Global bifurcations in generic one-parameter families with a parabolic cycle on S^2

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Abstract

We classify global bifurcations in generic one-parameter local families of vector fields on S^2 with a parabolic cycle. The classification is quite different from the classical results presented in monographs on the bifurcation theory. As a by product we prove that generic families described above are structurally stable.

Key words: bifurcations, polycycles, structural stability, sparkling saddle connections
Mathematics subject classification: 34C23, 37G99, 37E35

1 Introduction

1.1 Main results

This article is a part of a larger investigation whose main goals are:

- To prove structural stability of generic one-parameter families of vector fields in the two-sphere;
- To give a complete classification of the bifurcations in these families with respect to the weak equivalence relation (the definition is recalled below).

These goals are achieved in [IS], [St], and the present paper.

Structural stability result is well expected. It was predicted in [S]; the sketch of the proof (of another but close result) was given in [MP] with the note: "Full proofs will appear in a forthcoming paper." To the best of our knowledge, that paper was not written. Here we prove structural stability for unfoldings of parabolic limit cycles, which constitutes the first of the two main results of our paper.

Complete classification of the bifurcations seems to be quite unexpected. Global bifurcations in generic families that unfold a vector field with a separatrix loop are characterized by a finite set on a circle considered up to a homeomorphism [IS]. Global bifurcations in generic families that unfold a vector field with a parabolic cycle are characterized by two finite sets on a coordinate circle $\mathbb{R}/\mathbb{Z}$, considered up to a certain equivalence relation (see Definition 12). The latter result is the second main theorem of our paper. The precise statements follow (see Theorem 1 and Theorem 6).

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Together with [IS] and [St], this paper achieves the goals stated at the beginning, and thus completes the study of global bifurcations in one-parameter families.

These results are a part of a large program of the development of the global bifurcation theory on the sphere outlined in [I16]. There was a belief, formulated by V.Arnold as a conjecture in [AAIS, Sec. I.3.2.8], that a generic family is structurally stable up to a weak equivalence: close finite-parameter families are weakly equivalent. This natural conjecture turns out to be false for 3-parameter families, as was proved in a recent work [IKS] (with weak equivalence replaced by moderate equivalence, which is a technical difference). Namely, the authors prove that the moderate classification of families with "tears of the heart" polycycle has numerical moduli, and generic family of this class is *not* structurally stable. The effect is due to *sparkling saddle connections* that accumulate to the polycycle; their order is different for close families, which implies the statement.

Now the following problems arise:

- Find out whether 2-parameter families of vector fields on S^2 are structurally stable (up to the weak equivalence);
- Classify their bifurcations;
- Distinguish structurally stable three-parameter families from the unstable ones, and find new examples of structurally unstable three-parameter families.

These problems are natural steps that follow the present paper. Let us pass to the detailed presentation.

By default, *vector fields* below are infinitely smooth vector fields on S^2 with isolated singular points, and *families* are families of vector fields on S^2. The sphere is oriented, and all the homeomorphisms of S^2 under consideration preserve orientation.

1.2 Vector fields of class PC

Definition 1. *A C^2 vector field of class PC* is a vector field with a parabolic limit cycle γ and no other degeneracies. Namely, the following assumptions hold:

- all the singular points and limit cycles of the vector field except for γ are hyperbolic;
- the vector field has no saddle connections;
- the parabolic cycle γ is of multiplicity 2, that is, its Poincaré map has the form $x \mapsto x + ax^2 + \dots,\ a \neq 0$.

Vector fields of class PC form an immersed Banach manifold of codimension one in the space of C^r-smooth vector fields on S^2, $r \geq 3$, see [S, Proposition 2.2].

1.3 Structural stability

Let us recall some basic definitions.

Definition 2. Two vector fields v and w on S^2 are called *orbitally topologically equivalent*, if there exists a homeomorphism $S^2 \to S^2$ that links the phase portraits of v and w, that is, sends orbits of v to orbits of w and preserves their time orientation.

In this article, we consider one-parameter families of vector fields on $B \times S^2$. Here $B \subset \mathbb{R}$ is the base of the family. We work with local families with bases $(\mathbb{R}, 0)$, in the following sense.

Definition 3. A *local family* of vector fields at $\varepsilon = 0$ is a germ at $\{0\} \times S^2$ of a family on $B \times S^2$, where $B \ni 0$, $B \subset \mathbb{R}$ is open.

Definition 4. An *unfolding* of a vector field v is a local family for which v corresponds to zero parameter value. We say that this family *unfolds* the vector field.

The following definition lists the notions of equivalence for local families of vector fields that we will use in this paper.

Definition 5. Let B, B' be topological balls in $\mathbb{R}$ that contain 0. Two local families of vector fields $\{v_\alpha, \alpha \in B\}$, $\{w_\beta, \beta \in B'\}$ on S^2 are called

- *weakly topologically equivalent* if there exists a map

$$H \colon B \times S^2 \to B \times S^2, \qquad H(\alpha, x) = (h(\alpha), H_\alpha(x))$$

  such that h is a homeomorphism of the bases, $h(0) = 0$, and for each $\alpha \in B$ the map $H_\alpha : S^2 \to S^2$ is a homeomorphism that takes the phase portrait of v_α to the phase portrait of $w_{h(\alpha)}$.

- *sing*-equivalent if H is continuous on the union of all the singular points and hyperbolic limit cycles of the vector field v_0.

- *strongly topologically equivalent* provided that the map H above is continuous.

Weak equivalence is also called mild equivalence in some sources.

Definition 6. A local family of vector fields is called *weakly structurally stable* if it is weakly topologically equivalent to any nearby family.

Theorem 1. *A generic one-parameter unfolding of a generic vector field of class PC is weakly structurally stable.*

Vector fields from this theorem have to satisfy an extra genericity assumption in addition to those included in the definition of class PC. This assumption is presented in Sec. 1.7, where an improved version of Theorem 1 is stated.

The genericity assumption for the unfolding in Theorem 1 is *transversality to PC*.

The previous theorem is wrong if the weak equivalence is replaced by the strong equivalence, see [MP].

Remark 1. Sing-equivalence has the following property. Let $V = \{v_\varepsilon\}$ and $W = \{w_\delta\}$ be two sing-equivalent families. For any singular point O of v_0, let $O(\varepsilon)$ be a singular point of v_ε depending continuously on ε and such that $O(0) = O$. Put $\tilde{O} = H_0(O)$ and let $\tilde{O}(\delta)$ be a similarly defined singular point of w_δ. Then

$$H_\varepsilon(O(\varepsilon)) = \tilde{O}(h(\varepsilon)). \tag{1}$$

The same holds for limit cycles of v_ε, w_δ.

Sing-equivalence is designed to imply this property.

1.4 Time function

The following arguments are based on the heuristic principle: local dynamics near an equilibrium point usually determines a canonical chart at this point.

Let v be a vector field of class PC, γ be its parabolic limit cycle. Let Γ be a cross-section to γ, x a smooth chart on it with $x(\gamma \cap \Gamma) = 0$. Let P be a germ of the corresponding Poincaré map, $P(x) = x + ax^2 + \dots$. By assumption, $a \neq 0$. Rescaling x and changing sign if needed we will make $a = 1$; so we will assume that

$$P(x) = x + x^2 + \dots \tag{2}$$

Theorem 2 (Takens, [T])**.** *Let P be a C^∞-smooth parabolic germ of the form* (2)*. Then it has an infinitely smooth generator: there exists a germ of a vector field $u(x) = x^2 + \dots$ at zero, whose time one phase flow transformation equals P:*

$$P = g_u^1.$$

The smooth generator u of P is unique.

Let Γ be a cross-section to γ, put $O = \Gamma \cap \gamma$, and let x be a chart on Γ with $x(O) = 0$ in which P has the form (2). Let Γ^+ and Γ^- be the parts of Γ where $x \geq 0$ and $x \leq 0$ respectively. Define the *time functions* on Γ^+ and Γ^-, unique up to adding a constant, in the following way. Choose two small numbers $b^- < 0 < b^+$, and let $T^+(b)$ be the time of the motion from the point b^+ to the point $b \in \Gamma^+ \setminus \{0\}$ along the solution of the equation $\dot{x} = u(x)$; let $T^-(b)$ be the time of the motion from the point b^- to the point $b \in \Gamma^- \setminus \{0\}$ along the solution of the equation $\dot{x} = u(x)$. In other words,

$$T^+(b) = \int_{b^+}^{b} \frac{dx}{u(x)} \text{ for } b > 0,\ b \in \Gamma^+,$$

$$T^-(b) = \int_{b^-}^{b} \frac{dx}{u(x)} \text{ for } b < 0,\ b \in \Gamma^-.$$

1.5 Large bifurcation support

The bifurcations in a local family that unfolds a vector field of class PC are not only reduced to splitting and vanishing of the limit cycle γ. They also produce so called *sparkling saddle connections* discovered by Malta and Palis [MP].

Suppose that the vector field $v \in PC$ has two saddles E and I on different sides of γ whose separatrices wind towards γ in the positive and negative time respectively. The saddle E lies outside, and the saddle I inside γ; E and I stand for "exterior" and "interior". Let $V = \{v_\varepsilon\}$ be an unfolding of v transversal to PC, $v_0 = v$, $E(\varepsilon)$ and $I(\varepsilon)$ be the saddles of v_ε continuous in ε and such that $E(0) = E, I(0) = I$. Let γ disappear for $\varepsilon > 0$. Then there exists a sequence $\varepsilon_n \searrow 0$ such that the vector fields v_{ε_n} have saddle connections between the saddles $E(\varepsilon_n), I(\varepsilon_n)$. These connections are called *sparkling saddle connections.*

This motivates the following definition.

Definition 7. Let $v \in PC$. *The large bifurcation support of* v is the union of the parabolic cycle γ and the closures of all the separatrices of the hyperbolic saddles that wind towards γ in the negative or positive time.

Remark 2. Large bifurcation supports are defined in a much more general setting in [I16].

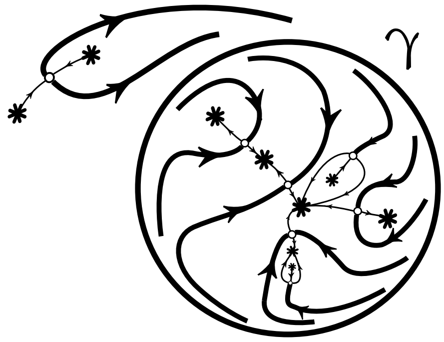

Figure 1: Large bifurcation support of a PC vector field (shown in thick curves). Here and below asterisks show sinks and sources of a vector field

The term is motivated by the heuristic statement that all the bifurcations that occur in the generic unfolding of v are determined by those in a neighborhood of the large bifurcation support of v. For vector fields of class PC this follows from Theorem 6 below. In the general setting it is proved in [GI*], work in progress.

The large bifurcation supports for the vector fields of class PC are characterized by two so called marked finite sets on a circle.

1.6 Marked finite sets

Large bifurcation supports may be rather complicated, see Figure 1.

Yet they admit a simple combinatorial description. The Poincaré map on Γ^-, as well as on Γ^+, in the charts $T^\pm$, is the mere translation by 1:

$$T^\pm(P(x)) = T^\pm(x) + 1.$$

Hence, the set of orbits of P on $\Gamma^\pm$ is a coordinate circle $S^1_\pm = \mathbb{R}^+/\mathbb{Z}$, the coordinate is $T^\pm \pmod{\mathbb{Z}}$. Note that this coordinate is defined uniquely up to an additive constant that depends on the choice of $b^\pm$ in Sec. 1.4.

Denote by D^+ the set of all intersection points of the separatrices that wind toward γ with the half open segment $[b^+, P(b^+)),\ b^+ \in \Gamma^+$. In the same way the set D^- is defined for $b^- \in \Gamma^-$. Let

$$A^\pm = T^\pm(D^\pm) \pmod{\mathbb{Z}},\ A^\pm \subset S^1_\pm. \tag{3}$$

Let us define the equivalence relations on D^+ and D^-. Namely, two points of D^+ (or D^-) are equivalent if they belong to the separatrices of the same saddle. This induces equivalence relations on A^+ and A^-. Note that any two equivalence classes $(a, b), (c, d)$ are not intermingled on the oriented circle: either both points c, d belong to an arc from a to b, or none of them.

Definition 8. The equivalence relation on a finite set on a circle is called *proper* if each equivalence class consists of one or two points, and any two classes of two points each are not intermingled in the sense explained above.

A finite set on a circle with a proper equivalence relation is called *marked*.

Thus for any vector field $v \in PC$ a pair of marked sets $A^\pm(v)$ on coordinate circles is defined.

Definition 9. The marked sets $A^{\pm}(v)$ are called the *characteristic pair* (of sets) for the vector field $v \in PC$.

Remark 3. Recall that the time coordinates on coordinate circles are defined modulo additive constants that depend on the choice of $b^{\pm}$ in Sec. 1.4. So the characteristic sets $A^{\pm}(v)$ are defined modulo additive constants.

1.7 Non-synchronization condition

Definition 10. Two finite sets $A^+, A^- \subset S^1$ are *non-synchronized* provided that for any $\alpha \in \mathbb{R}$,

$$\#\left((A^+ + \alpha) \cap A^-\right) \leq 1. \tag{4}$$

We can now give an explicit form of Theorem 1.

Theorem 3. *Suppose that characteristic sets $A^{\pm}(v)$ for a vector field $v \in PC$ are non-synchronized. Then any local one-parameter unfolding of v transversal to the Banach manifold PC is structurally stable in the space of one-parameter families with the C^1 metric on it.*

This theorem is proved in Sec. 4.

Definition 11. One-parameter local families described by this theorem are called $\mathcal{PC}$ *families.*

The following realization theorem holds.

Theorem 4. *Let $A^{\pm}$ be a pair of marked non-synchronized set on the coordinate circles. Then there exists a vector field v of class PC such that $A^{\pm}$ are characteristic sets of v.*

It is proved in Sec. 2.4. The vector field set v whose characteristic sets coincide with $A^{\pm}$ is in no way unique.

1.8 Topologically equivalent vector fields of class PC with non-equivalent one-parameter unfoldings

Theorem 5. *There exist vector fields mentioned in the title of this section. More precisely, there exist topologically equivalent vector fields of class PC whose generic one-parameter unfoldings are not sing-equivalent.*

We prove this theorem in Sec. 5.

We conjecture that this is the only result of this kind: bifurcations in other generic one-parameter unfoldings are determined by the topology of the phase portrait of an unperturbed vector field.

Conjecture 1. *Consider two generic one parameter local families of vector fields $\{v_\varepsilon\}$ and $\{w_\delta\}$ such that v_0, w_0 are not of class PC. Suppose that v_0, w_0 are orbitally topologically equivalent. Then these local families $\{v_\varepsilon\}$ and $\{w_\delta\}$ are weakly equivalent.*

1.9 Classification of $\mathcal{PC}$ families

To any pair of finite sets on coordinate circles ordered counterclockwise and enumerated:

$$A^+ = \{a_1^+, ..., a_K^+\},\ A^- = \{a_1^-, ..., a_M^-\}$$

a set of pairwise differences corresponds:

$$\tau_{km} := \{a_k^+ - a_m^-\}, \quad \Lambda(A^\pm) := \{\tau_{km} \mid k = 1, ..., K; m = 1, ..., M\}, \tag{5}$$

where $\{a_k^+ - a_m^-\} \in [0, 1)$ stands for the fractional part of $a_k^+ - a_m^-$; that is, τ_{km} is the length of the positively oriented arc from a_m^- to a_k^+. A pair $A^\pm$ is non-synchronized iff all the elements of the set $\Lambda(A^\pm)$ are pairwise distinct.

Remark 4. If one of two sets A^-, A^+ is empty, the set $\Lambda(A^\pm)$ is empty.

Remark 5. If we add a shift to one of the sets $A^\pm$, then the shift is added to the set of differences $\Lambda(A^\pm)$: $\Lambda(A^\pm) + \alpha = \Lambda(A^+ + \alpha, A^-)$. This is the reason for considering $\Lambda(B^\pm) + \alpha$ in the following definition.

Definition 12. Two non-synchronized pairs of marked finite sets $A^\pm$ and $B^\pm$ on two circles are *equivalent* if $|A^-| = |B^-|$, $|A^+| = |B^+|$, and for some shift $x \mapsto x + \alpha$, the sets $\Lambda(A^\pm)$ and $\{\Lambda(B^\pm) + \alpha\}$ are ordered in the same way on $[0, 1)$. In more detail, let

$$B^+ = \{b_1^+, ..., b_K^+\},\ B^- = \{b_1^-, ..., b_M^-\}$$

be ordered counterclockwise, and put

$$\lambda_{km} := \{b_k^+ - b_m^-\}, \quad \Lambda(B^\pm) = \{\lambda_{km} \mid k = 1, \dots, K; m = 1, \dots, M\}.$$

Then there exists $\alpha \in \mathbb{R}$ such that

$$\tau_{km} > \tau_{k'm'} \Rightarrow \{\lambda_{km} + \alpha\} > \{\lambda_{k'm'} + \alpha\}. \tag{6}$$

We will use this definition for characteristic pairs of sets, see Definition 8 of Sec. 1.6. Recall that the characteristic sets $A^\pm(v)$ for a vector field v are well-defined up to additive constants, so the equivalence of characteristic sets for two vector fields is well-defined.

Definition 13. Let two vector fields v, w of class PC be orbitally topologically equivalent. Enumerate the sets $A^\pm(v)$ counterclockwise along coordinate circles. This enumeration and the orbital topological equivalence of v, w induce the enumeration of the sets $\tilde{A}^\pm(w)$: the intersections of transversals with corresponding separatrices of v, w will have the same numbers.

Now, the vector fields v and w are said to have *non-synchronized and equivalent characteristic sets* if $A^\pm(v)$ and $A^\pm(w)$ are non-synchronized and equivalent in the sense of Definition 12 (with the numbering described above).

Theorem 6. *1. Let two $\mathcal{PC}$ families be sing-equivalent. Then their characteristic sets on the coordinate circles are equivalent in the sense of Definition 13.*

2. Let two vector fields of class PC be orbitally topologically equivalent. Let their characteristic pairs be non-synchronized and equivalent in the sense of Definition 13. Then the generic unfoldings of these two vector fields are sing-equivalent.

Any pair of non-synchronized marked finite sets determines the bifurcation scenario (sequence of bifurcations) in the corresponding class of $\mathcal{PC}$ families. This scenario will be explicitly described, see Sec. 2.6.

2 Bifurcations in the $\mathcal{PC}$ families

2.1 Embedding theorem for families

Takens embedding Theorem 2 for parabolic germs may be extended to their unfoldings.

Theorem 7. *[IYa] Let P_ε be a generic one-parameter C^∞ unfolding of a parabolic germ*

$$P_\varepsilon(x) = x + x^2 + \varepsilon + \dots .$$

Then in the domain $\{\varepsilon \geq 0\} \setminus \{0, 0\}$, the family P_ε is C^∞ equivalent to the time one phase flow transformation of the field

$$u_\varepsilon(x) = \frac{x^2 + \varepsilon}{1 + a(\varepsilon)x}, \tag{7}$$

where $a(\varepsilon)$ is a C^∞ function; the equivalence is infinitely smooth both in x and ε.

The coordinate x_ε that brings P_ε to the time-one shift of the vector field u_ε is called *normalizing.* From now on, the coordinate on the cross-section Γ for $\varepsilon \geq 0$ is the normalizing coordinate x_ε.

Remark 6. Since the normalizing coordinate x_ε is C^∞ smooth on the set $\{\varepsilon = 0\} \setminus \{0, 0\}$, it may be smoothly extended to some neighborhood of any point of this set. As a corollary, all the derivatives of x_ε at $\varepsilon = 0, x_\varepsilon \neq 0$ exist and are finite.

2.2 Transversal loops and canonical coordinates on them

Consider a vector field v of class PC; let γ be its parabolic cycle, Γ be a cross-section to γ, $O = \Gamma \cap \gamma$, and

$$P : (\Gamma, O) \to (\Gamma, O)$$

be the germ of the Poincaré map corresponding to γ. Consider a generic unfolding $V = \{v_\varepsilon\}$ of v, $v_0 = v$. Let P_ε be the corresponding Poincaré map of v_ε, and x_ε the corresponding normalizing chart on Γ provided by Theorem 7. Let C^+ and C^- be two transversal loops around γ, $C^\pm \cap \Gamma = b^\pm$. For $\varepsilon > 0$, the cycle γ vanishes, and the Poincaré map $\Delta_\varepsilon \colon C^- \to C^+$ is well-defined. We will now choose coordinates $\varphi_\varepsilon^\pm$ on $C^\pm$ such that Δ_ε becomes a rotation in these coordinates.

For $\varepsilon > 0$ consider a one form ω_ε on Γ dual to the vector field u_ε:

$$\omega_\varepsilon = \frac{dx_\varepsilon}{u_\varepsilon(x_\varepsilon)}.$$

For small $b \in \Gamma$ and $\varepsilon > 0$ let

$$T_\varepsilon^\pm(b) = \int_{x_\varepsilon(b^\pm)}^{x_\varepsilon(b)} \omega_\varepsilon. \tag{8}$$

Let

$$\tau(\varepsilon) = T_\varepsilon^-(b^+).$$

This function may be explicitly calculated:

$$\tau(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}(\arctan \frac{x_\varepsilon(b^+)}{\sqrt{\varepsilon}} - \arctan \frac{x_\varepsilon(b^-)}{\sqrt{\varepsilon}}) + \frac{a(\varepsilon)}{2} \log \frac{x_\varepsilon^2(b^+) + \varepsilon}{x_\varepsilon^2(b^-) + \varepsilon}. \tag{9}$$

Note that for $\varepsilon > 0$,

$$T_\varepsilon^-(b) = T_\varepsilon^+(b) + \tau(\varepsilon).$$

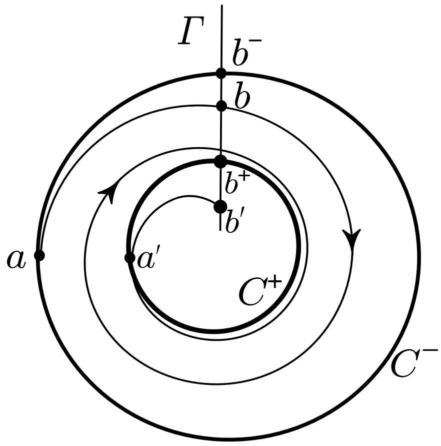

Figure 2: Canonical coordinates and the Poincaré map on the transversal loops

Note that formula (8) works for $\varepsilon = 0$ also, with the following restriction:

$$T_0^- = T^- \text{ on } \Gamma^- \setminus \{0\};$$

$$T_0^+ = \lim_{\varepsilon \to 0} T_\varepsilon^- - \tau(\varepsilon).$$

The time functions $T_\varepsilon^\pm$ induce coordinates $\varphi_\varepsilon^\pm$ on $C^\pm$ in the following way. Consider first C^-. Take a point $a \in C^-$ and emerge a forward orbit of v_ε from it, see Figure 2. Let $b \in \Gamma^-$ be its first intersection point with Γ. Take

$$\varphi_\varepsilon^-(a) = T_\varepsilon^-(b).$$

Note that $\varphi_\varepsilon^-(b^-) = 0$; as a tends to b^-, one of the one-sided limits of φ_ε^- at b^- is 0 and the other is 1. Thus φ_ε^- maps C^- onto the coordinate circle S_-^1. The same construction provides a function

$$\varphi_\varepsilon^+ : C^+ \to S_+^1,\ a' \mapsto T_\varepsilon^+(b'),$$

see Figure 2. These ε-dependent coordinates $\varphi_\varepsilon^\pm$ on the transversal loops $C^\pm$ are called *canonical.*

Without loss of generality we may assume that the cycle γ is time oriented clockwise. Then the transversal loops $C^\pm$ are oriented counterclockwise by the canonical coordinates.

2.3 The Poincaré map of the transversal loops

Consider a small $\varepsilon > 0$. The orbit of the vector field v_ε that starts at a point $a \in C^-$ eventually reaches C^+ at a unique point a'. This defines the Poincaré map

$$\Delta_\varepsilon : C^- \to C^+,\ a \to a'$$

along the orbits of v_ε, see Figure 2 again.

Proposition 1. *In the coordinates $\varphi_\varepsilon^\pm$, the Poincaré map $\Delta_\varepsilon : C^- \to C^+$ is a mere rotation:*

$$\varphi_\varepsilon^+(\Delta_\varepsilon(a)) = \varphi_\varepsilon^-(a) - \tau(\varepsilon) \pmod{\mathbb{Z}}. \tag{10}$$

In what follows, the map Δ_ε in $\varphi_\varepsilon^\pm$-coordinates (i.e. the rotation by $-\tau(\varepsilon)$) will be denoted by the same symbol.

Proof. Let $a \in C^-$, and b be the same as above, that is, the first intersection point with Γ of the forward orbit of v_ε emerging from a. Let $a' \in C^+$ be the image of a: $a' = \Delta_\varepsilon(a)$, and b' be the first intersection point with Γ of the forward orbit of v_ε that emerges from a'. Then, by definition of canonical coordinates,

$$\varphi_\varepsilon^-(a) = T_\varepsilon^-(b), \ \varphi_\varepsilon^+(a') = T_\varepsilon^+(b').$$

On the other hand,

$$b' = P_\varepsilon^n(b)$$

for some n. Hence,

$$T_\varepsilon^-(b') - T_\varepsilon^-(b) = n,$$

because in the chart T_ε the Poincaré map P_ε is a mere shift by 1. Moreover,

$$T_\varepsilon^+(b') = T_\varepsilon^-(b') - \tau(\varepsilon).$$

Recall that $a' = \Delta_\varepsilon(a)$. Hence,

$$\varphi_\varepsilon^+(\Delta_\varepsilon(a)) = \varphi_\varepsilon^+(a') = T_\varepsilon^+(b') = T_\varepsilon^-(b') - \tau(\varepsilon) = T_\varepsilon^-(b) - \tau(\varepsilon) \pmod{\mathbb{Z}} = \varphi_\varepsilon^-(a) - \tau(\varepsilon) \pmod{\mathbb{Z}},$$

see Figure 2. This proves the proposition. □

2.4 Characteristic sets on the transversal loops

Let $S^- = \{s_m^-\}$ be the set of all intersections of C^- with separatrices of v, enumerated counterclockwise along C^-. Let l_m^- be the corresponding separatrices and E_m be the corresponding saddles of v. The canonical coordinate φ_0^- maps the set S^- to the characteristic set

$$A^- = \{a_1^-, ..., a_M^-\}, \ a_m^- = \varphi_0^-(s_m^-),$$

see Figure 3.

If two points $a_m^- = \varphi_0^-(s_m^-)$ and $a_{m'}^- = \varphi_0^-(s_{m'}^-)$ are equivalent as the points of the marked set A^-, then $E_m = E_{m'}$.

Similarly, separatrices l_k^+ of saddles I_k are all separatrices of v that intersect C^+, $S^+ = \{s_k^+\}$ are intersection points, and $a_k^+ = \varphi_0^+(s_k^+)$. This determines another characteristic set

$$A^+ = \{a_1^+, ..., a_K^+\}, \ a_k^+ = \varphi_0^-(s_k^+).$$

2.5 The connection equation

In this and the next sections we describe the bifurcations in $\mathcal{PC}$ families.

Let $V = \{v_\varepsilon\}$ be a $\mathcal{PC}$-family. The vector fields v_ε have saddles $E_m(\varepsilon), \ I_k(\varepsilon)$ continuously depending on ε, $E_m(0) = E_m, \ I_k(0) = I_k$. The germs of their separatrices at these saddles depend continuously on ε. Denote by $l_m^-(\varepsilon)$ the separatrix with a germ $(l_m^-(\varepsilon), E_m(\varepsilon))$ that is continuous in ε and coincides with (l_m^-, E_m) for $\varepsilon = 0$. In the same way the separatrices $l_k^+(\varepsilon)$ of the saddles $I_k(\varepsilon)$ are defined. Let $s_m^-(\varepsilon)$ be the (unique for ε small) intersection of $l_m^-(\varepsilon)$ and C^-; let $s_k^+(\varepsilon)$ be the intersection of $l_k^+(\varepsilon)$ and C^+. Define $S^\pm(\varepsilon) = \{s_i^\pm(\varepsilon)\}$. Put

$$a_m^-(\varepsilon) := \varphi_\varepsilon^-(s_m^-), \quad a_k^+(\varepsilon) := \varphi_\varepsilon^+(s_k^+),$$

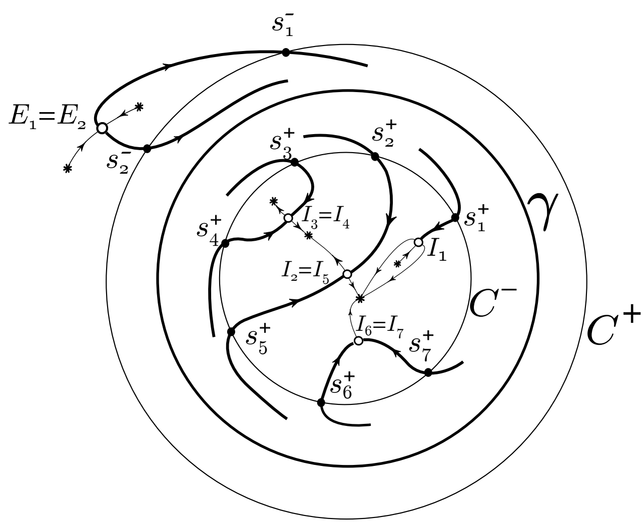

Figure 3: Characteristic sets on the transversal loops for the PC vector field

and let $A^-(\varepsilon) := \{a_m^-(\varepsilon)\}$, $A^+(\varepsilon) := \{a_k^+(\varepsilon)\}$. The sets $A^\pm(\varepsilon)$ depend continuously on ε and coincide with $A^\pm$ for $\varepsilon = 0$. In particular, if $A^\pm$ is a pair of non-synchronized sets, then $A^\pm(\varepsilon)$ also is for ε small.

Let

$$\tau_{km}(\varepsilon) = \{a_k^+(\varepsilon) - a_m^-(\varepsilon)\};\ \tau_{km}(\varepsilon) \in [0,1). \tag{11}$$

If none of τ_{km} is equal to zero, $\tau_{km}(\varepsilon)$ depends continuously on ε and coincides with τ_{km} for $\varepsilon = 0$. Without loss a generality, we may and will assume that none of τ_{km} is 0. Elsewhere, we will slightly change b^- keeping b^+ unchanged. This will rotate A^- and preserve A^+, thus change τ_{km} by an additive constant.

As the pair $A^\pm(\varepsilon)$ is non-synchronized, the values of $\tau_{km}(\varepsilon)$ are pairwise distinct.

A saddle connection between the saddles $E_m(\varepsilon)$ and $I_k(\varepsilon)$ occurs iff

$$a_k^+(\varepsilon) = \Delta_\varepsilon(a_m^-(\varepsilon)).$$

By Proposition 1, this is equivalent to

$$a_k^+(\varepsilon) = a_m^-(\varepsilon) - \tau(\varepsilon) \pmod{\mathbb{Z}}$$

or equivalently,

$$\tau_{km}(\varepsilon) = -\tau(\varepsilon) \pmod{\mathbb{Z}}.$$

Another form of this equation is:

$$\tau_{km}(\varepsilon) = -\tau(\varepsilon) + n,\ n \in \mathbb{N}. \tag{12}$$

This is a *connection equation.*

Proposition 2. *For ε small and n large enough, equation* (12) *has a unique solution $\varepsilon = \varepsilon_{kmn}$.*

When $\varepsilon = \varepsilon_{kmn}$, the vector field v_ε has a separatrix connection between the saddles $E_k(\varepsilon)$ and $I_m(\varepsilon)$. This separatrix connection makes n winds around γ between C^- and C^+.

Proof. Take $\varepsilon_0 > 0$ so small that the function $\varepsilon \mapsto \tau_{km}(\varepsilon)$ is well-defined on $[0, \varepsilon_0]$. By Theorem 7, this function has a bounded derivative on the whole segment. On the other hand, $\tau'(\varepsilon) \to -\infty$ as $\varepsilon \to 0$. Indeed, by (9)

$$\tau(\varepsilon) = F(x,\varepsilon)\Big|_{x = x_\varepsilon(b^-)}^{x = x_\varepsilon(b^+)}$$

where

$$F(x,\varepsilon) = \frac{1}{\sqrt{\varepsilon}} \arctan \frac{x}{\sqrt{\varepsilon}} + \frac{a(\varepsilon)}{2} \log(x^2 + \varepsilon).$$

Then

$$\tau'(\varepsilon) = F_\varepsilon(x,\varepsilon)|_{x=x_\varepsilon(b^-)}^{x=x_\varepsilon(b^+)} + F_x(x_\varepsilon(b^+),\varepsilon) \cdot D_\varepsilon x_\varepsilon(b^+) - F_x(x_\varepsilon(b^-),\varepsilon) \cdot D_\varepsilon x_\varepsilon(b^-).$$

For any $b \neq 0$, the functions $F(b, \varepsilon)$ and $x_\varepsilon(b)$ are well defined in a neighborhood of $\varepsilon = 0$ and have bounded derivatives. Hence, the second and the third terms in the expression for $\tau'(\varepsilon)$ are bounded near $\varepsilon = 0$. On the other hand,

$$F_\varepsilon(x,\varepsilon) = -\frac{1}{2\varepsilon^{3/2}} \arctan \frac{x}{\sqrt{\varepsilon}} - \frac{1}{2\varepsilon(x^2+\varepsilon)} + \dots$$

dots stand for bounded terms. Hence,

$$F_\varepsilon(x,\varepsilon)|_{x=x_\varepsilon(b^-)}^{x=x_\varepsilon(b^+)} \to -\infty \text{ as } \varepsilon \searrow 0.$$

Take ε so small that

$$(\tau(\varepsilon) + \tau_{km}(\varepsilon))' < 0.$$

Then for any $n > \tau(\varepsilon_0) + \tau_{km}(\varepsilon_0)$, the connection equation (12) has a unique solution $\varepsilon_{kmn} < \varepsilon_0$. □

2.6 The bifurcation scenario

The bifurcation scenario in a one-parameter family is a sequence of bifurcations that occur as ε changes. The previous section shows that for $\varepsilon > 0$, the sparkling saddle connections between saddles $E_k(\varepsilon)$ and $I_m(\varepsilon)$ occur for $\varepsilon = \varepsilon_{kmn}$.

Each bifurcation of the sparkling saddle connection goes in the same way as for usual saddle connections: the incoming separatrix of one saddle changes its α-limit set, and the outgoing separatrix of another saddle changes its ω-limit set, see Figure 4a. The only additional feature of the sparkling saddle connection is that for the critical parameter value $\varepsilon = \varepsilon_{kmn}$ the connection between the saddles $E_m(\varepsilon)$ and $I_k(\varepsilon)$ winds around γ many times, see Figure 4b. Yet topologically these pictures are the same: the second one may be transformed to the first one by the iterated Dehn twist.

To finish the description of the bifurcation scenario, we need to describe the order in the set $\{\varepsilon_{kmn}\}$. Recall that we assume without loss a generality that none of τ_{km} is equal to zero. The order of numbers ε_{kmn} is described by the following proposition.

Proposition 3. *For sufficiently large n and any k, m, k', m', we have $\varepsilon_{kmn} < \varepsilon_{k'm'(n-1)}$.*

For sufficiently large fixed n, the order of ε_{kmn} does not depend on n and coincides with the order of τ_{km}. In more detail, if for some natural k, m, k', m', we have $0 < \tau_{km} < \tau_{k'm'} < 1$, then for sufficiently large n, $\varepsilon_{kmn} < \varepsilon_{k'm'n}$.

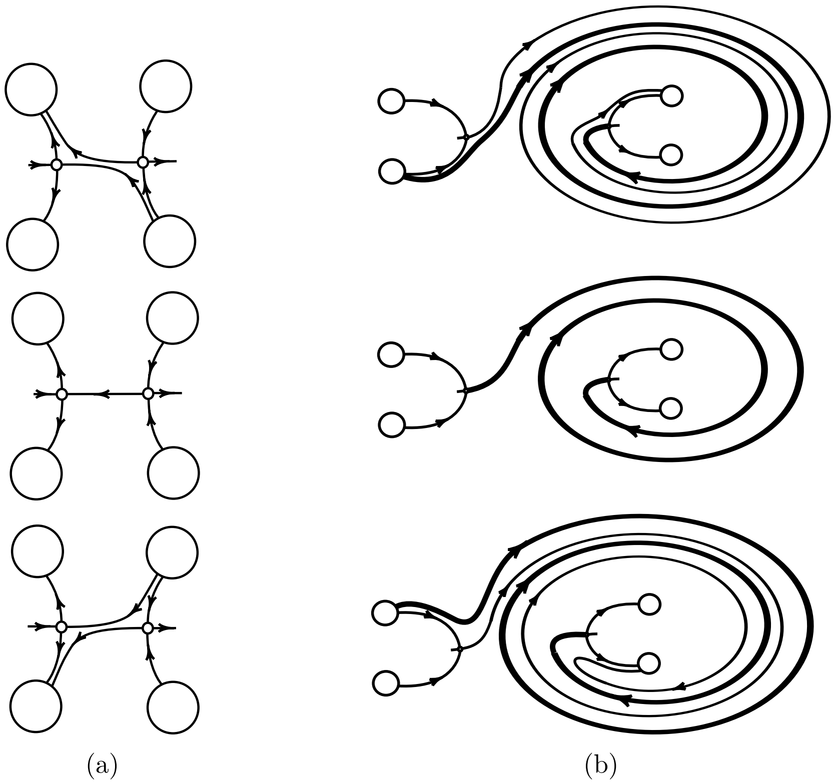

Figure 4: The bifurcation of a saddle connection (left) and the bifurcation of the sparkling saddle connection (right). Large circles are transversal loops around α-and ω-limit sets of saddle separatrices, small circles are saddles

Proof. The first statement follows directly from the connection equation (12), the inequality $0 < \tau_{km}(\varepsilon) < 1$ and monotonicity of τ: this function tends monotonically to infinity as positive ε decreases.

Since $\tau_{km} = \tau_{km}(0)$ and $\tau_{km}(\varepsilon)$ depends continuously on ε, we have $\tau_{km}(\varepsilon_{kmn}) < \tau_{k'm'}(\varepsilon_{k'm'n})$ for large n. Now (12) implies $-\tau(\varepsilon_{kmn}) < -\tau(\varepsilon_{k'm'n})$. However, the function τ decreases in a small neighborhood of zero, so $\varepsilon_{kmn} < \varepsilon_{k'm'n}$. □

Recall that the numbers τ_{km} are well-defined modulo an additive constant that depends on the choice of $b^{\pm}$, and the set of bifurcation parameters $\{\varepsilon_{kmn}\}$ does not depend on this choice. However, that does not contradict the proposition above, because when we change $b^{\pm}$, the same bifurcation value $\varepsilon = \varepsilon_{kmn}$ will obtain a different number n. Here we describe this change in more detail.

Definition 14. Let ε_{kmn} be as above. Let $i, 1 < i < K$, and $j, 1 < j < M$ be two indices, N be an integer number.

A cyclical shift of n in the set ε_{kmn} is the change of numeration: the bifurcation parameter ε_{kmn} obtains indices kmn', where $n' = n + N$ for $\varepsilon_{kmn} < \varepsilon_{ijn}$ and $n' = n + N - 1$ otherwise.

Suppose that we change our choice of $b^{\pm}$ replacing these points by $\tilde{b}^{\pm}$. The number $\tau_{km}(\varepsilon)$ is replaced by $\tilde{\tau}_{km}(\varepsilon) = \{\tau_{km}(\varepsilon) + \alpha(\varepsilon)\}$, where $\alpha(\varepsilon) = T^{-}_{\varepsilon}(\tilde{b}^{-}) - T^{+}_{\varepsilon}(\tilde{b}^{+})$; the charts $T^{\pm}(\varepsilon)$ correspond to $b^{\pm}$, see (8). We assume that none of $\tilde{\tau}_{km}(0)$ is zero. Similarly, $\tau(\varepsilon)$ is replaced by $\tau(\varepsilon) - \alpha(\varepsilon)$.

Proposition 4. *The change of $b^{\pm}$ described above results in a cyclical shift of n in the set $\{\varepsilon_{kmn}\}$; any cyclical shift may be achieved.*

Proof. Recall that ε_{kmn} is the solution of the connection equation $\tau_{km}(\varepsilon) = -\tau(\varepsilon) + n$. Then it also solves the equation $\tau_{km}(\varepsilon) + \alpha(\varepsilon) = -\tau(\varepsilon) + \alpha(\varepsilon) + n$, i.e.

$$\{\tau_{km}(\varepsilon) + \alpha(\varepsilon)\} = -(\tau(\varepsilon) - \alpha(\varepsilon)) + n - [\tau_{km}(\varepsilon) + \alpha(\varepsilon)].$$

This is a connection equation in new coordinates, but n is replaced by $n - [\tau_{km}(\varepsilon) + \alpha(\varepsilon)]$. So the bifurcation parameter ε_{kmn} will obtain another number $n' = n - [\tau_{km}(\varepsilon) + \alpha(\varepsilon)]$.

Note that the integer number $[\tau_{km}(\varepsilon) + \alpha(\varepsilon)]$ does not depend on ε for small ε; this follows from the fact that none of $\tilde{\tau}_{km}(0)$ is zero. However it may depend on k, m: if it equals N for $0 < \tau_{km}(\varepsilon) < \{1 - \alpha(\varepsilon)\}$, then it equals $N + 1$ for $\{1 - \alpha(\varepsilon)\} < \tau_{km}(\varepsilon) < 1$. These inequalities on $\tau_{km}(\varepsilon)$ are equivalent to $\varepsilon_{kmn} < \varepsilon_{ijn}$ and $\varepsilon_{kmn} \geq \varepsilon_{ijn}$ for some i, j, because the order in the set $\{\varepsilon_{kmn}, n \text{ fixed}\}$ is the same as the order of τ_{km}; so we have a cyclical shift of n in the set $\{\varepsilon_{kmn}\}$. One can easily show that the choice of $\tilde{b}^{\pm}$ may give any prescribed value of $\alpha(0)$, thus it may result in arbitrary cyclical shift of $\{\varepsilon_{kmn}\}$. This completes the proof. □

We conclude that the bifurcation scenario in $\mathcal{PC}$ families repeats cyclically as $\varepsilon \to 0$. As ε decreases, in the family v_{ε} we have several saddle connections that make n winds in a small neighborhood of γ between C^{-} and C^{+}, then several saddle connections that make $n + 1$ winds, etc. The n-wind saddle connections occur in one and the same order for all n, and this order is determined by the order of τ_{km}. However, the number n of winds of a particular saddle connection depends on the choice of b^{-}, b^{+}.

One of our main goals is achieved: the bifurcation scenario is described and justified.

Note that the bifurcating separatrices of the vector fields v_{ε} are located not only inside a small neighborhood of the large bifurcation support (see Definition 7), but also outside it.

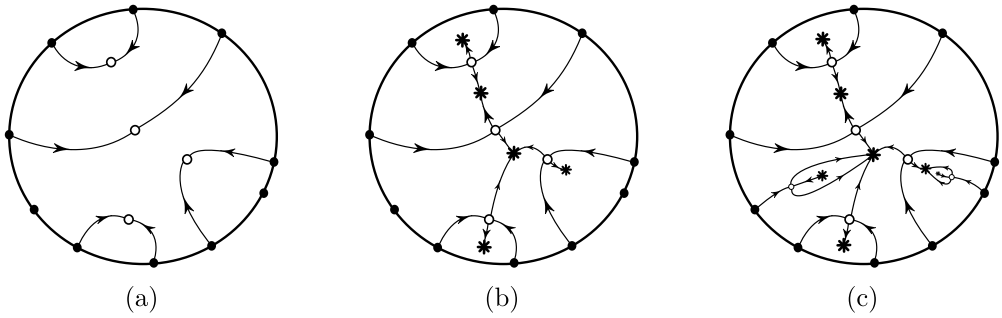

Figure 5: Realization of characteristic sets: steps a, b, c.

Yet this bifurcation is predicted by what happens in a neighborhood of the large bifurcation support.

2.7 Realization Lemma for discs

In this and the next sections Theorem 4 is proved, skipping the routine details.

Begin with a Realization Lemma that will be used not only here but in the study of the global bifurcations of the vector fields with a separatrix loop. Consider a Morse – Smale vector field in a disc D with the boundary C transversal to v. No special coordinate on C is considered. Let A be the set of the intersection points of the separatrices of v with C; two points of A are equivalent iff they belong to the separatrices of the same saddle. Thus A is a marked set. Let us call it *the characteristic set* of v.

Lemma 1. *Consider a marked set A on a circle C that is a boundary of a disc D. Then there exists a C^∞ vector field v in D such that A is a characteristic set of v.*

Proof. Without loss of generality, we assume that the vector field v on C points inside D.

a. Take any two equivalent points of the set A and connect them by a smooth simple arc transversal to C at its endpoints, whose interior part lies in D. The arcs should be pairwise disjoint. Take a point in each arc different from its endpoints; this will be a saddle of the vector field v to be constructed. The parts of the arcs from the endpoints to the saddles should be arcs of the *incoming separatrices of* v, see Figure 5a. This operation is proceeded for all the pairs of equivalent points of A.

b. The disc D is split by the arcs just constructed to topological disks. Let us choose a point inside each of these disks; this will be an attractor of v. Connect each saddle on the boundary of the topological disk above to the attractor inside it; this will be an outgoing separatrix of the saddle, see Figure 5b. Now each saddle has four separatrices: two ingoing and two outgoing.

c. The domains to which D is split have one of the two shapes shown on Figure 6.

On any arc of the curve C which is an edge of the domains mentioned above, take all the points of the set A; each of them is a one-point equivalence class in the marked set A. Add to this point a graph shown at Figure 7a; the attractor a on this figure should coincide with the attractor a in the domain, see Figures 7b, 7c. The construction of the separatrices of v is over, see Figure 5c. We can now construct the vector field v with these separatrices in D. This may be easily done in any connected component of the complement to these separatrices in D. It is easy to make the vector field v smooth. □

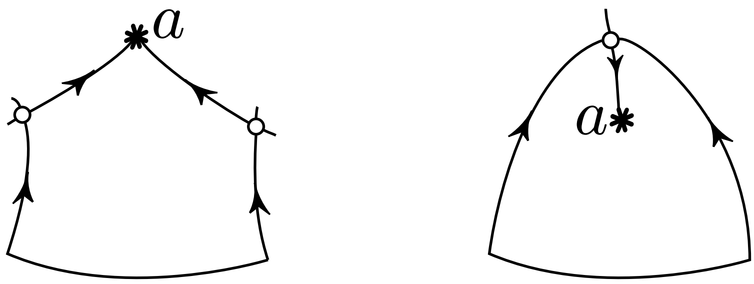

Figure 6: Domains constructed in Step b

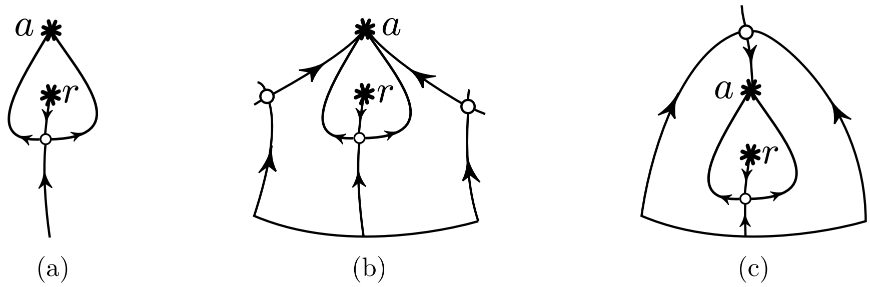

Figure 7: Constructing unstable separatrices in Step c

2.8 Realization Theorem for the sphere

Here we prove Theorem 4. The vector field will be constructed separately in the annular neighborhood of the parabolic cycle, and in the components of its complement to the sphere. Begin with the annulus.

Let U be the annulus $1 \le r \le 3$ in the polar coordinates r, α. Consider a vector field v_U in U given by

$$\dot{\alpha} = 1, \ \dot{r} = -(r-2)^2.$$

This system has a parabolic cycle γ given by $\{r = 2\}$. Let Γ be a cross-section to γ that belongs to the ray $\alpha = 0$. Let $C^{\pm}$ be the boundary circles of $U : C^- = \{r = 3\}, \ C^+ = \{r = 1\}$. Let $\varphi^{\pm} = \varphi_0^{\pm}$ be the coordinates on $C^{\pm}$ constructed for $v|_U$ as in Section 2.2. Let $A^{\pm} \subset C^{\pm}$ be the characteristic sets given in Theorem 4.

Let $D^{\pm}$ be the disc on the sphere disjoint from the interior of U and bounded by $C^{\pm} : S^2 = D^+ \cup U \cup D^-$. By Lemma 1, there exists a smooth vector field $v^{\pm}$ on $D^{\pm}$ such that $A^{\pm}$ is a characteristic set of $v^{\pm}$ in $D^{\pm}$. Thus we have constructed a piecewise C^{∞} vector field W on S^2 that is discontinuous on $C^{\pm}$. Let us change the smooth structure on S^2 in such a way that W will become a C^{∞} smooth vector field. We will get a smooth vector field W on a smooth manifold M^2 homeomorphic to a sphere. But there exists only one smooth structure on S^2. Hence, there exists a C^{∞} diffeomorpism $H : M^2 \to S^2$. The vector field $v = H_* W$ is the desired one. This proves Theorem 4.

3 Classification of $\mathcal{PC}$-families

In this section we prove Theorem 6.

3.1 Theorem 6 part 1

Our goal is to prove that any pair of sing-equivalent local families gives rise to two pairs of characteristic sets that are equivalent in the sense of Definition 13. The heuristic proof is straightforward: as the families V and W are sing-equivalent, the homeomorphism h identifies the values of ε and δ that correspond to saddle connections, thus identifies $\{\varepsilon_{kmn}\}$ and $\{\delta_{kmn}\}$. Proposition 3 implies that the sets $\Lambda(A^{\pm})$ and $\Lambda(B^{\pm})$ are ordered in the same way. This implies the statement.

Let us pass to the detailed proof.

Let $V = \{v_\varepsilon\}$ and $W = \{w_\delta\}$ be two families satisfying assumptions of Theorem 6 part 1. Let $H = (h, H_\varepsilon)$ be the sing-equivalence of V, W. Following the previous sections, we denote by γ the parabolic cycle of v_0, by $C^{\pm}$ its transversal loops, by $l_j^{\pm}(\varepsilon)$ the separatrices of hyperbolic saddles $E_j(\varepsilon)$, $I_j(\varepsilon)$ of v_ε that cross $C^{\pm}$. Let $\tilde{\gamma}, \tilde{C}^{\pm}, \tilde{l}_j^{\pm}(\delta), \tilde{E}_j(\delta), \tilde{I}_j(\delta)$ be analogous objects for the family W such that $\tilde{l}_j^{\pm} = H_0(l_j^{\pm})$, $\tilde{E}_j = H_0(E_j), \tilde{I}_j = H_0(I_j)$. We assume that the parabolic cycles disappear for $\varepsilon > 0, \delta > 0$, otherwise we reverse the parameter.

We will need the following lemma on sing-equivalence.

Lemma 2. *In the above assumptions,* $H_\varepsilon(l_i^{\pm}(\varepsilon)) = \tilde{l}_i^{\pm}(h(\varepsilon))$.

This lemma implies Theorem 6 part 1. Indeed, let $A^{\pm}$ be characteristic sets for v_0, and let $B^{\pm}$ be characteristic sets for w_0. Since the vector fields v_0 and w_0 are topologically conjugate, we have $|A^+| = |B^+|$ and $|A^-| = |B^-|$. Let ε_{kmn} be bifurcation parameters for the family V and δ_{kmn} be bifurcation parameters for W.

Now, for $\varepsilon = \varepsilon_{kmn}$, the separatrices $l_k^+(\varepsilon)$ and $l_m^-(\varepsilon)$ coincide, thus their images $H_\varepsilon(l_k^+(\varepsilon)) = \tilde{l}_k^+(h(\varepsilon))$ and $H_\varepsilon(l_m^-(\varepsilon)) = \tilde{l}_m^-(h(\varepsilon))$ coincide; here we use Lemma 2. So the separatrices $\tilde{l}_k^+(h(\varepsilon))$ and $\tilde{l}_m^-(h(\varepsilon))$ of $w_{h(\varepsilon)}$ form a saddle connection, and we conclude that $h(\varepsilon_{kmn}) = \delta_{kmn'}$ for any n and some n'.

Finally, the homeomorphism h takes the set $\{\varepsilon_{kmn}\}$ to the set $\{\delta_{kmn}\}$ and preserves k, m. However it may change n.

Note that $\{\varepsilon_{kmn} \mid n = n_0\}$ is the set of $|A^-| \cdot |A^+|$ subsequent numbers in the set $\{\varepsilon_{kmn}\}$, so h takes them to $|A^-| \cdot |A^+| = |B^-| \cdot |B^+|$ subsequent numbers among $\{\delta_{kmn}\}$. Recall that changing $b^{\pm}$, we may achieve any cyclical shift of n in the set $\{\varepsilon_{kmn}\}$, see Proposition 4. So with a suitable choice of $b^{\pm}$, we may and will assume that $h(\varepsilon_{kmn}) = \delta_{kmn}$. By Proposition 3, this implies that the sets $\Lambda(A^{\pm})$ and $\Lambda(B^{\pm})$ are equivalent in the sense of Definition 13.

Proof of Lemma 2. We only prove the lemma for l_i^-; for l_i^+, the proof is analogous. We have two topologically different cases: l_i^- is the only separatrix of E_i that intersects C^-, or both unstable separatrices of E_i intersect C^-. We start with the second case.

Case 1. Consider a saddle E whose two unstable separatrices wind towards γ. Let L_1, L_3 be these separatrices of E (so $L_1 = l_i^-$ and $L_3 = l_j^-$ for some i, j; $E = E_i = E_j$), and let L_2, L_4 be two its stable separatrices. Let $R_2 = \alpha(L_2)$ and $R_4 = \alpha(L_4)$ be the α-limit sets of these separatrices; both of them are either hyperbolic singular points, or hyperbolic limit cycles. Note that $R_2 \neq R_4$ because these sets are on two different sides with respect to $\gamma \cup L_1 \cup L_3$, see Fig. 8a.

Let $E^\varepsilon, (L_i^\varepsilon, E^\varepsilon)$, and R_i^ε be continuous families of singular points, germs of separatrices, and repellors of v_ε such that $E^0 = E, L_i^0 = L_i, R_i^0 = R_i$. Let $\tilde{E}, \tilde{L}_i, \tilde{R}_i$ be the images of E, L_i, and R_i under H_0. Let $\tilde{E}^\delta, (\tilde{L}_i^\delta, \tilde{E}^\delta), \tilde{R}_i^\delta$ be continuous families of singular points, germs of separatrices, and repellors of w_δ.

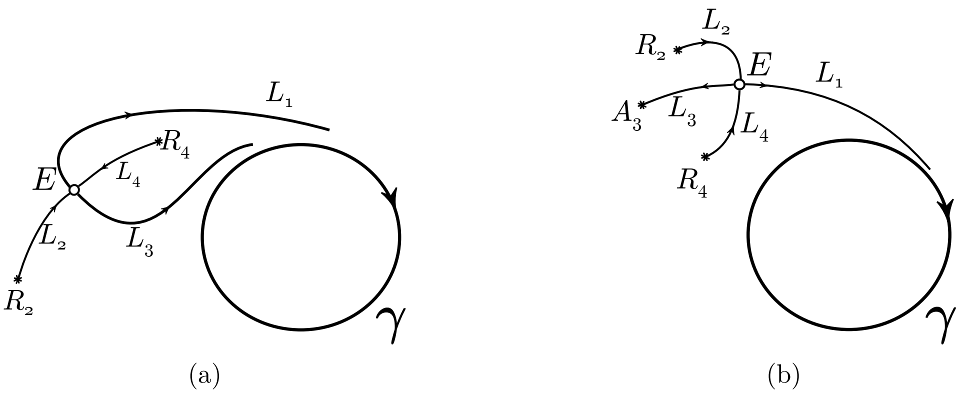

Figure 8: Two cases for Lemma 2

The definition of sing-equivalence implies that $H_\varepsilon(E^\varepsilon) = \tilde{E}^{h(\varepsilon)}$ and $H_\varepsilon(R_i^\varepsilon) = \tilde{R}_i^{h(\varepsilon)}$. We should prove that the analogous statement holds for separatrices: $H_\varepsilon(L_i^\varepsilon)$ coincides with $\tilde{L}_i^{h(\varepsilon)}$. Clearly, $H_\varepsilon(L_1^\varepsilon)$ is an unstable separatrix of $\tilde{E}^{h(\varepsilon)}$; two possibilities are $H_\varepsilon(L_1^\varepsilon) = \tilde{L}_1^{h(\varepsilon)}$ and $H_\varepsilon(L_1^\varepsilon) = \tilde{L}_3^{h(\varepsilon)}$. Our goal is to prove that the second case is impossible. Since H_ε preserves orientation, it preserves a cyclical order of separatrices at E^ε, so in this case, we must have $H_\varepsilon(L_2^\varepsilon) = \tilde{L}_4^{h(\varepsilon)}$: in comparison with H_0, the map H_ε rotates separatrices of E^ε.

Since H_ε is a homeomorphism that conjugates v_ε to $w_{h(\varepsilon)}$, it respects α-, ω- limit sets. So the α-limit set of $H_\varepsilon(L_2^\varepsilon)$ must be $H_\varepsilon(R_2^\varepsilon) = \tilde{R}_2^{h(\varepsilon)}$. But the α-limit set of $\tilde{L}_4^{h(\varepsilon)}$ is $\tilde{R}_4^{h(\varepsilon)}$. The contradiction shows that $H_\varepsilon(L_2^\varepsilon) \neq \tilde{L}_4^{h(\varepsilon)}$. Thus $H_\varepsilon(L_i^\varepsilon) = \tilde{L}_i^{h(\varepsilon)}$ for all i, which proves the lemma in Case 1.

Case 2. Suppose that E has only one unstable separatrix L_1 that winds towards γ, and other its separatrices L_2, L_3, L_4 have hyperbolic α- and ω-limit sets (see Fig. 8b). Then similar arguments as in Case 1 apply to L_3. Namely, if the ω-limit set of L_3 is A_3, then the separatrix $H_\varepsilon(L_3^\varepsilon)$ must have the ω-limit set $\tilde{A}_3^{h(\varepsilon)}$, while $\tilde{L}_1^{h(\varepsilon)}$ has its ω-limit set inside γ. So $H_\varepsilon(L_3^\varepsilon) \neq \tilde{L}_1^{h(\varepsilon)}$, thus $H_\varepsilon(L_i^\varepsilon) = \tilde{L}_i^{h(\varepsilon)}$ for all i. This proves the lemma in Case 2.

□

3.2 Theorem 6 part 2

Let $V = \{v_\varepsilon\}$ and $W = \{w_\delta\}$ be two families satisfying assumptions of Theorem 6. Let $\hat{H}$ be an orbital topological equivalence of v_0 and w_0.

Let γ be a parabolic cycle of v_0; let C^+ and C^- be its transversal loops. We assume that C^- is outside γ and C^+ is inside it. Let U be the open annulus bounded by C^- and C^+. Let $D^\pm$ be the disc on the sphere disjoint from the interior of U and bounded by $C^\pm$, so that $S^2 = D^- \cup U \cup D^+$.

Let $\tilde{\gamma}, \tilde{C}^\pm, \tilde{U}, \tilde{D}^\pm$ be analogous sets for the family W. We may and will modify $\hat{H}$ so that $\hat{H}(C^\pm) = \tilde{C}^\pm$. Recall that the choice of $\tilde{C}^\pm$ implies that the trajectories starting on $\tilde{C}^-$ wind towards $\tilde{\gamma}$, and the trajectories starting on $\tilde{C}^+$ are repelled from $\tilde{\gamma}$. We assume that $\gamma, \tilde{\gamma}$ are oriented clockwise by time orientation.

We will use the notation of Section 2.5 for the family V: $l_j^\pm(\varepsilon)$ are separatrices of hyperbolic saddles $E_j(\varepsilon)$, $I_j(\varepsilon)$ of v_ε that cross $C^\pm$. Let $\tilde{l}_j^\pm(\delta), \tilde{E}_j(\delta), \tilde{I}_j(\delta)$ be analogous objects for the family W such that $\tilde{l}_j^\pm = \hat{H}(l_j^\pm)$, $\tilde{E}_j = \hat{H}(E_j), \tilde{I}_j = \hat{H}(I_j)$.

As in Section 2.5, $a_j^\pm(\varepsilon)$ are intersection points of $l_j^\pm(\varepsilon)$ with $C^\pm$ in $\varphi_\varepsilon^\pm$-chart, and

$\Delta_\varepsilon\colon C^- \to C^+$ is the Poincaré map along v_ε. Let $\tilde{a}_j^\pm(\delta)$ and $\tilde{\Delta}_\delta$ be the corresponding objects for $\{w_\delta\}$.

3.3 Construction of the homeomorphism of bases $h\colon (\mathbb{R},0) \to (\mathbb{R},0)$

Assume that for families $V = \{v_\varepsilon\}$ and $W = \{w_\delta\}$, the parabolic cycle disappears for $\varepsilon > 0$, $\delta > 0$; otherwise we reverse the parameter.

The equivalence of the characteristic sets for V and W (see Definition 13) implies that the numbers τ_{km} are ordered on $[0,1)$ in the same way as $\{\lambda_{km}+\alpha\}$ for some α. Recall that λ_{km} are well-defined modulo an additive constant that depends on the choice of coordinates on coordinate circles. Let us add a shift by α to the coordinate on S^1_+; this will add α to all numbers λ_{km}. Finally, we may and will assume that the numbers τ_{km} and λ_{km} are ordered in the same way.

Let $\{\varepsilon_{kmn}\}$ be the sequence of the bifurcation parameter values defined in Section 2.5 for the family V, and $\{\delta_{kmn}\}$ be the analogous sequence for the family W. Proposition 3 implies that the order of numbers ε_{kmn} and δ_{kmn}, n fixed, is the same. This implies that the sets $\{\varepsilon_{kmn}\}$ and $\{\delta_{kmn}\}$ may be identified by some homeomorphism $h\colon (\mathbb{R},0) \to (\mathbb{R},0)$ with $h(\varepsilon_{kmn}) = \delta_{kmn}$.

In more detail, the homeomorphism h is defined in the following way. We put $h(0) = 0$ and $h|_{\varepsilon<0} = id$, and for $\varepsilon > 0$, we take any homeomorphism that satisfies $h(\varepsilon_{kmn}) = \delta_{kmn}$. We will need the following lemma.

Lemma 3. *In assumptions of Theorem 6 part 2, for a homeomorphism $h\colon (\mathbb{R},0) \to (\mathbb{R},0)$ constructed above and for any small positive $\varepsilon > 0$, the points $a_j^+(\varepsilon)$ and $\Delta_\varepsilon(a_j^-(\varepsilon))$ are ordered along C^+ in the same way as the corresponding points $\tilde{a}_j^+(\delta)$ and $\tilde{\Delta}_\delta(\tilde{a}_j^-(\delta))$ on $\tilde{C}^+$ for w_δ, where $\delta = h(\varepsilon)$.*

Proof. Note that if for small ε, two points $a_m^+(\varepsilon)$ and $\Delta_\varepsilon(a_k^-(\varepsilon))$ coincide, this implies $\varepsilon = \varepsilon_{kmn}$ for some n (cf. Sec. 2.5). Due to the construction of h, $h(\varepsilon) = \delta_{kmn}$, thus two corresponding points $\tilde{a}_m^+(\delta)$ and $\tilde{\Delta}_\delta(\tilde{a}_k^-(\delta))$ coincide.

We must also prove that for all k_1, k_2, m, the condition on ε

$$\Delta_\varepsilon(a_m^-(\varepsilon)) \in (a_{k_1}^+(\varepsilon), a_{k_2}^+(\varepsilon)) \tag{13}$$

is equivalent to the condition on δ

$$\tilde{\Delta}_\delta(\tilde{a}_m^-(\delta)) \in (\tilde{a}_{k_1}^+(\delta), \tilde{a}_{k_2}^+(\delta)) \tag{14}$$

for small ε and $\delta = h(\varepsilon)$ (on the right-hand side, we have oriented arcs of the oriented coordinate circles). This will finish the proof of the lemma.

Proposition 1 implies that (13) is equivalent to $(a_m^-(\varepsilon) - \tau(\varepsilon) \mod 1) \in (a_{k_1}^+(\varepsilon), a_{k_2}^+(\varepsilon))$, i.e.

$$(-\tau(\varepsilon) \mod 1) \in (\tau_{k_1 m}(\varepsilon), \tau_{k_2 m}(\varepsilon)) \subset \mathbb{R}/\mathbb{Z}.$$

Recall that $\tau(\varepsilon)$ is monotonic for small ε with $\tau'(\varepsilon) \to -\infty$ as $\varepsilon \to 0$, and the derivatives of $\tau_{km}(\varepsilon)$ are bounded. So the values of ε that satisfy the last inclusion are between the solutions $\varepsilon_{k_1 mn}$, $\varepsilon_{k_2 mn}$ of the connection equation (12).

In more detail, if the arc $(\tau_{k_1 m}(\varepsilon), \tau_{k_2 m}(\varepsilon))$ of the circle does not contain zero (or, equivalently, the segment $(\tau_{k_1 m}, \tau_{k_2 m})$ does not contain zero), (13) is equivalent to $-\tau(\varepsilon) + n \in (\tau_{k_1 m}(\varepsilon), \tau_{k_2 m}(\varepsilon))$ for some n, and the solutions are $\varepsilon \in (\varepsilon_{k_1 mn}, \varepsilon_{k_2 mn})$ for some n. If the segment $(\tau_{k_1m}, \tau_{k_2m})$ contains zero, (13) is equivalent to $-\tau(\varepsilon) + n \in (\tau_{k_1m}(\varepsilon), \tau_{k_2m}(\varepsilon) + 1)$ for some n, i.e. to $\varepsilon \in \bigcup_n (\varepsilon_{k_1mn}, \varepsilon_{k_2m(n-1)})$.

Finally, for small ε, the condition (13) is equivalent to

$$\varepsilon \in \bigcup_n (\varepsilon_{k_1mn}, \varepsilon_{k_2mn}) \text{ if } 0 \in (\tau_{k_1m}, \tau_{k_2m})$$

$$\varepsilon \in \bigcup_n (\varepsilon_{k_1mn}, \varepsilon_{k_2m(n-1)}) \text{ if } 0 \notin (\tau_{k_1m}, \tau_{k_2m})$$

Similarly, the condition (14) is equivalent to

$$\delta \in \bigcup_n (\delta_{k_1mn}, \delta_{k_2mn}) \text{ if } 0 \in (\lambda_{k_1m}, \lambda_{k_2m})$$

$$\delta \in \bigcup_n (\delta_{k_1mn}, \delta_{k_2m(n-1)}) \text{ if } 0 \notin (\lambda_{k_1m}, \lambda_{k_2m})$$

Since τ_{km} and λ_{km} are ordered in the same way, the construction of h above implies that the conditions (13) and (14) are equivalent for $\delta = h(\varepsilon)$.

□

3.4 Construction of equivalence of V, W in the neighborhoods of parabolic cycles

In this section, we construct a required homeomorphism conjugating v_ε to $w_{h(\varepsilon)}$ in the neighborhoods of the parabolic cycles γ, $\tilde{\gamma}$. We will extend it to the whole sphere in the next section.

Lemma 4. *In assumptions of Theorem 6 part 2, for h constructed above and for each small ε, there exists a homeomorphism $H_\varepsilon^{mid} \colon U \to \tilde{U}$ such that*

$$H_\varepsilon^{mid}(U \cap l_j^-(\varepsilon)) = \tilde{U} \cap \tilde{l}_j^-(\delta) \quad \textit{and} \quad H_\varepsilon^{mid}(U \cap l_j^+(\varepsilon)) = \tilde{U} \cap \tilde{l}_j^+(\delta), \tag{15}$$

where $\delta = h(\varepsilon)$.

Proof. For $\varepsilon > 0$, the parabolic cycle disappears, and both vector fields $v_\varepsilon|_U$, $w_\delta|_{\tilde{U}}$ are conjugate to the radial vector field in the annulus $1 < r < 2$. So the statement follows from Lemma 3.

For $\varepsilon = 0$, we may take $H_\varepsilon^{mid} = \hat{H}|_U$.

For $\varepsilon < 0$, the parabolic cycle splits into two. Both vector fields $v_\varepsilon|_U$ and $w_\delta|_{\tilde{U}}$ have two hyperbolic cycles in $U, \tilde{U}$, the outer one is attracting and the inner one is repelling; both oriented clockwise by time orientation. We conclude that $v_\varepsilon|_U$ and $w_\delta|_{\tilde{U}}$ are conjugate. It is easy to modify the conjugacy so that it takes characteristic sets on $C^\pm$ to characteristic sets on $\tilde{C}^\pm$, i.e. the conditions (15) are satisfied. □

3.5 Proof of Theorem 6 part 2 modulo Lemma 5 on flexibility of authomorphisms

Recall that v_0 is topologically conjugate to w_0, and the conjugacy $\hat{H}$ takes $l_j^\pm$ to $\tilde{l}_j^\pm$.

Now, $v_0|_{D^\pm}$ is structurally stable, so $v_\varepsilon|_{D^\pm}$ is conjugate to it. Similarly, $w_0|_{\tilde{D}^\pm}$ is conjugate to $w_{h(\varepsilon)}|_{\tilde{D}^\pm}$. Thus there exist two homeomorphisms $H_\varepsilon^+ \colon D^+ \to \tilde{D}^+$ and $H_\varepsilon^- \colon D^- \to \tilde{D}^-$ that conjugate $v_\varepsilon|_{D^\pm}$ to $w_\delta|_{\tilde{D}^\pm}$ and take $l_j^\pm(\varepsilon)$ to $\tilde{l}_j^\pm(\delta)$, where $\delta = h(\varepsilon)$.

Now let us construct H_ε.

For $\varepsilon = 0$, we simply take $H_\varepsilon = \hat{H}$. Our goal for $\varepsilon \neq 0$ will be to agree $H_\varepsilon^\pm$ with H_ε^{mid} constructed in Lemma 4.

We will use the following lemma. Let $\operatorname{Sep} v$ be the union of all separatrices of v, $\operatorname{Per} v$ be the union of all limit cycles of v, $\operatorname{Sing} v$ be the union of all singular points of v.

Lemma 5 (On flexibility of automorphisms)**.** *Let v be a smooth Morse-Smale vector field in a closed disc D with smooth boundary. Let v be transversal to ∂D.*

Then any orientation-preserving homeomorphism $g\colon \partial D \to \partial D$ such that g is identical on $\partial D \cap \operatorname{Sep} v$ extends to an orbital topological automorphism $G\colon D \to D$ of the vector field v, and $G|_{\operatorname{Sing} v \cup \operatorname{Per} v} = id$.

The proof of this lemma constitutes Section 3.6 below. Here we finish the proof of Theorem 6 modulo Lemma 5.

In order to agree H_ε^+ to H_ε^{mid} on C^+, we apply this lemma to the vector field $v_\varepsilon|_{D^+}$ and the homeomorphism $g = (H_\varepsilon^+)^{-1} \circ H_\varepsilon^{mid}$ on C^+, and get an automorphism G^+ of $v_\varepsilon|_{D^+}$. Now $H_\varepsilon^+ \circ G^+$ is a sing-equivalence of v_ε and w_δ that takes D^+ to $\tilde{D}^+$, coincides with H_ε^{mid} on C^+ and conjugates v_ε to $w_{h(\varepsilon)}$ in D^+ and $\tilde{D}^+$. Similarly, we construct a homeomorphism $H_\varepsilon^- \circ G^-$ in D^- that coincides with H_ε^{mid} on C^- and conjugates v_ε to $w_{h(\varepsilon)}$. These two homeomorphisms $H_\varepsilon^\pm \circ G^\pm$ and H_ε^{mid} glue into the sing-equivalence H_ε on the whole sphere, and the proof of Theorem 6 is complete.

3.6 Flexibility of automorphisms

This section is devoted to the proof of Lemma 5. We start with some general definitions.

Definition 15. Let v be a Morse-Smale vector field. The union of all singular points, separatrices and limit cycles of v is called a *separatrix skeleton* of v. The connected components of the complement to the separatrix skeleton are called *canonical regions*.

Clearly, each canonical region R has a common α- and ω-limit set; we denote them by $\alpha(R)$ and $\omega(R)$. The following statement is formulated in [DLA].

Proposition 5. *Any canonical region R of a C^2-smooth vector field on S^2 is parallel, i.e. $v|_R$ is topologically equivalent to one of the following:*

- *strip flow: $\partial/\partial x$ in the strip $\Pi := \mathbb{R} \times (0, 1)$;*
- *spiral flow: $\partial/\partial r$ in $\mathbb{R}^2 \setminus 0$, where (r, ϕ) are polar coordinates;*
- *annular flow: $\partial/\partial \phi$ in $\mathbb{R}^2 \setminus 0$.*

The last case does not appear for Morse-Smale vector fields. The possible shapes for canonical regions of Morse-Smale fields are shown on Fig. 9, see also [N, Sec. 1.2]. In particular, each strip canonical region is bounded by four or three separatrices.

Proposition 5 provides us by continuous charts in R that conjugate v to standard vector fields. This will enable us to prove the following propositions.

Proposition 6. *Let R be a **strip** canonical region of a Morse-Smale vector field v on the sphere. Let T be a transversal to v that intersects all trajectories of $v|_R$. Let endpoints of T be located on separatrices that bound R.*

Then given a homeomorphism $g\colon T \to T$ of a transversal that fixes endpoints of T, we may extend it to an automorphism $G\colon \overline{R} \to \overline{R}$ of v such that $G|_{\partial R} = id$.

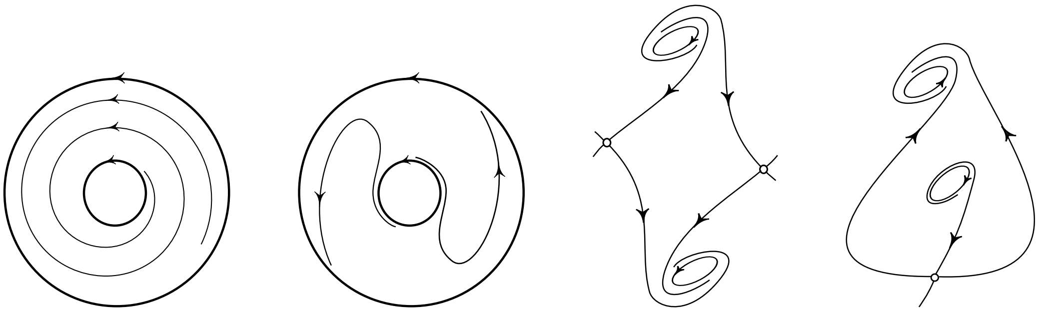

Figure 9: Possible shapes of strip and spiral canonical regions for Morse-Smale vector fields. Each limit cycle may be replaced by a singular point

Proposition 7. *Let R be a **spiral** canonical region of a Morse-Smale vector field v on the sphere. Let T be a closed transversal to v that intersects all trajectories of $v|_R$.*

Then given an orientation-preserving homeomorphism $g\colon T \to T$ of a transversal, we may extend it to an automorphism $G\colon \overline{R} \to \overline{R}$ of v such that $G|_{\partial R} = id$.

The idea of their proofs is to construct the extension G in the canonical charts provided by Proposition 5. We may take G that preserves the trajectories of v in the canonical chart and coincides with g on T. The continuity of G on ∂R requires a little more caution.

Now we prove Lemma 5 modulo these propositions.

The proof of Lemma 5. Without loss of generality, assume that on ∂D, the vector field v points inside D. For $v|_D$, we are going to refer to the general statements that hold for vector fields on the sphere. So we will extend v smoothly to the complement of D by a radial vector field having one hyperbolic source in $S^2 \setminus D$. The vector field $\hat{v}$ thus obtained is Morse-Smale.

Define the required homeomorphism G to be identical on the separatrix skeleton of $\hat{v}$. If a canonical region R of $\hat{v}$ does not intersect ∂D, define $G|_R := id$. If ∂D intersects several strip canonical regions, the map G in all such regions is provided by Proposition 6 above. Finally, if there exists a spiral canonical region R that contains the whole ∂D, the map G is provided by Proposition 7. □

We conclude by proving Propositions 6 and 7.

Proof of Proposition 6. We will need a continuous chart $\psi\colon \Pi \to R$ that conjugates $\partial/\partial x$ to v (as the one provided by Proposition 5), but with the following additional properties.

- ψ takes the vertical open segment $\tilde{T} = \{0\} \times (0,1) \subset \Pi$ to T.
- ψ extends continuously to $\partial\Pi$. ψ^{-1} extends continuously to $\psi(\mathbb{R}\times\{0\})$ and $\psi(\mathbb{R}\times\{1\})$ (however it is possible that ψ^{-1} is not well-defined on ∂R, see Fig. 9d);
- The diameter of a transversal $\psi(\{x\} \times (0,1))$ to v tends to zero as $x \to \pm\infty$.

To satisfy all properties, we act in the following way. Let $U(\alpha(R))$ and $U(\omega(R))$ be small neighborhoods of $\alpha(R), \omega(R)$. We take the chart ψ provided by Proposition 5, then shift and rescale it on each horizontal line in Π so that it takes $\tilde{T}$ to T and provides a natural parametrization on each trajectory of v in $R \setminus U(\alpha(R)) \setminus U(\omega(R))$. This ensures the first two requirements in $R \setminus U(\alpha(R)) \setminus U(\omega(R))$. The set $\overline{R} \cap U(\alpha(R))$ is a family of non-singular trajectories of v, and it is easy to choose ψ^{-1} on this set so that the second requirement is still satisfied; the same holds for $R \cap U(\omega(R))$. After all, ψ satisfies the first and the second requirement.

Now if both $\alpha(R)$ and $\omega(R)$ are singular points, the third requirement is automatically satisfied: $\psi(\{x\} \times (0,1))$ is in a small neighborhood of $\alpha(R)$ or $\omega(R)$ for x close to $\pm\infty$, so has a small diameter. If $\alpha(R)$ is a limit cycle c, we further modify ψ in $U(\alpha(R))$. Namely, we choose a continuous family of transversals to v: one transversal at each point of c. Then we modify ψ^{-1} near c so that it takes the intersections of these transversals with R to vertical segments in Π. We do the same in a small neighborhood of $\omega(R)$ if this set is a limit cycle as well. Clearly, all the three requirements are satisfied after such modifications.

Now, let $\hat{g} := \psi^{-1} g \psi$ be the map g in ψ-chart. Then it extends to the map $\hat{G} \colon \Pi \to \Pi$ given by $\hat{G}(x, y) = (x, \hat{g}(y))$; note that $\hat{G}$ preserves vertical segments in Π. Since g fixes endpoints of T, $\hat{g}$ fixes endpoints of $\hat{T}$. So $\hat{G}$ is identical on the upper and the lower border of Π.

Let $G \colon R \to R$ be the map $\hat{G}$ in ψ^{-1}-chart, i.e. $G = \psi \hat{G} \psi^{-1}$; clearly, $G|_T = g$. Now, G is identical on the separatrices that bound R, due to the continuity of ψ, ψ^{-1} and the corresponding property of $\hat{G}$. Moreover, if we put $G|_{\alpha(R)} = id, G|_{\omega(R)} = id$, the map G is still continuous. Indeed, since $\hat{G}$ preserves vertical segments in Π, the map G near $\alpha(R), \omega(R)$ takes transversals $\psi(\{x\} \times (0,1))$ into themselves, and the diameter of these transversals tends to 0 (see the third requirement on ψ); this implies the statement.

Finally, $G|_{\partial R} = id$. This finishes the proof. □

Proof of Proposition 7. The proof is analogous to the proof of Proposition 6 but simpler, because we only need the continuity of G on $\alpha(R), \omega(R)$. Note that the vector field d/dr in $\mathbb{R}^2 \setminus \{0\}$ is topologically equivalent to the vector field d/dx in the strip with identified borders $\Pi^* := \mathbb{R}^2/((x, y) \sim (x, y + 2\pi))$. We will find a continuous chart $\psi \colon \Pi^* \to R$ that conjugates $\partial/\partial x$ to v and has two additional properties:

- ψ takes the vertical segment $\tilde{T} = \{0\} \times [0, 1] \subset \Pi^*$ to T.
- The diameter of a transversal $\psi(\{y = x + n\})$ to v tends to zero as $n \to \pm\infty$.

As before, if $\alpha(R)$ and $\omega(R)$ are singular points, the second requirement is trivial. To satisfy this requirement near a limit cycle, we choose a family of small transversals to this cycle and require that ψ takes slanted segments $\{y = x + n\}$, n large, to these transversals. The definition of $\hat{g}$ is as before. When we extend $\hat{g}$ to $\hat{G}$, we choose $\hat{G}$ so that it preserves horizontal lines and also preserves slanted segments $\{y = x + n\}$ for large n. As before, $G = \psi \hat{G} \psi^{-1}$ is a continuous map in R that satisfies $G|_T = g$. Moreover, we may extend G continuously to $\alpha(R), \omega(R)$ by the identity map, because the map G preserves short transversals $\psi(\{y = x + n\})$ to v. This finishes the proof. □

4 Structural stability

In this section Theorem 3, hence, Theorem 1, is proved.

4.1 Reduction to the Classification theorem.

Proposition 8. *For C^4-close vector fields v, w of class PC, the corresponding characteristic pairs of sets $A^\pm(v), A^\pm(w)$ are respectively close.*

This proposition is proved in the next two sections. Let us now check that two close local $\mathcal{PC}$ families satisfy the assumptions of Theorem 6 part 2. This will imply Theorem 3.

Let V be an unfolding of v: $V = \{v_\varepsilon | \varepsilon \in (\mathbb{R}, 0)\}$, $v_0 = v$. Let $w \in PC$ be so close to v that the Sotomayor theorem is applicable: w is orbitally topologically equivalent to v. Moreover, let v and w be so close that the corresponding pairs of characteristic sets $A^\pm(v), A^\pm(w)$ are close, see Proposition 8. Then they are equivalent in the sense of Definition 13. Now let W be a $\mathcal{PC}$-family that unfolds w. All the assumptions of Theorem 6 part 2 for the families V and W are justified. Hence, they are equivalent. Therefore, the family V is structurally stable. Theorem 3 is proved modulo Proposition 8.

4.2 Takens theorem with a parameter

The main step in the proof of Proposition 8 is to check that canonical coordinates on the transversal loops for close vector fields of class PC are also close. Equivalently, we should prove that the time functions $T^\pm$ defined in Sec. 1.6 are close for close vector fields. It is sufficient to check that the generators of close parabolic germs are close.

Proposition 9. *Suppose that two parabolic germs are C^4-close. Then their generators are C-close. In more detail, for any parabolic germ there exists a representative P with the following property. Let U be the domain of P. Then there exists a neighborhood $V \subset U$ of 0 such that any map Q which is sufficiently C^4-close to P in U has a generator C-close to that of P in V.*

Proof. First, recall the main steps of the proof of Takens Theorem (see Theorem 2), according to [M] and [I90]. Let $P(x) = x + x^2 + (a+1)x^3 + \dots$ be a real smooth germ. Then P is formally equivalent to the time one shift P_0 along the vector field u_a:

$$P_0 = g^1_{u_a},\ u_a = \frac{x^2}{1 - ax}. \tag{16}$$

The maps P and P_0 have the same 3-gets at 0. Hence,

$$P = P_0 + R,\ |R| \le C|x^4| \text{ and } |R'| \le C'|x^3| \tag{17}$$

for some $C, C' > 0$. In what follows, different constants depending on the functions considered are denoted by C with subscripts and superscripts. The chart

$$t = -\frac{1}{x} - a \ln x$$

rectifies the vector field u_a and brings a neighborhood of 0 to a neighborhood of infinity. The maps P and P_0 written in the chart t are denoted by $\hat{P}, \hat{P}_0$. Clearly, $\hat{P}_0$ is a mere translation by 1: $\hat{P}_0(t) = t + 1$. Let $\hat{P} = \hat{P}_0 + \hat{R} = t + 1 + \hat{R}$. Then $|\hat{R}| < C|t^{-2}|$, see Proposition 10 below. Let us find a map $H = id + h$ defined near infinity that conjugates $\hat{P}$ and $\hat{P}_0$. This map is found separately in two half-neighborhoods $(\mathbb{R}^-, \infty)$, $(\mathbb{R}^+, \infty)$. Let us find it in $(\mathbb{R}^+, \infty)$. Note that h satisfies $\hat{P}_0 \circ (id + h) = (id + h) \circ \hat{P}$, which implies the Abel equation on h:

$$h = h \circ \hat{P} + \hat{R}.$$

The solution of this equation in $(\mathbb{R}^+, \infty)$ has the form:

$$h^+ = \sum_{k=0}^{\infty} \hat{R} \circ \hat{P}^k. \tag{18}$$

Due to Proposition 11 below, this series converges on $\mathbb{R}^+$ near infinity; moreover, if C, C' in (17) are small, then h is C^1-small. Finally, we have found the desired generator u of P. In the coordinate $H = id + h^+$, this generator is a unit vector field $\mathbf{e}$. In the coordinate t, it equals $(H^{-1})_*\mathbf{e}$. In the initial coordinate, it equals $u = (t^{-1} \circ H^{-1})_*\mathbf{e}$. Together with (18), this provides the desired formula for u in $(\mathbb{R}^+, 0)$.

A similar formula holds in $(\mathbb{R}^-, 0)$ with the only difference that in this case, the solution of the Abel equation is

$$h = h^- = -\sum_{k=1}^{\infty} \hat{R} \circ \hat{P}^{-k}. \tag{19}$$

We stop here and do not check the assertion of Theorem 2 that u is infinitely smooth. See [M] and [I90] for the rest of the proof of Takens theorem.

Proposition 10. *In the above assumptions,* $|\hat{R}| < C_1|t^{-2}|$ *and* $|\hat{R}'| \leq C_1'|t^{-3}|$*; if the constants* C, C' *in* (17) *are small, then* C_1, C_1' *are also small.*

Proof. By definition of $\hat{R}$, $\hat{R} = \hat{P} - \hat{P}_0$. So

$$\hat{R} \circ t = t \circ P - t \circ P_0 = t \circ (P_0 + R) - t \circ P_0.$$

Hence, for some $\theta \in [0, 1]$,

$$|\hat{R}(t(x))| \leq t' \circ (P_0 + \theta R) \cdot |R(x)| \leq \tilde{C}x^{-2} \cdot x^4 = \tilde{C}x^2 < C_1 t^{-2}.$$

Moreover, by the same argument, for some $\theta \in [0, 1]$,

$$\left|\frac{d}{dx}\hat{R} \circ t\right| = |t' \circ (P_0 + R) \cdot (P_0' + R') - t' \circ (P_0) \cdot P_0'| \leq |t'' \circ (P_0 + \theta R) \cdot R \cdot P_0'| + |t' \circ (P_0 + R) \cdot R'| \leq \tilde{C}_1|x|;$$

dependence on x in the left and the middle part of the display is skipped for brevity. Hence,

$$|\frac{d}{dt}\hat{R}| \leq \frac{|\frac{d}{dx}\hat{R} \circ t|}{|t'(x)|} \leq \tilde{C}_1'|x|^3 \leq C_1'|t|^{-3}. \tag{20}$$

Note that if C, C' in (17) are small, then C_1 and C_1' are also small. □

Proposition 11. *In the above assumptions, the series for* h^+ *converges in* $(\mathbb{R}, +\infty)$*; if the constants* C, C' *in* (17) *are small, then* h^+ *is small in* C^1 *metric.*

Proof. The series (18) converges because $\hat{R}$ decreases as $C_1 t^{-2}$, and $\hat{P}^k$ increases as an arithmetic progression. Moreover, for small C, C' in (17), C_1 is also small, thus h is small in C metric near infinity.

To estimate h', note that $h' = \sum(\hat{R}' \circ \hat{P}^k) \cdot (\hat{P}^k)'$. The function $\hat{R}'$ is small by (20). The sequence $\hat{P}^k$ increases as an arithmetic progression. It remains to prove that the sequence $(\hat{P}^k)'$ is bounded; this will imply that h' is small in C-metric.

Let us estimate $\hat{P}^k(t)'$ for $t \in \mathbb{R}^+$ large. This derivative is a product of values of the function $\hat{P} = 1 + \hat{R}'$ along the first k points of the orbit of t under the map $\hat{P}$. This orbit grows as an arithmetic progression. The logarithm of the product mentioned above is no greater than the sum $c\sum_{k=1}^{\infty} |\hat{R}' \circ \hat{P}^k(t)|$. By (20), this sum is uniformly bounded in a neighborhood of infinity. Hence, the sequence $(\hat{P}^k)'$ is uniformly bounded for large t as required. □

Let us now prove Proposition 9. Takens theorem implies that the normalizing chart that conjugates P and P_0 is infinitely smooth. From now on, we switch to this chart; this reduces the general case to the case when $P = g^1_{u_a}$ for some a, and Q is C^4-close to it.

Put $Q = Q_0 + R$ where $Q_0 = g^1_{u_b}$ for some b. Since $a = \frac{P^{(3)}(0)}{6} - 1,\ b = \frac{Q^{(3)}(0)}{6} - 1$, we conclude that a and b are close. As above, $|R| \leq Cx^4$, $|R'| \leq C'|x^3|$; moreover, C, C' are small, because $P - P_0$ is zero and Q, Q_0 are C^4-close to P, P_0.

Now, let us repeat the arguments from the proof of Takens theorem for $Q = Q_0 + R$. Let $t_b = -\frac{1}{x} + b\ln x$ be the rectifying chart for the vector field u_b, and $\hat{Q} = t + 1 + \hat{R}$ be the map Q written in this chart. Let h be the map given by (18) and (19) for this $\hat{R}$; put $H = id + h$. Then the generator u_Q for the map Q is given by $u_Q = (t_b^{-1}H^{-1})_*\mathbf{e}$. Due to Proposition 11, h is C^1-small, thus H is C^1-close to identity; so u_Q is close to $(t_b^{-1})_*\mathbf{e} = u_b$. Finally, u_b is close to u_a because a and b are close as we showed above. So u_Q is close to u_a. This proves Proposition 9.

□

4.3 Proximity of the characteristic sets

Here we complete the proof of Proposition 8.

Let v and w be two close vector fields of class PC, γ and $\tilde{\gamma}$ be their (close) parabolic cycles, and $C^{\pm}$ be their common transversal loops that separate the parabolic cycles from the rest of the sphere. The vector fields v and w are orbitally topologically equivalent as explained above. Let $E_m, \tilde{E}_m,\ m = 1, \dots, M$ and $I_k, \tilde{I}_k,\ k = 1, ..., K$ be the saddles of v and w respectively, whose separatrices wind to γ and $\tilde{\gamma}$ as described above; E_m and $\tilde{E}_m$, I_k and $\tilde{I}_k$ are close to each other.

Then the intersection points of the separatrices of these saddles with the transversal loops $C^{\pm}$ are close. But we have to prove that these points are close on the coordinate circles with the canonical coordinates $\varphi_0^{\pm}, \tilde{\varphi}_0^{\pm}$. The latter statement follows from Proposition 9. This proves Proposition 8, and completes the proof of the Structural stability Theorems 1 and 3.

5 Bifurcation support vs large bifurcation support

In [AAIS], Arnold introduced a notion of a *bifurcation support.* Begin with the quotation from Arnold.

Although even local bifurcations in high codimensions (at least three) on a disc are not fully investigated, it is natural to discuss nonlocal bifurcations in multiparameter families of vector fields on a two-dimensional sphere. For their description, it is necessary to single out the set of trajectories defining perestroikas in these families.

Definition 16. A finite subset of the phase space is said to *support a bifurcation* if there exists an arbitrarily small neighborhood of this subset and a neighborhood of the bifurcation values of the parameter (depending on it) such that, outside this neighborhood of the subset, the deformation (at values of the parameter from the second neighborhood) is topologically trivial.

Definition 17. The *bifurcation support* of a bifurcation is the union of all minimal sets supporting a bifurcation ("minimal" means not containing a proper subset that supports a bifurcation).

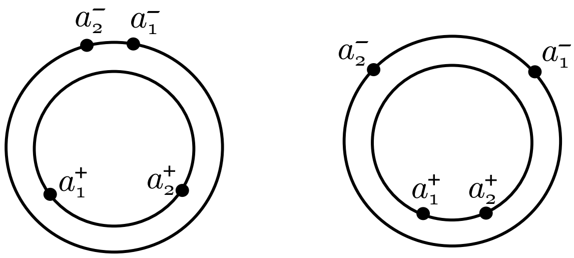

Figure 10: Non-equivalent pairs of characteristic sets on coordinate circles

Definition 18. Two deformations of vector fields with bifurcation supports Σ_1 and Σ_2 are said to be *equivalent on their supports* if there exist arbitrarily small neighborhoods of the supports, and neighborhoods of the bifurcation values of the parameters depending on them, such that the restrictions of the families to these neighborhoods of the supports are topologically equivalent, or weakly equivalent, over these neighborhoods of bifurcation values.

The quotation ends here. The following theorem shows that the bifurcation support is insufficient for the description of the bifurcations.

Theorem 8. *There exist two orbitally topologically equivalent vector fields of class PC, whose generic unfoldings in one-parameter families are equivalent on their supports, but not sing-equivalent on the whole sphere.*

This is an improved version of Theorem 5.

Proof. Consider a vector field v of class PC. A bifurcation carrier is an arbitrary point on the parabolic cycle γ of this field. The bifurcation support is the cycle γ itself. Under the unfolding of v, the cycle γ splits in two on one side of the critical value of the parameter, and vanishes on the other side. For any two vector fields of class PC their deformations are equivalent on their supports.

Consider now two vector fields of class PC with non-equivalent pairs of characteristic sets, see Figure 10 for example. By Theorem 6, the unfoldings of these fields are not sing-equivalent. □

This proves Theorem 8. Simultaneously Theorem 5 is proved.

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