Polynomial growth of cyclicity for elementary polycycles and Hilbert–Arnold problem

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Problem (Existential Hilbert Problem)

Prove that for any $n \geq 2$ there exists a number $H(n) < \infty$ such that any polynomial line field of degree $\leq n$ has at most $H(n)$ LC’s.

is a particular case of

Problem (Global Finiteness Conjecture)

For any analytic family of line fields on $S^2$ with a compact parameter base $B$ the number of LC’s is uniformly bounded.

Difficulties

- Analytical families are “rigid”
- “Bad” limit periodic sets (e.g. nonisolated singularities)
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“Bad” limit periodic sets

**Figure:** Limit periodic set containing a curve of singular points
Problem (Hilbert–Arnold Problem)

Prove that in a *generic* finite-parameter family of smooth vector fields on $S^2$, the number of LC’s is uniformly bounded.

Definition

- Let $\{v(x, \varepsilon)\}_{\varepsilon \in B^k}$ be a $k$-parameter family of vector fields on $S^2$ having a polycycle $\gamma$ for some $\varepsilon_* \in B$. The cyclicity $\mu$ of polycycle $\gamma$ is the maximal number of LC’s that can born near $\gamma$ for $\varepsilon$ close to $\varepsilon_*$.  
- The bifurcation number $B(k)$ is the maximal cyclicity of a nontrivial polycycle occurring in a generic $k$-parameter family.

Problem (Local Hilbert–Arnold Problem (LHAP))

Estimate $B(k)$ for any $k$. 

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Overview

Polynomial growth

Proof

Conclusion

Previous results

Results on LHAP

- $B(1) = 1$ (Andronov-Leontovich, 1930s; Hopf, 1940s)
- $B(2) = 2$ (Takens, Bogdanov, Leontovich–Cherkas, Mourtada, Grozovskii, early 1970s–1993)

Definition

- Polycycle is called elementary if its vertices are all elementary singular points (i.e. hyperbolic saddles and saddle-nodes)
- $E(k)$ is the same as $B(k)$ for elementary polycycles.

Results for $E(k)$ for any $k$

- $E(k) < \infty$ (Ilyashenko–Yakovenko, 1995)
- $E(k) \leq 2^{25k^2}$ (Kaloshin, 2003)
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As number of parameters $k$ increases, two effects contribute to growth of cyclicity:

1. Increasing of complexity of polycycles (e.g. more vertices);
2. Increasing of cyclicity of particular polycycle.

**Problem**: fix complexity of polycycle and estimate its cyclicity with respect to number of parameters $k$.

**Example**: cyclicity of simple separatrix loop $\leq k$ (Leontovich, 46; Roussarie, 86). This estimate is sharp (Ilyashenko–Yakovenko, 91).
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Main result

Definition

Let $E(n, k)$ be a maximal cyclicity of a polycycle with $n$ vertices in generic $k$-parameter family.

Theorem (P. Kaleda–I. S., 2010)

$$E(n, k) \leq C(n)k^{3n}.$$ 

- Good news: polynomial growth with respect to codimension $k$.
- Bad news: $C(n) = 2^{5n^2 + 20n}$. 
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1. Poincaré map and basic system
2. Normal forms near singular points
3. Khovanski reduction
4. Bézout–Kaloshin theorem
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Poincaré map and basic system

LC’s correspond to isolated solutions of the following system:

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\begin{align*}
    y_j &= \Delta_j(x_j; \varepsilon) \\
    x_{j+1} &= f_j(x_j; \varepsilon) \\
    x_{n+1} &= x_1
\end{align*}
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Note that $\Delta_j$’s are singular at 0, $f_j$’s are generic smooth.

Figure: Basic system
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Normal forms and Pfaffian equations

- **Good news**: there exists polynomial normal forms for “our” singular points.
- **Very good news**: these normal forms are integrable, so $\Delta_j$’s can be written explicitly.
- **Bad news**: $\Delta_j$’s are still singular (non-smooth at 0).
- **Good news**: they satisfy Pfaffian equations (differential equations with polynomial coefficients).

Functional-Pfaffian system:

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\begin{align*}
\omega_j &= 0, \\
F_j(x, y, \varepsilon) &= 0,
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Khovanski reduction

- A step of Khovanski reduction: replacing one Pfaffian equation with a functional one.
- Number of roots of initial system is estimated by the number of roots of a new system. (Version of Rolle’s Lemma.)
- Crucial observation: \( n = \) the number of vertices = the number of Pfaffian equations = the number of Khovanski reduction steps.
- Functional equations we obtain are of the form \( P \circ F(x) = a \), where \( P \) is a polynomial, \( F \) is a generic smooth function. Growth of degree of \( P \) on each step is under control.
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Bézout–Kaloshin theorem

**Theorem (Kaloshin, 2003)**

Let \( P = (P^1, \ldots, P^n) \) be some vector-polynomial, and \( F : \mathbb{R}^n \to \mathbb{R}^s \) is a generic smooth vector-function. Under some assumptions, the number of preimages \( \#\{x : P \circ F(x) = a\} \) can be estimated by the product \( \prod_{j=1}^n \deg P^j \) (like in Bézout Theorem).

Application of Bézout–Kaloshin Theorem to the functional system obtained as a result of Khovanski reduction gives an estimated for the number of roots of Basic system and thus for the number of LC’s.

Crucial point: we must control the growth of degree on every step of Khovanski reduction and the overall number of steps.
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**Problem**

*Estimate the cyclicity of polycycle with one cuspidal singular point.*
Details


Thank you for your attention!
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