Polynomial growth of cyclicity for elementary polycycles and Hilbert–Arnold problem

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Equadiff 2011
5th August 2011

From H16 to Hilbert–Arnold Problem

Problem (Existential Hilbert Problem)

Prove that for any $n \geq 2$ there exists a number $H(n) < \infty$ such that any polynomial line field of degree $\leq n$ has at most $H(n)$ LC's

is a particular case of

Problem (Global Finiteness Conjecture)

For any analytic family of line fields on $\mathbb{S}^2$ with a compact parameter base B the number of LC's is uniformly bounded.

Difficulties

- Analytical families are "rigid"
- "Bad" limit periodic sets (e.g. nonisolated singularities)

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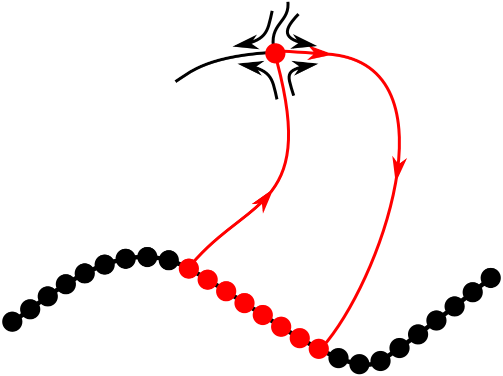

Figure: Limit periodic set containing a curve of singular points

Problem (Hilbert–Arnold Problem)

Prove that in a generic finite-parameter family of smooth vector fields on $\mathbb{S}^2$, the number of LC's is uniformly bounded.

Definition

- Let $\{v(x,\varepsilon)\}_{\varepsilon \in B^k}$ be k-parameter family of vector fields on $\mathbb{S}^2$ having a polycycle γ for some $\varepsilon_* \in B$. The cyclicity μ of polycycle γ is the maximal number of LC's that can born near γ for ε close to ε_*.
- The bifurcation number $B(k)$ is the maximal cyclicity of a nontrivial polycycle occuring in generic k-parameter family.

Problem (Local Hilbert–Arnold Problem (LHAP))

Estimate $B(k)$ for any k.

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Results on LHAP

- $B(1) = 1$ (Andronov-Leontovich, 1930s; Hopf, 1940s)
- $B(2) = 2$ (Takens, Bogdanov, Leontovich–Cherkas, Mourtada, Grozovskii, early 1970s–1993)

Definition

- Polycycle is called elementary if its vertices are all elementary singular points (i.e. hyperbolic saddles and saddle-nodes)
- $E(k)$ is the same as $B(k)$ for elementary polycycles.

Results for $E(k)$ for any k

- $E(k) < \infty$ (Ilyashenko–Yakovenko, 1995)
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Statement of the problem

As number of parameters k increases, two effects contribute to growth of cyclicity:

1. Increasing of complexity of polycycles (e.g. more vertices);
2. Increasing of cyclicity of particular polycycle.

Problem: fix complexity of polycycle and estimate its cyclicity with respect to number of parameters k.

Example: cyclicity of simple separatrix loop $\leq k$ (Leontovich, 46; Roussarie, 86). This estimate is sharp (Ilyashenko–Yakovenko, 91).

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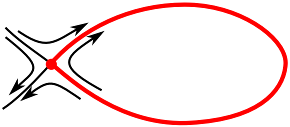

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Definition

Let $E(n, k)$ be a maximal cyclicity of a polycycle with n vertices in generic k-parameter family.

Theorem (P.Kaleda–I.S., 2010)

$$E(n, k) \leq C(n)k^{3n}.$$

- Good news: polynomial growth with respect to codimension k.
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2. Normal forms near singular points
3. Khovanski reduction
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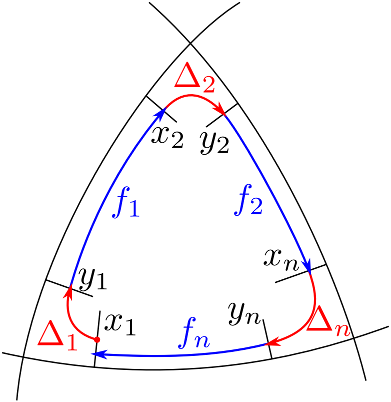
Poincaré map and basic system

LC's correspond to isolated solutions of the following system:

$$\begin{cases} y_j = \Delta_j(x_j;\varepsilon) \\ x_{j+1} = f_j(x_j;\varepsilon) \\ x_{n+1} = x_1 \end{cases}$$

Note that Δ_j's are singular at 0, f_j's are generic smooth.



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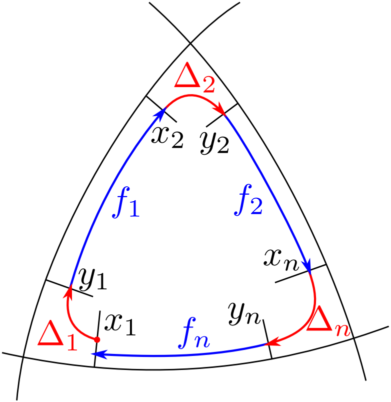

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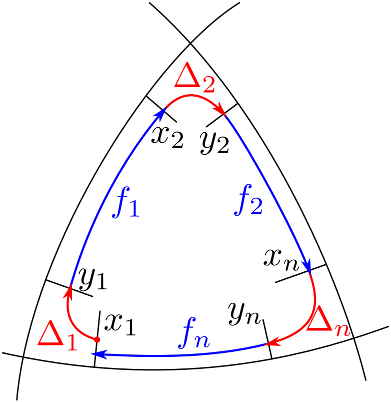

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Normal forms and Pfaffian equations

- Good news: there exists polynomial normal forms for "our" singular points.
- Very good news: these normal forms are integrable, so Δ_j's can be written explicitly.
- Bad news: Δ_j's are still singular (non-smooth at 0).
- Good news: they satisfy Pfaffian equations (differential equations with polynomial coefficients).

Functional-Pfaffian system:

$$\begin{cases} \omega_j = 0, \\ F_j(x, y, \varepsilon) = 0, \end{cases} \qquad j = 1, \ldots, n$$

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Khovanski reduction

- A step of Khovanski reduction: replacing one Pfaffian equation with a functional one.
- Number of roots of initial system is estimated by the number of roots of a new system. (Version of Rolle's Lemma.)
- Crucial observation: n = the number of vertices = the number of Pfaffian equations = the number of Khovanski reduction steps.
- Functional equations we obtain are of the form $P \circ F(x) = a$, where P is a polynomial, F is a generic smooth function. Growth of degree of P on each step is under control.

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Bézout–Kaloshin theorem

Theorem (Kaloshin, 2003)

Let $P = (P^1, \dots, P^n)$ be some vector-polynomial, and $F\colon \mathbb{R}^n \to \mathbb{R}^s$ is a generic smooth vector-function. Under some assumptions, the number of preimages $\#\{x\colon P \circ F(x) = a\}$ can be estimated by the product $\prod_{j=1}^{n} \deg P^j$ (like in Bézout Theorem).

Application of Bézout–Kaloshin Theorem to the functional system obtained as a result of Khovanski reduction gives an estimated for the number of roots of Basic system and thus for the number of LC's

Crucial point: we must control the growth of degree on every step of Khovanski reduction and the overall number of steps.

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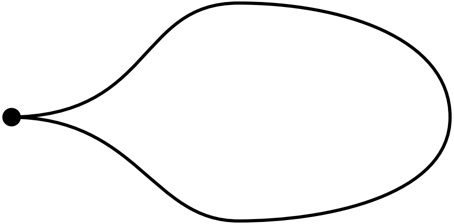

Figure: Polycycle with cuspidal point

Problem

Estimate the cyclicity of polycycle with one cuspidal singular point.

Details

- Yu. Ilyashenko, S. Yakovenko, *Finite Cyclicity of Elementary Polycycles in Generic Families*, Concerning the Hilbert 16th Problem, Amer. Math. Soc. Transl. Ser. 2, **165**, Amer. Math. Soc., Providence, RI, 1995, 21-96
- V. Kaloshin, *The Existential Hilbert 16-th Problem and an Estimate for Cyclicity of Elementary Polycycles*, Invent. math, **151** (2003), 451-512
- P. I. Kaleda, I. V. Shchurov. *Cyclicity of elementary polycycles with fixed number of singular points in generic k-parameter families.* St. Petersburg Math. J. **22** (2011), 557-571.

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