Duck farming on the two-torus: multiple canard cycles in generic slow-fast systems

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Abstract

Generic slow-fast systems with only one (time-scaling) parameter on the two-torus have attracting canard cycles for arbitrary small values of this parameter. This is in drastic contrast with the planar case, where canards usually occur in two-parametric families. In present work, general case of nonconvex slow curve with several fold points is considered. The number of canard cycles in such systems can be effectively computed and is no more than the number of fold points. This estimate is sharp for every system from some explicitly constructed open set.

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1 Introduction

Consider a generic slow-fast system:

$$\begin{cases} \dot{x} = f(x,y,\varepsilon) \\ \dot{y} = \varepsilon g(x,y,\varepsilon) \end{cases} \qquad \varepsilon \in (\mathbb{R}, 0). \tag{1}$$

For the planar case (i.e. $(x,y) \in \mathbb{R}^2$), there is a rather simple description of its behavior for small ε. It consists of interchanging phases of slow motion along stable parts of the slow curve $M := \{(x,y) \mid f(x,y,0) = 0\}$ and fast jumps along straight lines $y = \text{const}$. (See e.g. [5].) Given additional parameters, which depend on ε, one can observe more complicated behavior: appearance

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of *duck* (or *canard*) solutions (particularly limit cycles), i.e. solutions, whose phase curves contain an arc of length bounded away from 0 uniformly in ε, that keep close to the unstable part of the slow curve [4]. .

In [1], Yu. S. Ilyashenko and J. Guckenheimer discovered a new kind of behavior of slow-fast systems on the two-torus. It was shown that for some particluar family with no auxiliary parameters there exists a sequence of intervals accumulating at 0, such that for any ε from these intervals, the system has exactly two limit cycles, both of which are canards, where one is stable and the other unstable. Yu. S. Ilyashenko and J. Guckenheimer conjectured that there exists an open domain in the space of slow-fast systems on the two-torus with similar properties. Here we proof this conjecture, and provide almost complete description for bifurcations of canard cycles on the two-torus. Particularly, we give sharp estimate for the number of canard cycles in such systems.

Our main results follow. Consider system (1) and assume that the phase space is the two-torus:

$$(x, y) \in \mathbb{T}^2 \cong \mathbb{R}^2/(2\pi\mathbb{Z}^2). \tag{2}$$

Assume that the speed of the slow motion is bounded away from zero ($g > 0$), the slow curve M is a smooth connected curve, and its lift to the covering coordinate plane is contained in the interior of the fundamental square $\{|x| < \pi,\ |y| < \pi\}$. We also assume that all fold points (i.e. the points of M where the tangent to M is parallel to x-axis) are nondegenerate (the tangency rate is quadratic). In this case, the number of fold points is finite and even: let us denote it by $2N$.

Theorem 1.1. *For any generic slow-fast system on the two-torus with the described properties, under some additional nondegenericity assumptons, the following properties hold. There exists a positive number $k \leq N$ and a sequence of intervals accumulating to zero, such that for every ε that belongs to these intervals the system has exactly k attracting and k repelling limit cycles, which make one rotation along y-axis during the period. All these cycles are canards. The measure of their basins is bounded from below uniformly for $\varepsilon \to 0^+$. For any small $\varepsilon > 0$, the number of limit cycles that make one rotation along y-axis is bounded by $2k$.*

Theorem 1.2. *There exists an open set in the space of slow-fast systems on the two-torus for which the number k of pairs of canard cycles is maximal and equal to N.*

The paper is organized as follows. In section 2 we provide heuristic description of the phenomena discussed. In section 3 preliminary results about

Poincaré map are stated. Section 4 gives an overview of the proof of theorem 1.1. This proof rely on auxiliary results, which are discussed in sections 5 and 6. Section 7 is devoted to construction of system with maximal number of canards and proof of theorem 1.2.

Due to size limitations, we omit technical details from presented proofs. We refer the reader to works [2, 3] for more detailed discussion.

2 Description of the phenomena

In this section we provide heuristic description of the phenomena discovered by Ilyashenko and Guckenheimer. In what follows, we will assume that x-axis of fast motion is vertical, and y-axis is horizontal. The slow motion is directed from the left to the right.

We consider first the simplest case: M is a *convex curve* and therefore it have exactly two fold points (i.e. $N = 1$).[1] The right one is called *jump point* and the left one is *reverse jump point*. Consider a strip B in the phase space that contain M and bounded by vertical circles which pass through fold points. (See fig. 1.) We will call it *base strip*. In more generic (nonconvex) case the base strip is defined as the minimal vertical strip which contains M.

Fix some vertical cross-section $\Gamma = \{y = \text{const}\}$ that does not interset M. We will assume without loss of generality that $\Gamma = \{y = -\pi\} = \{y = \pi\}$. Consider some point $w \notin M$ from the interior of the base strip $\mathring{B}$. Trajectory, which pass through this point, in forward time attracts quickly to the stable part of the slow curve, then move slowly to the right until reaches jump point, then "jumps" and continue slow motion along the y-axis, making about $1/\varepsilon$ rotations along the x-axis before it intersects Γ (call this phase *after-jump rotations*). For given ε, denote the point of first intersection with Γ by $R(\varepsilon)$.

In backward time, the trajectory quickly attracts to *repelling* part of the slow curve, moves slowly to the left until reaches reverse jump point, jumps, and continue slow motion along y-axis while rotating along x-axis, up to intersection with Γ. Denote the point of intersection by $L(\varepsilon)$. This trajectory is *canard*, because it has a segment which is close to repelling part of the slow curve.

As $\varepsilon > 0$ decreases, "fast" parts of the trajectory become more vertical, and the number of rotations during the after-jump motion increases. Therefore, point $R(\varepsilon)$ moves upwards, and $L(\varepsilon)$ moves downwards. By continuity, there exists ε_1 such that for $\varepsilon = \varepsilon_1$ these two points coincide: $R(\varepsilon_1) = L(\varepsilon_1)$.

[1] In fact, only the latter condition matters: any system with $N = 1$ can be considered as a system with convex slow curve.

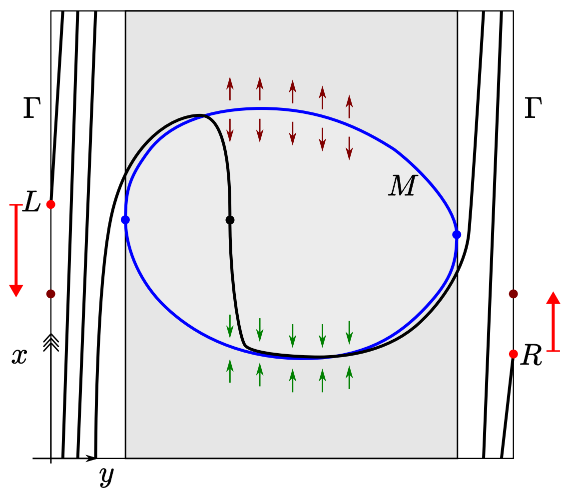

Figure 1: Canard solution of the system with convex slow curve: the base strip is shadowed

This gives us canard limit cycle. As $\varepsilon > 0$ continue decreasing, new coincidense occurs for some $\varepsilon = \varepsilon_2$, $0 < \varepsilon_2 < \varepsilon_1$, and so on. Therefore, for the sequence of parameter value $\varepsilon = \varepsilon_k$, accumulating at 0, the system has canard limit cycles. By choosing initial point close enough to the reverse jump point, it is possible to make these cycles stable. When we perturb initial point slightly, corresponding values of ε_k also perturb slighly, giving us "canard intervals", whose existence is stated in theorem 1.1.

When we consider more generic case of nonconvex M and $N > 1$, the description becomes more complicated, but main arguments work. Let us assume that no fold points lie on one vertical circle, and w does not lie above or below any fold point. In this case, the trajectory which starts in w, in forward (backward) time falls to attracting (repelling) segment of M, moves slowly to the right (left) until reaches the fold point, jumps and either leaves the base strip or falls to the other attracting (repelling) segment of M, and the process repeats until trajectory leaves the base strip (see fig. 2).

The main difference with convex case here is the possibility of several jumps, which does not affect heuristic arguments presented above, because they deal mostly with after-jump rotations. However, to provide rigorous proof of main results and particularly to calculate the number of limit cycles, it is not enough to use only the discussed ideas. Instead, we have to perform accurate analysis of the Poincaré map from Γ to itself, which is discussed in the next sections.

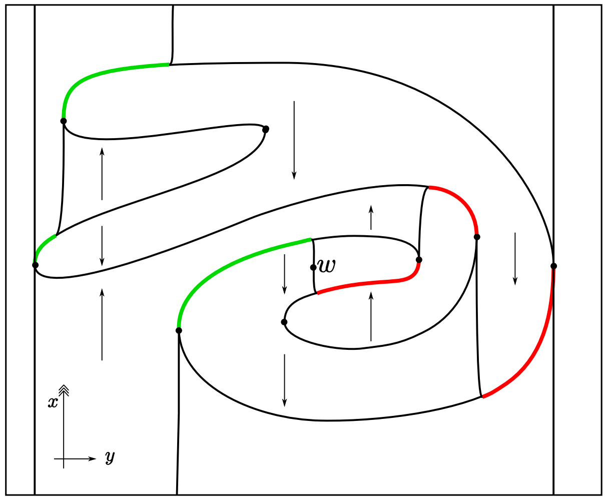

Figure 2: Nonconvex case: several jumps

3 Poincaré map

Note that the function g is bounded away from zero, so we can divide the system (1) by g, thus re-scaling the time: this does not change the desired properties of its solutions (we are interested only in phase curves), and the system with new function f will satisfy the same nondegenericity assumtions. Thus without loss of generality we can assume $g = 1$ in (1).

Consider Poincaré map $P_\varepsilon : \Gamma \to \Gamma$. The slow motion is constant (and bounded away from 0), so P_ε is well-defined diffeomorphism of a circle. Its periodic (particularly, fixed) points correspond to closed solutions of the system. Denote the graph of P_ε by γ_ε. Fixed points of Poincaré map correspond to intersection points of the graph with diagonal $\mathcal{D} := \{y = x\}$. Note, that in terms of previous section, $P_\varepsilon(L(\varepsilon)) = R(\varepsilon)$.

The derivative of Poincaré map in point $x_0 \in \Gamma$ can be easily calculated by integrating equation of variations. If $x = x(y; x_0, \varepsilon)$ is a phase curve with initial condition $x(-\pi; x_0, \varepsilon) = x_0$, it follows immediately that $P_\varepsilon(x_0) = x(\pi; x_0, \varepsilon)$ and

$$P'_\varepsilon(x_0) = \left.\frac{\partial P_\varepsilon(x)}{\partial x}\right|_{x=x_0} = \frac{\partial x}{\partial x_0}(\pi; x_0) = \exp \frac{1}{\varepsilon} \int_{-\pi}^{\pi} f'_x(x(y; x_0, \varepsilon), y, \varepsilon)\, dy. \quad (3)$$

Near attracting (repelling) parts of the slow curve M, the function under the intergral sign is negative (positive), and trajectories attract (repell) each other while moving in these areas. Corresponding parts of trajectories contribute contraction (expansion) to the derivative of Poincaré map. In “most of cases" $\int f'_x(x, y, \varepsilon)dy \neq 0$ and either contraction or expansion dominates, thus giving either exponentially small or exponentially big derivative with respect to ε.

It occurs that it is possible to replace actual trajectory in the right-hand side of (3) with so-called *singular trajectory* (or *contour*), which is defined as follows. For every point $w \in B \setminus M$, which does not lie above or below any fold point of M, recall the description of the trajectory which pass through w traced in backward and forward time up to exit from B (see section 2). Assume that all phases of fast motion in this description are strictly vertical. Then we obtain a picewise-smooth curve in the base strip which consists of vertical segments and arcs of the slow curve M, interchanging each other. Call this curve singular trajectory (or contour) of w and denote it by $Z(w)$. This curve is in a sense a limit (as $\varepsilon \to 0$) of trajectories with initial condition w. The part of the contour to the right of w (which corresponds to the trajectory in forward time) is denoted by $Z^-(w)$, and the part to the left of w (which correspond to backward time) is denoted by $Z^+(w)$.

The following lemma represents the fact that the derivative of Poincaré map is controlled (with given precision) by contour of the corresponding trajectory. It means that main contribution to the derivative is made by segments of slow motion near arcs of the slow curve, which dominates over contributions of jumps and after-jump rotations.

Lemma 3.1. *Fix some $\delta > 0$. Fix some vertical interval J, which intersects attracting part of M and δ-bounded from repelling part of M (and therefore from fold points). Let u be coordinate on J. Consider Poincaré map $Q : J \to \Gamma$ in forward time. Then for any $w \in J$,*

$$\log \frac{dQ(u)}{du}\bigg|_J = \frac{1}{\varepsilon}\left[\int_{Z^-(w)} f'_x(x, y, 0)\, dy + o(1)\right]. \tag{4}$$

Obviously, $Z^-(w)$ does not depend on choice of $w \in J$. The remainder term $o(1) \to 0$ as $\varepsilon \to 0$ uniformly in choice of J provided δ is fixed.

Lemma 3.2. *Let $w \in B \setminus M_\delta$, where M_δ is δ-neighborhood of M, and $y(w)$ (y-coordinate of w) is δ-far from $y(G)$ for any fold point G. Consider actual trajectory that pass through w, and denote its initial condition on Γ by x_0. Then*

$$\log P'_\varepsilon(x_0) = \frac{1}{\varepsilon}\left[\int_{Z(w)} f'_x(x, y, 0)\, dy + o(1)\right]. \tag{5}$$

The proof of these lemmas is given in [3] (see Lemma 5.4 and Lemma 7.3 there), and rely heavily on the analysis of after-jump rotations from [2] (see Theorem 4.3 there, which is reformulated as Theorem 4.6 in [3]).

4 Shape of the graph of P_ε

Our goal is to describe the shape of γ_ε and its dependency on ε. The following description goes back to Shape Lemma in [1], where it is proved for particular example.

Fix some vertical interval J^+ (resp. J^-) in the phase space, which intersects repelling (attracting) part of the slow curve close enough to far left (far right) fold point and bounded from attracting (repelling) part of the slow curve. (See fig. 3.)

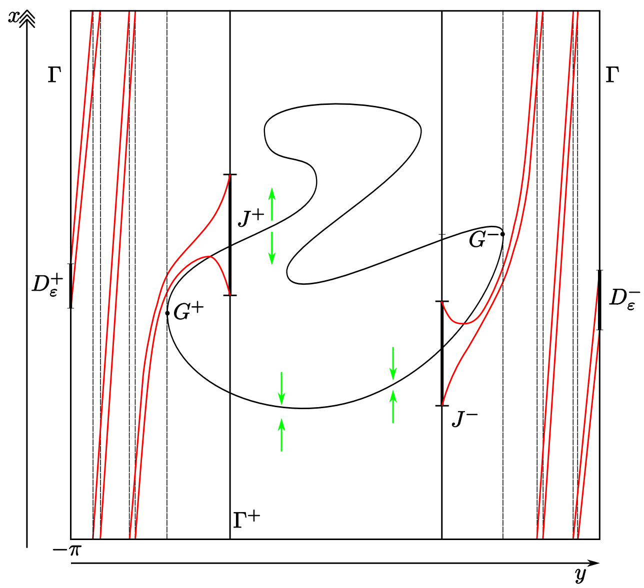

Figure 3: Phase curves near main jump points

Definition 4.1. A trajectory is called a *duck* (*canard*) if and only if it intersects J^+.

Consider the projection of J^+ (resp., J^-) to Γ along phase curves in backward (forward) time. Denote it by D_ε^+ (resp, D_ε^-). Note that all trajectories that intersect D_ε^+ are ducks. Lemma 3.1 (applied to the system with time reversed if necessary) implies immediately that $|D_\varepsilon^+| = O(\exp(-C_1/\varepsilon))$ and $|D_\varepsilon^-| = O(\exp(-C_2/\varepsilon))$ for some positive $C_1,\ C_2$. The trajectory with inital condition $x_0 \in \Gamma \setminus D_\varepsilon^+$ does not intersect J^+. Therefore, it attracts to attracting part of the slow curve rather quickly (it is controlled by the distance between J^+ and reverse jump point), and then moves near attracting parts of the slow curve only, accumulating contraction. It also have to intersect

J^- and therefore D_ε^-. Lemma 3.2 implies that in this case the derivative of Poincaré map is exponentially small. On the other hand, appliyng the same arguments to the system with time reversed, we have that outside of D_ε^-, the inverse Poincaré map P_ε^{-1} has exponentially small derivative. Informally speaking, it means that almost all circle in the pre-image (with except of very small interval) is mapped into very small interval in the image, while the exceptional interval in the pre-image is mapped into almost all circle in the image.

Geometrically, this means that the graph γ_ε belongs to union $\Pi^+ \cup \Pi^-$ of exponentially thin strips: vertical $\Pi^+ = D_\varepsilon^+ \times S^1$ and horizontal $\Pi^- = S^1 \times D_\varepsilon^-$. Outside of the rectangle $K_\varepsilon = \Pi^+ \cap \Pi^-$, the slope of γ_ε is either exponentially big or exponentially small (see fig. 4).

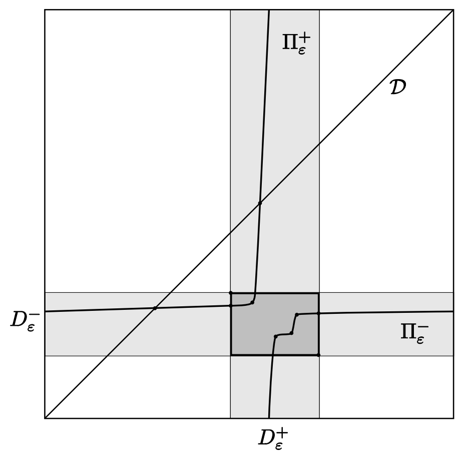

Figure 4: Graph of Poincaré map

Monotonicity arguments similar to discussed in section 2 show that as $\varepsilon \searrow 0$, rectangle K_ε moves from bottom-right to top-left corner, making infinitely many rotations. (See Monotinicity Lemmas in [1] and [2] for details.)

In this paper, we are interested only in limit cycles that correspond to fixed points of Poincaré map (i.e. making 1 rotation along y-axis). They born (or die) when the diagonal $\mathcal{D}$ tangents the graph γ_ε. Such a tangency is possible only in points where the slope of γ_ε is equal to 1. We will call such points *neutral*, applying this term both to points on the graph γ_ε and corresponding values of argument (i.e. roots of the equation $P_\varepsilon'(x) = 1$). Note that all neutral points belong to K_ε, and therefore fixed points can born only inside of K_ε, thus giving us pairs of repelling and attracting canard cycles. (All points in K_ε correspond to canard solutions because they lie over D_ε^+.)

For every neutral point x, consider second derivative $P''_\varepsilon(x)$, and call x *bearing* (resp., *killing*) if $P''_\varepsilon(x) < 0$ (resp., $P''_\varepsilon(x) > 0$). We may impose nondegenicity conditions, such that $P''_\varepsilon(x) \neq 0$ in every neutral point x. Consider a projection $\Delta(x, y) = x - y$ along the diagonal $\mathcal{D}$. It appears (this will be discussed below) that for particlar system for ε small enough the number of neutral points is fixed (does not depend on ε; see section 5) and the order of their projections under Δ is fixed as well (see section 6). In this case, actual maximal number of canard cycles is defined by the order of births and dearths, which is controlled by the order of bearing and killing neutral points under projection Δ, and thus does not depend on ε. This gives us the number k from theorem 1.1. Rolle's theorem implies that $k \leq N$. Neutral points depend on ε continiously, therefore on every turn of K_ε there exist some open interval of ε's, such that maximal number (which is $2k$) of canard cycles born. Such intervals accumulate at 0, and their existence is the main result of theorem 1.1.

The rest of paper is devoted to analysis of neutral points. We first demonstrate that the number of neutral points is bounded by the number of folds of M and show how it can be calculated explicitly (see section 5). Then we discuss the order of births and deaths of canard cycles (see section 6). Finally, we will contstruct an open set of systems with maximal number of limit cycles $k = N$ (section 7).

5 Neutral points

Consider trajectory, which pass through some point $w \in \mathring{B} \setminus M$ (see the description in section 2). The part of the trajectory to the left from w lies near repelling arcs of the slow curve; the part to the right from w lies near attracting parts of the slow curve. We will say that in w the trajectory pass through *duck (or canard) jump*: the transition from unstable part of the slow curve to the stable one. Consider some arc S of the slow curve M between two consequent fold points (*maximal arc*). It is well-known [6] that there exists invariant curve S_ε which tends to S as $\varepsilon \to 0$. This curve is called *(maximal) true slow curve*. It is not unique, but all such curves are exponentially close to each other and we can pick a suitable one.

For every maximal arc of M, consider corresponding maximal true slow curve (see fig. 5).

Extend them in backward time to Γ, and denote corresponding intersection points by $u_1, \ldots, u_{2N}$ (enumeration is consequent, even numbers correspond to repelling curves and odd to attracting ones; obviously, they should interchange). Put by definition $u_{2N+i} \equiv u_i$. Enumerate corresponding slow

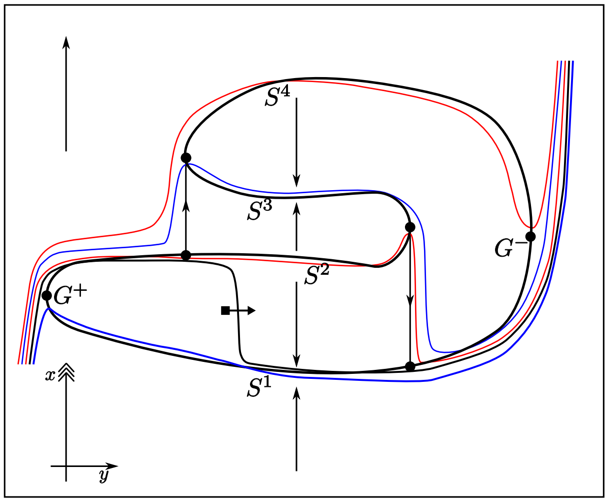

Figure 5: Maximal true slow curves and duck jump

curves as $S^1, \ldots, S^{2N}$, and true slow curves as $S_\varepsilon^1, \ldots, S_\varepsilon^{2N}$ respectively.

The trajectory, which pass through canard jump from S_{2l} can fall after the jump either to S_{2l-1} or to S_{2l+1}. Consider first case. It becomes possible if the initial condition u belongs to interval (u_{2l-1}, u_{2l}), i.e. lies below u_{2l}. When we move u a little bit upward (closer to u_{2l}), the whole trajectory moves closer to S_ε^{2l}. Thus the duck jump moves to the right. It means that the trajectory will spend more time near repelling part of the slow curve and less time near attracting part. Therefore, it will accumulate more expansion and less contraction, and the derivative of Poincaré map increase monotonically on this interval.[2]

Similar arguments show that the derivative of Poincaré map decreases monotonically on interval (u_{2l}, u_{2l+1}). It follows that Poincaré map has picewise-monotonic derivative with exactly N intervals of growth and N intervals of decrease. Therefore, equation $P'(x) = 1$ can have not more than $2N$ roots. This proves the estimate for the number of neutral points, and therefore canard cycles.

It also follows from this analysis, that actual number of neutral points can be calculated as the number of sign changes of logarithmic derivative $\log P_\varepsilon'(x)$ on maximal true slow curves. Lemma 3.2 (with some modifications) implies that this number can be calculated as the number of sign changes

[2]To be honest, we are cheating here a little bit: we can prove monotonicity only for some smaller interval. Frankly, it contains all neutral points, so the proof works. See [3] for details.

for integrals over special contours, which contain maximal arcs of the slow curve. Thus the number of neutral points does not depend on ε and can be effectively calculated.

6 Order of neutral points

Lift the rectangle K_ε to the universal cover of the two-torus continuosly with respect to ε. Pick two arbitrary neutral points $\xi, \eta \in \gamma_\varepsilon$ from this lifted rectangle. Then we can define the difference between their projections $\Delta(\xi) - \Delta(\eta)$, assuming that $\Delta = x - y$, where x and y are coordinates on universal cover. (We need these precuations, because in general case the difference between two points on a circle is not defined.) In this section, we show that for any two neutral points the sign of this difference does not depend on ε.

The main idea is to show that the segment $[\xi, \eta] \subset K_\varepsilon$ is either "almost vertical" or "almost horizontal". In the first case, if ξ is top end of the segment and η is bottom end, then $\Delta(\xi) - \Delta(\eta) > 0$. In the second cae, if ξ is left end and η is right end, then $\Delta(\xi) - \Delta(\eta) > 0$, and so on.

Consider two trajectories which correspond to neutral points (call them neutral too). Due to lemma 3.2, they should lie near some contours with zero integrals (call such contours neutral as well).[3] Consider first forward-time parts of these contours (which are denoted by Z^+). They both contain the far right fold point and therefore have some nonempty intersection. Denote far left point of this intersection by T. To the left of T, the corresponding contours (and therefore actual trajectories) are bounded from each other. To the right of T, the trajectories follow the same attracting arcs of the slow curve, and therefore attract each other. Consider Poincaré map from some interval $J' \ni T$ to Γ in forward time. Then the rate of the attraction is given by lemma 3.1 and is defined by the integral f'_x over intersection of the contours. This integral does not depend on ε and can be calculated explicitly. The distance between these trajectories when they approach Γ is the distance between y-coordinates of corresponding neutral points. It follows immediately, that $|y(\xi) - y(\eta)| = O^*(\exp(-C^-/\varepsilon))$ for some $C^- > 0$.

Applying the same arguments to the system with time reversed, we obtain similar statement for x-coordintes: $|x(\xi) - x(\eta)| = O^*(\exp(-C^+/\varepsilon))$ for some $C^+ > 0$.

[3]Note, that this implies that the measure of basins of limit cycles is bounded away from 0: these cycles lie in different areas of the phase space, which are separated by neutral solutions which are close to fixed neutral contours.

Consider the slope of the segment $[\xi, \eta]$, which is equal to

$$\frac{|y(\xi) - y(\eta)|}{|x(\xi) - x(\eta)|} = O^*\left(\exp \frac{C^+ - C^-}{\varepsilon}\right). \tag{6}$$

Again, we may impose additional nondegenericity conditions and assume that $C^+ \neq C^-$. This means that either numerator or denominator in the slope dominates, and therefore the slope is either exponentially big or exponentially small. This implies necessary assertion immediately.

This demonstrates that the order of neutral points under the projection Δ is fixed and thefore the maximal number of canard cycles k is well-defined. It finishes the proof of theorem 1.1.

7 Duck farm

The discussion above shows that we can translate "dynamical" questions (e.g. about limit cycles, Poincaré map and so on) into geometrical/combinatorial language which involves the shape of the slow curve M and values of integrals of f'_x over some arcs of M. As an application of this approach, we pick arbitrary $N > 1$ and construct the system with maximal number of canard cycles: $k = N$. In fact, this example provides an open set of such systems, because all conditions exposed on the system during construction of this example are open. To simplify the notation, we consider only case $N = 3$, but extension of these arguments to general case is strightforward.

The key ingredient of the construction is a shape of the slow curve, see fig. 6, top part. We demand here that depicted contours be neutral, and integrals of f'_x over corresponding arcs be equal to corresponding values (e.g. $\int_{H_1 F_1^-} f'(x, y, 0)\, dy = -4$, and so on).

This system has $2N$ neutral contours, and therefore $2N$ neutral points on the graph γ_ε. It follows from previous results, that for such a system, the graph looks like a "staircase", where "lengths" and "heights" of the steps monotonically decrease (see fig. 6, bottom part). This can be shown by explicit calculation of corresponding exponential rates which control "lengths" and "heights" of the steps (see the description in previous section). They depend only on integrals over the arcs which we control.

Due to this shape of the graph of Poincaré map, it follows that the order of bifurcations of limit cycles is the following: first we have N births and then we have N deaths. During every birth a pair of cycles appear, therefore the number of canard cycles here is maximal and equal to $2N$. Thus we constructed the desired example. This proves theorem 1.2.

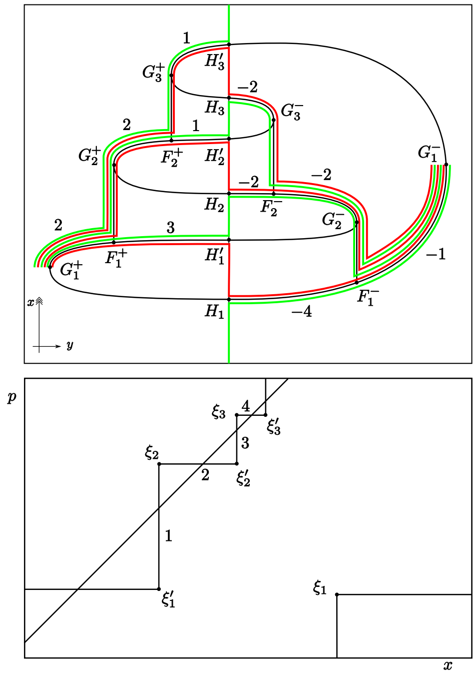

Figure 6: The system with maximal number of ducks

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