cycles or on transversal loops. Each edge is labelled depending whether it is a stable or unstable (truncated) separatrix, or a truncation arc. For arcs of transversal loops, we put labels showing if this loop is absorbing (this loop corresponds to the ω -limit set of its points) or outgoing.

2.2 Faces of LMF-graphs

Theorem 7. *All the faces of LMF-graph of a vector field of the type AH, SN, HC and SC or their small perturbations are topological disks or annuli. All the annuli-faces that appear in the graph have as the boundaries:*

- *a singular point and its transversal loop;*
- *a limit cycle and one of its transversal loops;*
- *two "empty" transversal loops.*

This statement follows from a theorem proved in [GI*].

Theorem 8. *For vector fields with a finite number of singular points and limit cycles counted with multiplicity, the previous theorem holds with the only difference: an annuli shaped face of an LMF graph may have a monodromic polycycle and its transversal loop as a boundary.*

The vector fields of classes AH, SN, HC, SC and their small perturbations have no polycycles; so Theorem 7 follows from Theorem 8.

2.3 LMF-graphs as invariants of orbital topological equivalence.

Theorem 9 (R. Fedorov, [F]). *If two LMF graphs $\Gamma_1 = LMF(v)$, $\Gamma_2 = LMF(w)$ of two vector fields v, w are isotopic on the sphere (i.e. there exists an orientation-preserving homeomorphism of the sphere which maps one to another, preserves orientation on edges and matches labels on edges and vertices), then v and w are orbitally topologically equivalent.*

2.4 Isotopy of spherical graphs.

Theorem 10 (Adkisson-McLane theorem, [MA]). *Images of two embeddings of the same connected graph on a sphere S^2 are isotopic iff the corresponding isomorphism of images preserves the counterclockwise order of the edges at each vertex.*

We will call the requirement of this theorem the star condition (**S** condition for brevity).

As it was mentioned above, the faces of *LMF*-graphs of vector fields of types *AH*, *SN*, *HC* and *SC* and their perturbations are topological disks or annuli. We provide a convenient criterion for isotopy of these graphs which is a generalization of the theorem above.

Theorem 11. *Suppose that two oriented planar graphs Γ_1, Γ_2 are embedded in \mathbb{S}^2 by maps $\varphi_1 : \Gamma_1 \rightarrow \mathbb{S}^2$, $\varphi_2 : \Gamma_2 \rightarrow \mathbb{S}^2$, and the (open) faces of the embedded graphs are topological discs or annuli. Choose an orientation in \mathbb{S}^2 . Suppose that these graphs are isomorphic as oriented graphs, and the isomorphism, denote it by g , satisfies the star condition. Suppose that the map $\varphi_2 \circ g \circ \varphi_1^{-1}$ extends to an orientation-preserving homeomorphism of all the annuli-shaped faces of the graph Γ_1 . Then the map $\varphi_2 \circ g \circ \varphi_1^{-1}$ can be extended to the orientation-preserving homeomorphism of \mathbb{S}^2 , so $\varphi_1(\Gamma_1)$ is isotopic to $\varphi_2(\Gamma_2)$.*

A simple proof of this fact is given in [GI*]. We will call the requirement of this theorem the annuli faces condition (**A** condition for brevity)

Corrolary 1. *If a family of spherical graphs (which are embeddings of the same abstract planar graph) depends continuously on a parameter then these graphs are pairwise isotopic on the sphere.*

Proof. Both **S** and **A** conditions persist under the homotopy of the graphs. □

3 Proof of the structural stability criterion: description of the LMF graphs

In this and the next section we prove Theorem 6.

3.1 Idea of the proof

Consider two generic one-parameter local families $V = \{v_\varepsilon | \varepsilon \in (\mathbb{R}, 0)\}$ and $W = \{w_\delta | \delta \in (\mathbb{R}, 0)\}$ with equivalent vector fields v_0 and w_0 that belong to classes *AH*, *SN*, *HC*, *SC*. Let M_ε and \tilde{M}_δ be the LMF graphs of the vector fields v_ε and w_δ . Let \hat{H} be a homeomorphism that links v_0 and w_0 (see def. 1). Without loss of generality we can assume it to preserve the orientation on the sphere. We will prove that, after a suitable reparametrization, the graphs M_ε and \tilde{M}_ε are isotopic on \mathbb{S}^2 . This will imply that the vector fields v_ε and w_ε are equivalent; thus, the families V and W are weakly equivalent.

We will prove that the graph M_ε depends continuously on ε for $\varepsilon < 0$ and $\varepsilon > 0$ separately. To do that we will represent the graph M_ε as a union of two:

$$M_\varepsilon = N_\varepsilon \cup X_\varepsilon.$$

The graph X_ε will depend continuously on ε in the whole neighbourhood of zero; the graph N_ε will be uniquely determined by the subgraph N_0 , and continuous in ε for $\varepsilon \neq 0$. We should think of the graph N_ε as a bifurcating part of M_ε and consider the graph X_ε as a non-bifurcating one. The graph \tilde{M}_ε may be represented in a similar way. Then we will prove that the graphs M_ε and \tilde{M}_ε are isotopic.

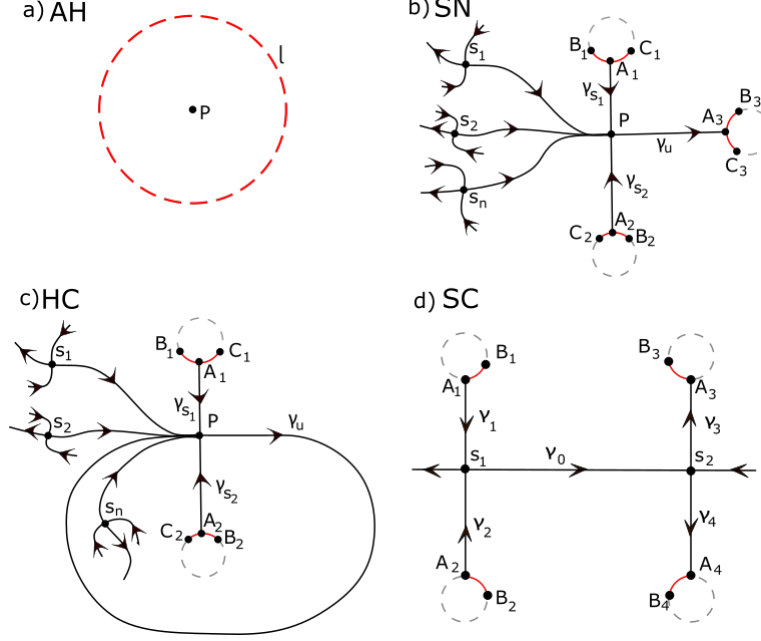


Figure 1: Subgraphs N_0 for vector fields of classes AH, SN, HC, SC

3.2 Subgraph N_0

We describe the bifurcating parts of the LMF graphs of the quasigeneric vector fields under consideration.

Class AH .

The graph N_0 consists of one non-hyperbolic singular point P , see Figure 1a.

Class SN .

Let P be a saddle-node of v_0 ; let $\gamma_s^1, \gamma_s^2, \gamma_u$ be its two stable and one unstable separatrices. We choose γ_s^1 to be the first ingoing separatrix if we encircle P in a counterclockwise direction starting from γ_u , see Figure 1b.

We will denote by $\omega(\gamma)$ ($\alpha(\gamma)$ respectively) the ω -limit (α -limit) set of an arbitrary trajectory γ .

By the Poincaré-Bendixson theorem and no saddle connections assumption, $\alpha(\gamma_s^i), i = 1, 2$ are hyperbolic repellers, $\omega(\gamma_u)$ is a hyperbolic attractor (a point or a cycle). Denote by $l_i, i = 1, 2, 3$ the corresponding transversal loops (l_3 for $\omega(\gamma_u)$). Let $A_i \in l_i$ be the truncation vertices of the separatrices $\gamma_s^1, \gamma_s^2, \gamma_u$ respectively. The curve l_i is crossed by at least one separatrix of a hyperbolic saddle of v_0 ; let us prove it for $i = 3$. In the opposite case the Poincaré map $l_3 \rightarrow l_1$ would be defined on the whole of l_3 , which is impossible. The same argument holds for $i = 1, 2$. If there is only one such vertex, denote it by B_i . If they are more than one, let B_i, C_i be the closest to A_i vertices of l_i ; these vertices follow on the oriented curve l_i in the order B_i, A_i, C_i . We will use these notation even when B_i is unique, just considering that B_i and C_i coincide.

Let $s_j, j = 1, \dots, n$ be the hyperbolic saddles whose separatrices γ_j enter the parabolic sector of P .

The elements of N_0 are:

- Vertices: $P; A_i, B_i, C_i, i = 1, 2, 3; s_j, j = 1, \dots, n;$

- Edges: arcs $A_i B_i, A_i C_i, i=1, 2, 3$; truncated separatrices $\gamma_s^1, \gamma_s^2, \gamma_u$, complete separatrices $\gamma_j, j=1, \dots, n$.

Class HC .

The graph N_0 in this case is constructed by the slight modification of the previous one, see Fig.1c. The truncated separatrices γ_s^i , the transversal loops l_1, l_2 and the truncation vertices A_i, B_i, C_i are the same as before. But now there may be no extra truncation vertices on l_i except for A_i , in this case we consider that $B_i = A_i = C_i$ and the arcs $A_i B_i, A_i C_i$ coincide with l_i . The unstable separatrix of P is now the homoclinic loop: $\omega(\gamma_u) = P$. The saddles s_j and their separatrices γ_j are defined as before.

Then N_0 consists of:

- Vertices: $P; A_i, B_i, C_i, i = 1, 2; s_j, j = 1, \dots, n$;
- Edges: arcs $A_i B_i, A_i C_i, i = 1, 2$; (truncated) separatrices $\gamma_{s_1}, \gamma_{s_2}, \gamma_u$, complete separatrices $\gamma_j, j=1, \dots, n$.

Class SC .

Let γ_0 be a saddle connection between two saddles s_1, s_2 of v_0 . We can assume that γ_0 is an unstable separatrix of s_1 . Let $\gamma_1, \gamma_2(\gamma_3, \gamma_4)$ be the stable separatrices of s_1 (unstable separatrices of s_2) that bound the same hyperbolic sectors as γ_0 , see Fig.1d. Let l_i be a transversal loop around $\alpha(\gamma_1), \alpha(\gamma_2), \omega(\gamma_3), \omega(\gamma_4), i = 1, \dots, 4$. Let B_j be the truncation vertices on l_j different from A_j (if exist); B_1 and B_2 are chosen in such a way that the points on the arcs $A_1 B_1$ and $A_2 B_2$ close to A_1 and A_2 are connected by orbits passing through the hyperbolic sectors of s_1 and s_2 ; same for B_3 and B_4 . If such a point B_j does not exist, we set $B_j = A_j$, and the arc $A_j B_j$ is the whole curve l_j .

The graph N_0 :

- Vertices: $A_i, B_i, i = 1, \dots, 4, s_1, s_2$;
- Edges: arcs $A_i B_i; \gamma_i, i = 1, \dots, 4; \gamma_0$.

3.3 Persistent subgraph X_ε .

In this section we construct a part of the graph M_ε that does not bifurcate. Let $X_0 = \overline{M_0 \setminus N_0}$.

Proposition 1. *The graph M_ε for any small ε contains a subgraph X_ε that depends continuously on ε and coincides with X_0 for $\varepsilon = 0$.*


Proof. For a vector field v_0 let P be some non degenerate singular point of v_0 and c be some hyperbolic limit cycle of v_0 . Then, by the implicit function theorem, there exists a map $(\mathbb{R}, 0) \rightarrow \mathbb{R}^2, \varepsilon \mapsto P_\varepsilon$ such that P_ε is a singular point of v_ε , and $P_0 = P$. Similarly there exists a smooth family of hyperbolic limit cycles c_ε of the vector fields v_ε such that $c_0 = c$.

Let π_ε be a map that brings all non degenerate singular points and all (hyperbolic) limit cycles of v_ε to those of v_0 defined as $\pi_\varepsilon(P_\varepsilon) = P_0, \pi_\varepsilon(c_\varepsilon) = c_0$, such that π_ε preserves the time orientation of the limit cycles.

Note that if P is an attractor or repeller of v_0 then the transversal loop l for P and v_0 is at the same time a transversal loop for P_ε and v_ε . The same holds for transversal loops l^+, l^- for the cycle c . Hence the transversal loops in X_ε do not depend on ε at all.

Note that in the class AH the non-hyperbolic singular point P is degenerate, and thus included in the previous construction. Its transversal loop l is not included in N_0 (while P is); thus $l \subset X_\varepsilon$.

By construction vertices of X_0 are:

- (all the) hyperbolic singular points of v_0 ; 
- truncation vertices of separatrices of the hyperbolic saddles;
- (all the) vertices on the limit cycles of v_0 (these cycles are all hyperbolic);
- (all the) vertices on the empty transversal loops.

Edges of X_0 are:

- truncated separatrices of hyperbolic saddles;
- arcs on the transversal loops formed by truncation vertices of separatrices of the hyperbolic saddles;
- (all the) limit cycles;
- (all the) empty transversal loops.

Let now P_0 be a hyperbolic saddle of v_0 , γ_0 be a separatrix of P_0 that belongs to X_0 , hence, not to N_0 . Then γ_0 has a hyperbolic limit cycle or a non degenerate singular point as an ω -limit set if γ_0 is unstable, or as an α -limit set if γ_0 is a stable separatrix. Hence, γ_0 intersects a transversal loop l of this limit set.

There exists a family of germs of separatrices $(\gamma_\varepsilon, P_\varepsilon)$ of vector fields v_ε continuously depending on ε such that (γ_0, P_0) is a germ of γ_0 at P_0 . By the implicit function theorem, the separatrices γ_ε intersect the loop l at the points $a(\varepsilon)$ that depend smoothly on ε . These points are truncation vertices of the separatrices γ_ε . Extend the maps π_ε to the separatrices and their truncated vertices:

$$\pi_\varepsilon(a_\varepsilon) = a_0 = \gamma_0 \cap l; \quad \pi_\varepsilon(\gamma_\varepsilon^t) = \gamma_0^t,$$

where γ_ε^t and γ_0^t are truncated separatrices γ_0 and γ_ε , that is, the arcs of these separatrices between the saddle and the truncation vertex.

Define the map π_ε on the transversal loops in such a way that it brings the transversal loop into itself and the truncation vertices on it corresponding to the vector field v_ε to those corresponding to v_0 .

Now the homeomorphism π_ε is defined on the whole of X_ε , and the map $\pi_\varepsilon^{-1} : X_0 \rightarrow X_\varepsilon$ is the required homotopy. \square

3.4 Bifurcating subgraph N_ε

In this section we will describe the bifurcating part N_ε of the LMF-graph M_ε in each of the classes considered. We will prove that these subgraphs depend continuously on ε in a punctured neighbourhood of zero and are completely determined by the graph N_0 .

Class AH.

Suppose that the nonhyperbolic singular point P of the vector field v_0 is Lyapunov stable (this may be achieved by the time reversal). The local bifurcation near P is described as follows.

After a suitable reparametrization (see [S]),

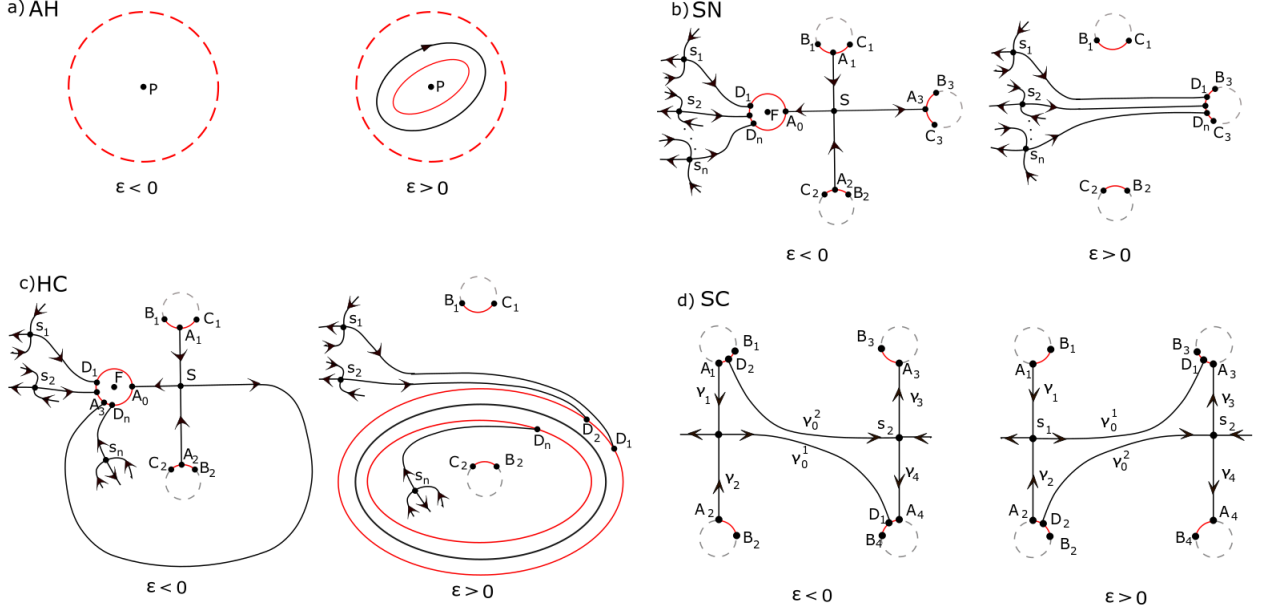


Figure 2: Subgraphs N_ε for vector fields of classes AH, SN, HC, SC

- for $\varepsilon < 0$, the vector field v_ε has an attracting hyperbolic fixed point P_ε ;
- for $\varepsilon > 0$, the vector field v_ε has a repelling focus P_ε surrounded by an attracting limit cycle c_ε .

For all positive ε close to 0 the singular point P_ε is surrounded by an absorbing transversal loop l that has this property for all the vector fields v_ε and does not depend on ε . For $\varepsilon \leq 0$ this transversal loop corresponds to the stable focus P_ε ; for $\varepsilon > 0$, it is an outer transversal loop of the limit cycle c_ε .

For $\varepsilon > 0$, by definition, M_ε contains a transversal loop l_1 , that corresponds to P_ε , and l_2 , an interior transversal loop of c_ε .

Hence, the graph N_ε has the following elements, see Fig.2a:

for $\varepsilon < 0$: vertices: P_ε ;

edges: none;

for $\varepsilon > 0$: vertices: P_ε , one vertex on the limit cycle c_ε ; one vertex on each of the empty closed curves l_1 and l_2 ;

edges: c_ε ; l_1 , l_2 .

This completes the description of the subgraph N_ε for the class AH .

Class SN.

The local bifurcation in a neighbourhood of the saddle-node P is the following:

- for $\varepsilon < 0$, P is split in two singular points, a node F_ε and a saddle S_ε . Recall that P has three separatrices met in the following order when P is surrounded in a counterclockwise direction: $\gamma_u, \gamma_s^1, \gamma_s^2$. The saddle S_ε has four separatrices $\gamma_u^1(\varepsilon), \gamma_s^1(\varepsilon), \gamma_u^2(\varepsilon), \gamma_s^2(\varepsilon)$ that are met in this order under a counterclockwise circuit of S_ε . The germs $(\gamma_u^1(\varepsilon), S_\varepsilon)$, $(\gamma_s^i(\varepsilon), S_\varepsilon)$ tend to (γ_u, P) , (γ_s^i, P) as $\varepsilon \rightarrow 0$, $i = 1, 2$; the germ $(\gamma_u^2(\varepsilon), S_\varepsilon)$ shrinks to P as $\varepsilon \rightarrow 0$.

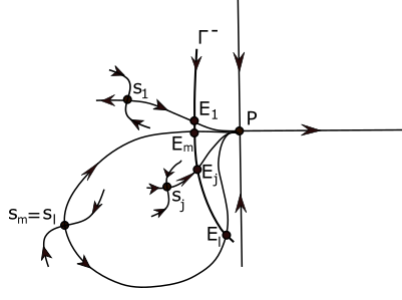


Figure 3: "Baobab" shape and the orientation on Γ^-

- For $\varepsilon > 0$, P vanishes. Moreover, for any cross-section Γ^- transversal to the orbits of v_0 in a parabolic sector of P , and any cross-section Γ^+ transversal to the orbits of v_0 and intersecting γ_u there exists a Poincaré map $\Delta_\varepsilon : \Gamma^- \rightarrow \Gamma^+$. For any neighbourhood U of $O = \Gamma^+ \cap \gamma_u$, $U \subset \Gamma^+$, and for any sufficiently small ε , we have $\Delta_\varepsilon(\Gamma^-) \subset U$.

Let us pass to the description of the graph N_ε .

For $\varepsilon \leq 0$ a transversal loop l_{F_ε} around F_ε should be chosen; it shrinks to P as $\varepsilon \rightarrow 0$.

Consider now the deformations of the separatrices of hyperbolic saddles s_j that enter the parabolic sector of P . Suppose that such separatrices exist. They may form a complicated "baobab shaped" figure, see Fig.3. To put it in order, let us choose a cross-section Γ^- mentioned above such that all the separatrices of the saddles of v_0 that enter P cross Γ^- . Let us take a point $a \in \Gamma^-$ and orient Γ^- by a tangent vector $\zeta \in T_a \Gamma^-$ in such a way that vectors $v_0(a)$ and ζ form a negatively oriented basis (Γ^- is oriented "upward") see Fig.3. Let us enumerate the intersection points of the separatrices that enter P with Γ^- from up to down: E_1, \dots, E_n . Let us enumerate the corresponding separatrices and saddles in the same way: $\gamma_1, \dots, \gamma_n; s_1, \dots, s_n; \gamma_i \ni E_j, \bar{\gamma}_j \ni s_j$. Two separatrices of the same saddle may intersect Γ^- say, at E_k and E_l ; in this case this saddle is enumerated twice, and we have: $s_k = s_l$.

For $\varepsilon < 0$, all the orbits of v_ε come from Γ^- to l_{F_ε} ; denote by Δ_ε the Poincaré map from Γ^- to l_{F_ε} along the orbits of v_ε .

Let $s_j(\varepsilon)$ and $(\gamma_j(\varepsilon), s_j(\varepsilon))$ be the saddles and the germs of the separatrices of v_ε , $\varepsilon \in (\mathbb{R}, 0)$, that depend continuously on ε and coincide with s_j and (γ_j, s_j) for $\varepsilon = 0$.

Let $E_j(\varepsilon)$ be the intersection points of $\gamma_j(\varepsilon)$ with Γ^- .

Let $D_j(\varepsilon) := \Delta_\varepsilon(E_j(\varepsilon)) \in l_{F_\varepsilon}$. The order of the points $D_j(\varepsilon)$ on the circle l_{F_ε} oriented counterclockwise is the same as for the points E_j on Γ^- .

Let $A_0(\varepsilon)$ be the truncation vertex of $\gamma_u^2(\varepsilon)$ on l_{F_ε} . By the Shoshitashvili reduction principle, the truncation vertices on l_{F_ε} are met in the order $A_0(\varepsilon), D_1(\varepsilon), \dots, D_n(\varepsilon)$, see Fig.2b.

For $\varepsilon < 0$, the graph N_ε consists of the following elements:

vertices: $s_j(\varepsilon), D_j(\varepsilon), F_\varepsilon, S_\varepsilon, A_0(\varepsilon), A_i(\varepsilon)$ (truncation vertices of $\gamma_s^i(\varepsilon), \gamma_u^1(\varepsilon)$ on l_i , $A_i(0) := A_i$), $i = 1, 2, 3$; $B_i(\varepsilon), C_i(\varepsilon)$ (truncation vertices on l_i continuous in ε , $B_i(0) := B_i, C_i(0) := C_i$), $i = 1, 2, 3$;

edges: arcs on transversal loops: $(B_i(\varepsilon), A_i(\varepsilon)), (A_i(\varepsilon), C_i(\varepsilon))$ on l_i , $i = 1, 2, 3$; $(A_0(\varepsilon), D_1(\varepsilon)), (D_1(\varepsilon), D_2(\varepsilon)), \dots, (D_{n-1}(\varepsilon), D_n(\varepsilon)), (D_n(\varepsilon), A_0(\varepsilon))$ on l_{F_ε} ; truncated separatrices: $(s_j(\varepsilon), D_j(\varepsilon)), j = 1, \dots, n$; $(S_\varepsilon, A_i(\varepsilon)), i = 0, 1, 2, 3$.

Let us define N_ε for $\varepsilon > 0$.

The points $s_j(\varepsilon), E_j(\varepsilon)$, $j = 1, \dots, n$, $B_i(\varepsilon), C_i(\varepsilon)$, $i = 1, 2, 3$ depend continuously on ε in $(\mathbb{R}, 0)$. Recall that for $\varepsilon > 0$ the Poincaré map $\Delta_\varepsilon : \Gamma^- \rightarrow l_3$ is well defined. The image of Δ_ε is close to A_3 for positive ε small enough. Let $D_j(\varepsilon) := \Delta_\varepsilon(E_j(\varepsilon)) \in l_3$.

The graph N_ε for $\varepsilon > 0$ is the following:

vertices: $s_j(\varepsilon), D_j(\varepsilon)$, $j = 1, \dots, n$, B_i, C_i , $i = 1, 2, 3$;

edges: $(s_j(\varepsilon), D_j(\varepsilon))$, $j = 1, \dots, n$, $(B_i(\varepsilon), C_i(\varepsilon))$, $i = 1, 2$; $(B_3(\varepsilon), D_1(\varepsilon))$, $(D_1(\varepsilon), D_2(\varepsilon)), \dots, (D_{n-1}(\varepsilon), D_n(\varepsilon)), (D_n(\varepsilon), C_3(\varepsilon))$.

This completes the description of the bifurcating subgraph N_ε of the vector field v_ε in the class SN.

Class HC.

The semilocal bifurcation occurs in a neighbourhood of the homoclinic loop of the saddle-node in the following way:

for $\varepsilon < 0$, if we hold the same notations as for SN, then the only difference with the previous case is: the deformation $\gamma_u^1(\varepsilon)$ of the outgoing separatrix γ_u of P tends to the node F_ε as $t \rightarrow \infty$. We note by $A_3(\varepsilon)$ the corresponding truncation point on l_{F_ε} . Recall that $A_0(\varepsilon)$ is the truncation vertex of $\gamma_u^2(\varepsilon)$ on l_{F_ε} . Then $A_3(\varepsilon)$ and $A_0(\varepsilon)$ divide l_{F_ε} in two parts. The truncation vertices $D_j(\varepsilon)$ on one of them correspond to the saddles s_j outside the homoclinic loop γ_u , the truncation vertices on the other part - to the saddles s_j inside γ_u . To define the notions "outside" and "inside" the loop γ_u , we consider γ_s^1 as the external separatrix of P with regard to γ_u .

For $\varepsilon > 0$, P vanishes and in a neighbourhood of γ_u a limit cycle c_ε occurs. Let us note its external and internal transversal loops by l^+ and l^- respectively. The external objects are defined in the following way. The deformations of separatrices tending to P tend to the limit cycle c_ε . The deformed separatrices of external saddles intersect l^+ , the separatrices of internal saddles intersect l^- .

Let us take a cross-section Γ^- as for SN. We can assume that Γ^- intersects γ_u , denote the point of intersection K_3 .

We enumerate the saddle separatrices coming to P , the corresponding saddles and truncation vertices as for the class SN, see Fig.2c.

There exists k , $0 \leq k \leq n$ such that the saddles s_1, \dots, s_k are outside γ_u , the saddles s_{k+1}, \dots, s_n are inside it. It means that the order of points on Γ^- from up to down is $E_1, \dots, E_k, K_3, E_{k+1}, \dots, E_n$, see Fig. 4

Let us consider $\varepsilon < 0$.

Let us note $K_3(\varepsilon)$ the point of intersection of $\gamma_u^1(\varepsilon)$ with Γ^- , $K_3(0) := K_3$.

Let $A_3(\varepsilon) := \Delta_\varepsilon(K_3(\varepsilon))$, where $\Delta_\varepsilon : \Gamma^- \rightarrow l_{F_\varepsilon}$ the Poincaré map along the orbits of v_ε .

Analogically to the case SN, the order of points on l_{F_ε} for $\varepsilon < 0$ in a counterclockwise direction will be $A_0(\varepsilon), D_1(\varepsilon), \dots, D_k(\varepsilon), A_3(\varepsilon), D_{k+1}(\varepsilon), \dots, D_n(\varepsilon)$.

Now we define N_ε for $\varepsilon < 0$:

vertices: $s_j(\varepsilon), D_j(\varepsilon), F_\varepsilon, S_\varepsilon, A_0(\varepsilon), A_i(\varepsilon)$ ($A_i(0) := A_i$), $i = 1, 2, 3$; $B_i(\varepsilon), C_i(\varepsilon)$ (truncation vertices on l_i continuous in ε , $B_i(0) := B_i, C_i(0) := C_i$), $i = 1, 2$;

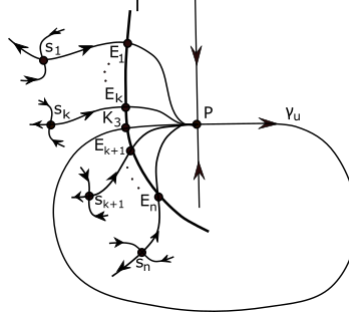


Figure 4: v_0 : the order of intersection points of separatrices with Γ^- in case HC

edges: arcs on transversal loops: $(B_i(\varepsilon), A_i(\varepsilon)), (A_i(\varepsilon), C_i(\varepsilon))$ on l^\pm , $i = 1, 2$;
 $(A_0(\varepsilon), D_1(\varepsilon)), (D_1(\varepsilon), D_2(\varepsilon)), \dots, (D_k(\varepsilon), A_3(\varepsilon)), \dots, (D_n(\varepsilon), A_0(\varepsilon))$ on l_{F_ε} ;
truncated separatrices: $(s_j(\varepsilon), D_j(\varepsilon))$, $j = 1, \dots, n$; $(S_\varepsilon, A_i(\varepsilon))$, $i = 0, 1, 2, 3$.

For $\varepsilon > 0$, let $K_3(\varepsilon)$ be the intersection point of c_ε with Γ^- .

Then the order of points on Γ^- is the following:
 $E_1(\varepsilon), \dots, E_k(\varepsilon), K_3(\varepsilon), E_{k+1}(\varepsilon), \dots, E_n(\varepsilon)$.

Let us take $U \subset \Gamma^-$ an open subset containing $E_1(\varepsilon), \dots, E_k(\varepsilon)$. If ε is small enough then U can be chosen so that the Poincaré map $\Delta_{\varepsilon, ext} : U \rightarrow l^+$ is well defined. For some neighbourhood $V \subset \Gamma^-$ of E_{k+1}, \dots, E_n the Poincaré map $\Delta_{\varepsilon, int} : V \rightarrow l^-$ is also well defined. Thus the map Δ (we skip the indication on its dependence on ε): $U \cup V \rightarrow l^+ \cup l^-$ is well defined.

Let $D_j(\varepsilon) := \Delta(E_j(\varepsilon)) \in l^+ \cup l^-$. The order of points $D_1(\varepsilon), \dots, D_k(\varepsilon)$ on l^+ is the same as the order of E_1, \dots, E_k on Γ^- over K_3 . The order of points $D_{k+1}(\varepsilon), \dots, D_n(\varepsilon)$ on l^- is the same as the order of E_{k+1}, \dots, E_n on Γ^- under K_3 .

Let us construct N_ε for $\varepsilon > 0$:

vertices: $s_j(\varepsilon), D_j(\varepsilon)$, $j = 1, \dots, n$, B_i, C_i , $i = 1, 2$; one vertex on the cycle c_ε ; one vertex on l^\pm if this transversal loop is empty;

edges: cycle c_ε ;

truncated separatrices: $(s_j(\varepsilon), D_j(\varepsilon))$, $j = 1, \dots, n$;

arcs on transversal loops: $(B_i(\varepsilon), C_i(\varepsilon))$, $i = 1, 2$; $(D_1(\varepsilon), D_2(\varepsilon)), \dots, (D_k(\varepsilon), D_1(\varepsilon))$;
 $(D_{k+1}(\varepsilon), D_{k+2}(\varepsilon)), \dots, (D_n(\varepsilon), D_{k+1}(\varepsilon))$.

This completes the description of the bifurcating subgraph N_ε of the vector field v_ε in the class HC.

SC.

The bifurcation in the neighbourhood of the saddle connection is the following: the separatrix connection γ_0 vanishes, and two new separatrices $\gamma_0^1(\varepsilon)$ of $s_1(\varepsilon)$ and $\gamma_0^2(\varepsilon)$ of $s_2(\varepsilon)$ appear. In our case $\gamma_0^1(\varepsilon)$ is unstable and $\gamma_0^2(\varepsilon)$ is a stable separatrix.

The saddles s_i , $i = 1, 2$ are the vertices of X_0 , hence their deformations are already defined. The deformations of their separatrices contained in X_0 are also defined. Let us define $\gamma_1(\varepsilon), \gamma_2(\varepsilon)$ the separatrices of $s_1(\varepsilon)$ such that $(\gamma_1(\varepsilon), s_1(\varepsilon))$ and $(\gamma_2(\varepsilon), s_1(\varepsilon))$ are the germs of (γ_1, s_1) and (γ_2, s_1) respectively. In the same way we define $\gamma_3(\varepsilon)$ and $\gamma_4(\varepsilon)$.

The deformations of points A_i, B_i , $i = 1, 2, 3, 4$, are already defined too.

- for $\varepsilon < 0$, the ω -limit set of $\gamma_0^1(\varepsilon)$ is $\omega(\gamma_4(\varepsilon))$, the α -limit set of $\gamma_0^2(\varepsilon)$ is $\alpha(\gamma_1(\varepsilon))$, see Fig.2d.
- for $\varepsilon > 0$, the ω -limit set of $\gamma_0^1(\varepsilon)$ is $\omega(\gamma_3(\varepsilon))$, the α -limit set of $\gamma_0^2(\varepsilon)$ is $\alpha(\gamma_2(\varepsilon))$.

Denote by $D_1(\varepsilon), D_2(\varepsilon)$ the truncation vertices of $\gamma_0^1(\varepsilon), \gamma_0^2(\varepsilon)$ respectively.

For $\varepsilon < 0$ $D_1(\varepsilon)$ is at the arc $(A_4(\varepsilon), B_4(\varepsilon))$ arbitrarily close to $A_4(\varepsilon)$. $D_2(\varepsilon)$ lies on the arc $(A_1(\varepsilon), B_1(\varepsilon))$ arbitrarily close to $A_1(\varepsilon)$.

We obtain the similar description for $\varepsilon > 0$ changing the indexes of transversal loops.

Now we construct N_ε :

For $\varepsilon < 0$:

vertices: $s_i(\varepsilon), D_i(\varepsilon)$, $i = 1, 2$; $A_j(\varepsilon), B_j(\varepsilon)$, $j = 1, 2, 3, 4$;

edges: truncated separatrices: $(s_i(\varepsilon), D_i(\varepsilon))$, $i = 1, 2$, $(s_1(\varepsilon), A_1(\varepsilon)), (s_1(\varepsilon), A_2(\varepsilon)), (s_2(\varepsilon), A_3(\varepsilon)), (s_2(\varepsilon), A_4(\varepsilon))$;

arcs on transversal loops: $(A_4(\varepsilon), D_1(\varepsilon)), (D_1(\varepsilon), B_4(\varepsilon)), (A_1(\varepsilon), D_2(\varepsilon)), (D_2(\varepsilon), B_1(\varepsilon)); (A_2(\varepsilon), B_2(\varepsilon)), (A_3(\varepsilon), B_3(\varepsilon))$;

For $\varepsilon < 0$:

vertices: $s_i(\varepsilon), D_i(\varepsilon)$, $i = 1, 2$; $A_j(\varepsilon), B_j(\varepsilon)$, $j = 1, 2, 3, 4$;

edges: truncated separatrices: $(s_i(\varepsilon), D_i(\varepsilon))$, $i = 1, 2$, $(s_1(\varepsilon), A_1(\varepsilon)), (s_1(\varepsilon), A_2(\varepsilon)), (s_2(\varepsilon), A_3(\varepsilon)), (s_2(\varepsilon), A_4(\varepsilon))$;

arcs on transversal loops: $(A_3(\varepsilon), D_1(\varepsilon)), (D_1(\varepsilon), B_3(\varepsilon)), (A_2(\varepsilon), D_2(\varepsilon)), (D_2(\varepsilon), B_2(\varepsilon)); (A_1(\varepsilon), B_1(\varepsilon)), (A_4(\varepsilon), B_4(\varepsilon))$.

This completes the description of the bifurcating subgraph N_ε of the vector field v_ε in all four cases of degeneracies.

4 Proof of the structural stability criterion: isotopy of the LMF graphs

In this section we will prove Theorem 6. Let us begin with general arguments that do not depend on the class considered.

4.1 General part

Let V and W be unfoldings of topologically orbitally equivalent vector fields v_0 and w_0 of classes AH, SN, HC, SC . Let \hat{H} be a homeomorphism $\mathbb{S}^2 \rightarrow \mathbb{S}^2$ that links v_0 and w_0 . In what follows, we will make some assumptions on \hat{H} that may be obtained without loss of generality.

Suppose that the families V and W are parametrized as in Section 3.4. We will prove that the LMF-graphs M_ε for v_ε and \tilde{M}_ε for w_ε are isotopic. For this we will construct a map $G_\varepsilon : M_\varepsilon \rightarrow \tilde{M}_\varepsilon$ and check star and annuli-faces condition (defined in section 2.4) for it. Recall that we call them **S** and **A** conditions respectively.

For all the elements and subgraphs of M_ε described in Section 3, let the same notation with tilde denote the corresponding object for the graph \tilde{M}_ε . Let $\tilde{\pi}_\varepsilon : \tilde{X}_\varepsilon \rightarrow \tilde{X}_0$ be a homeomorphism defined in Section 3.3.

On the graph X_ε the map G_ε is defined by a general formula:

$$G_\varepsilon|_{X_\varepsilon} := \tilde{\pi}_\varepsilon^{-1} \circ \hat{H}|_{X_\varepsilon} \circ \pi_\varepsilon : X_\varepsilon \rightarrow \tilde{X}_\varepsilon.$$

Let G_ε bring the following elements of N_ε - singular points, separatrices and limit cycles - to the corresponding elements of \tilde{N}_ε .

The map G_ε is now almost defined. We should check that it may be correctly extended to oriented transversal loops. Namely, we have to check that the corresponding truncation vertices of N_ε and \tilde{N}_ε follow on the corresponding oriented transversal loops in the same order.

After that we will check **S** and **A** conditions.

The map $G_\varepsilon|_{X_\varepsilon}$ can be extended to a homeomorphism of the sphere; so, it satisfies both **S** and **A** conditions. We have to check the **S** condition for the vertices of N_ε only, and the **A** condition for the faces whose boundary has a non-empty intersection with N_ε .

This is done for the classes AH, SN, HC, SC one by one.

4.2 Class AH

There are no truncation vertices on the transversal loops of the graph N_ε , so there is nothing to check: the map $G_\varepsilon : M_\varepsilon \rightarrow \tilde{M}_\varepsilon$ is well defined.

The only vertices of N_ε are: P_ε for $\varepsilon < 0$; P_ε , two vertices on the empty transversal loops l_1, l_2 , and one vertex on the limit cycle c_ε . The star condition is obviously satisfied for them.

Let us check the **A** condition. For $\varepsilon < 0$ the only one annuli-shaped face whose boundary intersects N_ε is bounded by P_ε and l . The map $G_\varepsilon : P_\varepsilon \mapsto \tilde{P}_\varepsilon, l \mapsto \tilde{l}$ is trivially extended to a homeomorphism $D \rightarrow \tilde{D}$, where D (\tilde{D}) is a disc bounded by l (\tilde{l}) and containing P_ε (\tilde{P}_ε).

For $\varepsilon > 0$, the union $N_\varepsilon \cup l$ is homeomorphic to a union of four concentric circles l_1, c, l_2, l_0 , each one inside the next, and their center O ; let H_ε be the corresponding homeomorphism. The same is true for the union $\tilde{N}_\varepsilon \cup \tilde{l}$; we note \tilde{H}_ε the corresponding homeomorphism. Let D_0 be the disk bounded by l_0 with the center O . The homeomorphisms H_ε and \tilde{H}_ε can be extended to homeomorphisms $D \rightarrow D_0$ and $\tilde{D} \rightarrow D_0$ still denoted by H_ε and \tilde{H}_ε .

Without loss of generality we may assume that $G_\varepsilon|_{N_\varepsilon \cup l} = \tilde{H}_\varepsilon^{-1} \circ H_\varepsilon|_{N_\varepsilon \cup l}$. This map clearly satisfies the **A** condition.

Hence, graphs M_ε and \tilde{M}_ε are isotopic and therefore the families V and W are weakly equivalent. Theorem 6 for the class AH is proved.

4.3 Class SN

Let us prove that the map $G_\varepsilon : M_\varepsilon \rightarrow \tilde{M}_\varepsilon$ is well defined. For this we have to prove that the truncation vertices of the graphs M_ε and \tilde{M}_ε follow in the same order on the corresponding oriented transversal loops.

The points $A_j(0), B_j(0)$ and $C_j(0)$ on l_j are mapped to $\tilde{A}_j(0), \tilde{B}_j(0)$ and $\tilde{C}_j(0)$ by \hat{H} ; the homeomorphism \hat{H} preserves the order of the points on l_j , $j = 1, 2, 3$. The points $A_j(\varepsilon), B_j(\varepsilon), C_j(\varepsilon)$ and $\tilde{A}_j(\varepsilon), \tilde{B}_j(\varepsilon), \tilde{C}_j(\varepsilon)$ are continuous in ε . Hence, they still follow in the same order on l_j and \tilde{l}_j .

Consider the curves l_{F_ε} and $\tilde{l}_{\tilde{F}_\varepsilon}$. Without loss of generality, we may assume that $\hat{H}(\Gamma^-) = \tilde{\Gamma}^-$. Then $\hat{H}(E_j) = \tilde{E}_j$. Hence, the order of points \tilde{E}_j on $\tilde{\Gamma}^-$ is the same as of E_j on Γ^- . The same holds for the points $E_j(\varepsilon)$ and $\tilde{E}_j(\varepsilon)$ because they depend continuously on ε for $\varepsilon \in (\mathbb{R}, 0)$. The Poincaré maps Δ_ε and $\tilde{\Delta}_\varepsilon$ preserve the order both for $\varepsilon < 0$

and $\varepsilon > 0$. Hence, the truncation vertices $\tilde{D}_1(\varepsilon), \dots, \tilde{D}_n(\varepsilon)$ follow on $\tilde{l}_{\tilde{F}_\varepsilon}$ for $\varepsilon < 0$ (on \tilde{l}_3 for $\varepsilon > 0$) in the same order as $D_1(\varepsilon), \dots, D_n(\varepsilon)$. For $\varepsilon < 0$, by the reduction principle, the point $\tilde{A}_0(\varepsilon) \in \tilde{l}_{\tilde{F}_\varepsilon}$ belongs to the arc from $\tilde{D}_n(\varepsilon)$ to $\tilde{D}_1(\varepsilon)$, like $A_0(\varepsilon)$ does. So the order of truncation vertices on the transversal loops of v_ε is preserved by G_ε .

Let us check **S** condition for G_ε . For the truncation vertices it is trivial. For the saddles s_j and the map $G_0 := \hat{H}|_{M_0}$ it follows from the fact that \hat{H} is a homeomorphism. The deformations of saddles s_j, \tilde{s}_j and the germs of their separatrices are continuous in ε . This implies **S** condition for G_ε at the saddles $s_j(\varepsilon)$.

Let us check this condition at the saddle S_ε for $\varepsilon < 0$. Note that the germs $(\gamma_s^1(\varepsilon), S_\varepsilon), (\gamma_s^2(\varepsilon), S_\varepsilon), (\gamma_u^1(\varepsilon), S_\varepsilon)$ tend to $(\gamma_s^1, P), (\gamma_s^2, P), (\gamma_u, P)$ respectively as $\varepsilon \nearrow 0$. For $\varepsilon = 0$, $G_0 := \hat{H}|_{M_0}$ and $\hat{H}((\gamma_u, P)) = (\tilde{\gamma}_u, \tilde{P})$, $\hat{H}((\gamma_s^i, P)) = (\tilde{\gamma}_s^i, \tilde{P})$, $i = 1, 2$. By continuity in ε , this implies that G_ε preserves the order of separatrices at S_ε . **S** condition for G_ε is checked.

Let us check the **A** condition. For both $\varepsilon < 0$ and $\varepsilon > 0$ there are annuli-shaped faces that have a common boundary with N_ε : the faces $F_i(\varepsilon)$ that have exterior boundaries l_i , $i = 1, 2, 3$ and hyperbolic attractors or repellers (hyperbolic singular points or limit cycles) as the interior boundaries. If this interior boundary is a singular point then G_ε can be trivially extended to the corresponding face. Let $F_i(\varepsilon)$ have the boundary $l_i \cup c(\varepsilon)$, where $c(\varepsilon)$ is a limit cycle. We construct the extension of $\pi_\varepsilon : X_\varepsilon \rightarrow X_0$ to the closure of $F_i(\varepsilon)$ which is well defined because all the vertices on l_i and $c(\varepsilon)$ for $\varepsilon \neq 0$ depend continuously on ε for $\varepsilon \rightarrow 0$. The extension will be also denoted by π_ε here. We construct the similar extension of $\tilde{\pi}_\varepsilon : \tilde{X}_\varepsilon \rightarrow \tilde{X}_0$. Then $\tilde{\pi}_\varepsilon^{-1} \circ \hat{H} \circ \pi_\varepsilon : cl(F_i) \rightarrow cl(\tilde{F}_i)$ is an extension of G_ε to the face $F_i(\varepsilon)$.

For $\varepsilon < 0$ there is an annular face of the graph M_ε , the one between F_ε and l_{F_ε} . For it the **A** condition is obvious. No other annuli-shaped faces of the graph M_ε whose boundary intersects N_ε exist. Indeed, for $\varepsilon < 0$ the graph N_ε contains saddle separatrices intersecting transversal loops; but the boundaries of the annuli-shaped faces of M_ε contain no saddles, see Theorem 7. This checks the **A** condition for G_ε , $\varepsilon < 0$.

For $\varepsilon > 0$ this argument does not work: the arcs $(B_i(\varepsilon), C_i(\varepsilon))$ on the topological circles l_i , $i = 1, 2$ are connected components of N_ε . But still l_i are not the inner components of the boundary of the annuli-shaped faces of M_ε : they contain the truncation vertices for $\varepsilon \in (\mathbb{R}, 0)$, hence, if the boundary of a face contains a part of l_i , it contains also the corresponding truncated saddle separatrix. This checks the **A** condition for G_ε in the class SN.

4.4 Class HC.

Let us prove that G_ε preserves the order of vertices on the transversal loops of M_ε . Below we use the notations from Section 3.4.

For $\varepsilon < 0$ we will prove it only for l_{F_ε} , for $\varepsilon > 0$ only for l^+ and l^- . All the other transversal loops are already considered in the case SN. As in the previous subsection we can assume that $\hat{H}(\Gamma^-) = \tilde{\Gamma}^-$, $\hat{H}(E_j) = \tilde{E}_j$, $j = 1, \dots, n$ and $\hat{H}(K_3) = \tilde{K}_3$. So the points E_j, K_3 are met on Γ^- in the same order as \tilde{E}_j, \tilde{K}_3 on $\tilde{\Gamma}^-$, namely, $E_1, \dots, E_k, K_3, E_{k+1}, \dots, E_n$.

In case $\varepsilon < 0$, as it was mentioned in the Section 3.4, the order of vertices on l_{F_ε} is $A_0(\varepsilon), D_1(\varepsilon), \dots, D_k(\varepsilon), A_3(\varepsilon), D_{k+1}(\varepsilon), \dots, D_n(\varepsilon)$. The same holds for $\tilde{l}_{\tilde{F}_\varepsilon}$. Hence, $G_\varepsilon : D_j(\varepsilon) \mapsto \tilde{D}_j(\varepsilon)$, $j = 1, \dots, n$, $A_i \mapsto \tilde{A}_i$, $i = 0, 3$, preserves the order of vertices on the transversal loop l_{F_ε} .

In case $\varepsilon > 0$, the order of points $D_1(\varepsilon), \dots, D_k(\varepsilon)$ on l^+ is the same as for the points E_1, \dots, E_k , the order of points $D_{k+1}(\varepsilon), \dots, D_n(\varepsilon)$ on l^- is the same as for the points

E_{k+1}, \dots, E_n ; the same holds for \tilde{l}^+ and \tilde{l}^- . Hence, G_ε that sends the vertices on l^\pm , to the vertices on \tilde{l}^\pm with the same indexes, keeps their order the same.

The **S** condition can be checked as in the case SN.

Let us check the annuli-faces condition. For $\varepsilon < 0$ the annuli-faces that have the common boundary with N_ε are: the face between F_ε and its transversal loop l_{F_ε} ; the face with the exterior boundary l_i , $i = 1, 2$. These faces are considered in the previous subsection.

For $\varepsilon > 0$, there are the annuli-faces with the exterior boundaries l_i , $i = 1, 2$, too. In addition two faces between the limit cycle c_ε and its transversal loops l^+ and l^- appear. The map G_ε preserves the orientation on transversal loops and the orientation on the cycle c_ε (with regard to the orientation on \mathbb{S}^2). Hence, it can be extended to the faces described.

Let us suppose that the transversal loop l_i is an interior boundary of an annular face $F_i(\varepsilon)$ of the graph M_ε , $i = 1, 2$. Without loss of generality we can assume $i = 1$. In this case there is no saddle separatrix of v_0 that intersects l_1 ; hence, P as the ω -limit set of all the trajectories that intersect l_1 . Then the annular face $F_1(\varepsilon)$ should be bounded by l_1 and l^+ and so v_ε is topologically equivalent to the north-south vector field on $F_1(\varepsilon)$. As G_ε preserves the orientation on the transversal loops, it can be extended from $l_1 \cup l^+$ to the closure of $F_1(\varepsilon)$.

Hence, the **A** condition is checked for the class HC.

4.5 Class SC

Let us prove that G_ε is well defined. We consider $\varepsilon < 0$, the case $\varepsilon > 0$ can be obtained changing the indexes.

As it was said in Section 3, the truncation vertices $D_1(\varepsilon), D_2(\varepsilon)$ belong to the arcs $(A_4(\varepsilon), B_4(\varepsilon))$, $(A_1(\varepsilon), B_1(\varepsilon))$ and can be arbitrarily close to the points $A_4(\varepsilon), A_1(\varepsilon)$ respectively.

The map G_ε brings the order of points of X_ε on l_i to that of \tilde{X}_ε on \tilde{l}_i . So it sends the arc between $A_i(\varepsilon)$ and $B_i(\varepsilon)$ to the arc between $\tilde{A}_i(\varepsilon)$ and $\tilde{B}_i(\varepsilon)$. \hat{H} sends the saddle connection γ_0 to the saddle connection $\tilde{\gamma}_0$, so it sends the arc (A_i, B_i) to the arc $(\tilde{A}_i, \tilde{B}_i)$, not to its complement - we recall that we denoted by (A_i, B_i) the arcs on l_i such that all the trajectories, that are close to γ_0 , intersect one of the arcs (A_i, B_i) . Hence, $G_\varepsilon : (A_i(\varepsilon), B_i(\varepsilon)) \mapsto (\tilde{A}_i(\varepsilon), \tilde{B}_i(\varepsilon))$, $i = 1, 2, 3, 4$, $D_j(\varepsilon) \mapsto \tilde{D}_j(\varepsilon)$, $j = 1, 2$, can be correctly prolonged on transversal loops.

Let us check the star condition. For truncation vertices it is trivial. Let us check it for the saddles $s_1(\varepsilon)$ and $s_2(\varepsilon)$. The order of separatrices $\gamma_0, \gamma_1, \gamma_2$ at s_1 is the same as the order of $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_2$ at \tilde{s}_1 ; the order of $\gamma_0, \gamma_4, \gamma_3$ at s_2 is the same as the order of $\tilde{\gamma}_0, \tilde{\gamma}_4, \tilde{\gamma}_3$ at \tilde{s}_2 . The same holds for deformations $\gamma_i(\varepsilon)$ because the germs $(\gamma_0^k(\varepsilon), s_k(\varepsilon))$, $k = 1, 2$, $(\gamma_j(\varepsilon), s_i(\varepsilon))$ (where $j = 1, 2, 3, 4$, $i = 1, 2$ and $\gamma_j(\varepsilon)$ is a separatrix of a saddle $s_i(\varepsilon)$) are continuous in ε . The same holds for the vector field w_ε , $\varepsilon \in (\mathbb{R}, 0)$. As G_ε maps the truncated separatrices of v_ε to the corresponding separatrices of w_ε , it preserves the order of three separatrices at $s_1(\varepsilon)$ and at $s_2(\varepsilon)$. Hence, it preserves the order of all the separatrices at $s_1(\varepsilon)$ and $s_2(\varepsilon)$.

Let us check the **A** condition. The only annuli-shaped faces of M_ε that have a common boundary with N_ε are the faces with the exterior boundaries l_i , $i = 1, 2, 3, 4$. G_ε can be extended to them similarly to the case of transversal loops l_j , $j = 1, 2, 3$ in SN.

Finally, we proved theorem 6.

5 Structural stability of local families.

In this section we prove Theorem 3 mentioned in section 1.4.

Proof. We use the following theorem.

Theorem 12 (Sotomayor theorem, [S]). *Quasi-generic vector fields form an open and dense subset of the set Σ of structurally unstable vector fields. Quasi-generic vector fields are structurally stable in the set Σ .*

Let $\{v_\varepsilon\}$ be a generic one-parameter vector field, let v_0 be of class AH, SN, HC or SC . Then, by the theorem below, v_0 is structurally stable in Σ and particularly it is structurally stable in the corresponding hypersurface of generic vector fields (AH, SN, HC or SC). For generality we will call this hypersurface \mathcal{H} . Let $U \subset \mathcal{H}$ be a neighbourhood of v_0 such that $\forall w \in U \ v_0 \sim w$.

Let V be a generic local family that unfolds a vector field from the class AH, SN, HC, SC . Let W be another such family close to V . If these two families are close enough, then, by the Sotomayor theorem, vector fields v_0 and w_0 are orbitally topologically equivalent. By Theorem 6, families V and W are weakly equivalent. \square

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