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Titre:

Des orbites périodiques et des attracteurs des systémes dynamiques

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1 Introduction

Cette thèse est consacrée à l'étude des systèmes dynamiques, un modèle mathématique pour décrire l'évolution de ce que ce passe dans le « vrai monde ». Informellement, on cherche à décrire une telle évolution soit par une application (qui associe à un état du système son état une minute plus tard), soit par une équation différentielle (qui donne l'évolution de l'état du système en temps continu).

Pour les systèmes à temps continu, on peut aussi considérer une famille d'applications φ_t, qui transforment l'état du système à un moment en son état t secondes plus tard.

Que se passe-t-il si les équations qui décrivent le système sont un peu plus compliquées que dans les exemples le plus basiques ? D'un côté, effectivement, on sait d'après le théorème de Cauchy qu'une solution de cette equation différentielle existe, est unique et dépend de manière lisse du point de départ. Mais de l'autre côté, une telle application ne peut prèsque jamais être écrite en termes de fonctions élementaires et de leurs intégrales. L'un des premiers exemples est donné par le théorème de Liouville : les solutions de l'équation $\dot{x} = x^2 - t$ ne peuvent pas être écrites sous une telle forme.

Quand-même, malgré l'absence d'une possibilité d'écrire une solution sous forme explicite, on peut établir certaines propriétés d'un système dynamique — ce qui fait l'objet de la théorie qualitative des systèmes dynamiques.

Voici quelques questions qu'on peut poser et auxquelles on peut eventuellement répondre sans résoudre le système correspondant.

- Combien de points d'équilibre et d'orbites périodiques possède le système ?
- Quels sous-ensembles de l'espace des phases attirent de nombreux points lorsque le temps tend vers l'infini ?
- Qu'est-ce qu'il arrive à une trajectoire du système après une petite perturbation de la condition initiale ?
- Qu'est-ce qu'il arrive au portrait de phase (c'est-à-dire à la partition de l'espace des phases en les orbites du système) après une petite perturbation de la loi d'évolution ?

La thèse est décomposée en deux parties, consacrées à deux problèmes différents. Dans la première partie de la thèse, qu'on expose au Chapitre 2 « Bony attractors » (« Attracteurs osseux »), on discute les attracteurs des systèmes dynamiques. Considérons un système dynamique à temps discret. Officieusement, on dit qu'un sous-ensemble fermé $A \subset X$ de l'espace des phases est *un attracteur* si

- les images d'un sous-ensemble suffisamment grand de l'espace des phases par les itérations F^n tendent vers A lorsque n tend vers l'infini ;
- A est le minimum fixé pour attirer son domaine d'attraction.

Il y a plusieurs formalisations de cette notion. Nous en allons présenter quelques-unes dans la Section 2.1 « Preliminaries » du Chapitre 2.

A quoi ressemble un attracteur d'un système dynamique ? Dans les cas les plus simples un attracteur d'un système dynamique est un ensemble discret (voire un seul point, par exemple, pour l'application $x \mapsto x/2$). Il existe des exemples bien connus de systèmes dynamiques dont les attracteurs ressemble localement à une variété lisse (par exemple, le produit Cartésien d'un difféomorphisme d'Anosov et une contraction), ou à un ensemble de Cantor (par exemple, le solénoïde de Smale-Williams), ou à un livre de Cantor (par exemple, l'attracteur de Lorenz).

Nous allons construire un ensemble ouvert de difféomorphismes $F\colon \mathbb{T}^3 \to \mathbb{T}^3$ du tore $\mathbb{T}^3$ ayant le comportement suivant. Tout d'abord, F possède une fibration invariante de l'espace des phases $\mathbb{T}^3$ en cercles. Ensuite, F a un attracteur unique qui croise la plupart des fibres sur un seul point (la *partie graphique* de l'attracteur), et coupe la reste des fibres sur les arcs (les *os*). Il n'y a rien de nouveau dans ces deux propriétés. Le fait intéressant à propos de cette attraction est que l'ensemble des os est grand mais pas trop gros. Plus précisément, les conditions suivantes sont satisfaites[1].

- Tant la partie graphique que la partie osseuse sont denses dans l'attracteur.
- L'ensemble des os n'est pas dénombrable.
- La mesure de l'attracteur est nulle (donc l'ensemble des os n'est pas trop grande).

Décrivons un système dynamique ayant un attracteur osseux. L'espace des phases de ce système n'est pas une variété : il est le produit cartésien de deux ensembles de Cantor C et de l'intervalle $I = [0, 1]$.

Formellement, un système dynamique agissant sur l'espace $C \times C \times I$ ne peut pas avoir un attracteur osseux dans le sens de la définition donnée ci-dessus. Nous allons donc remplacer la fibration invariante en cercles par la fibration $\{\mathrm{pt}_1\} \times \{\mathrm{pt}_2\} \times I$.

Considérons l'espace Σ^3 de toutes les suites bi-infinies $\omega = \ldots\omega_{-1}\omega_0\omega_1\ldots$ de symboles 0, 1, 2 : $\omega_i \in \{0, 1, 2\}$. Nous munissons Σ^3 de la topologie p-adique : deux suites ω et η sont proches si elles coincident sur un large segment $[-n, n]$: $\omega_i = \eta_i$ pour $|i| \leqslant n$. On peut vérifier facilement que cet espace est le produit cartésien de deux ensembles de Cantor, l'ensemble Σ^3_+ de toutes les queues à droite $\omega_0\omega_1\ldots\omega_n\ldots$ et l'ensemble Σ^3_- de toutes les queues à gauche $\ldots\omega_{-n}\ldots\omega_{-2}\omega_{-1}$.

Dans notre exemple, l'espace des phases est le produit cartésien $\Sigma^3 \times I$, et l'application est donnée par

$$F\colon \Sigma^3 \times I \to \Sigma^3 \times I, \quad (\omega, x) \mapsto (\sigma\omega, f_{\omega_0}(x)),$$

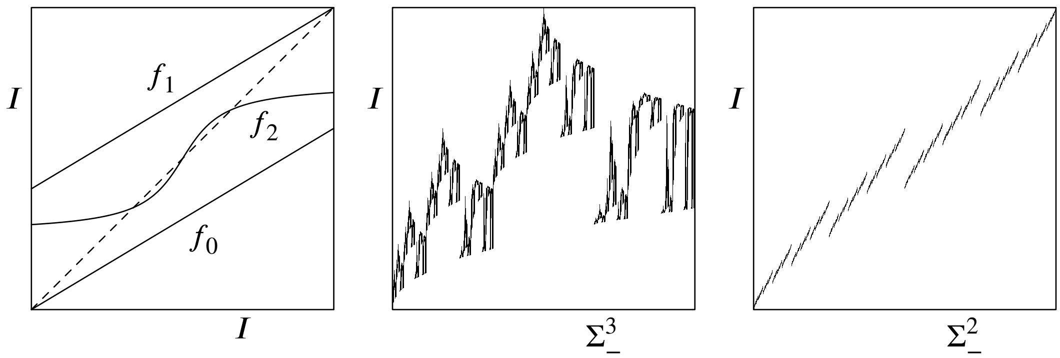
[1] Pour la définition précise d'un attracteur osseuse voir Section 2.2 « Definition of a bony attractor ».

où $\sigma\colon \Sigma^3 \to \Sigma^3$ est le décalage de Bernoulli, $(\sigma\omega)_i = \omega_{i+1}$, et les $f_i\colon I \to I$, $i = 0, 1, 2$ sont données par

$$f_0(x) = 0.6x, \quad f_1(x) = 1 - 0.6(1 - x), \quad f_2(x) = \frac{1}{2\pi}\arctan(10x - 5) + \frac{1}{2}. \tag{1.1}$$

Les graphes des applications f_i sont esquissés à la Figure 1.1 (a). L'attracteur de système dynamique correspondant est esquissé à la Figure 1.1 (b). La seconde figure est obtenue par un script en Ruby qui a calculé l'image de l'espace des phases par F^8. Dans la seconde figure l'axe horizontal correspond à l'espace Σ^3_- des toutes les queues à gauche $\dots\omega_{-n}\dots\omega_{-2}\omega_{-1}$ possible, et l'axe vertical correspond à l'intervalle I. Nous avons ignoré l'autre coordonnée qui paramétrise les valeurs possibles de la queue à droite $\omega_0, \omega_1, \dots$ parce que l'intersection de l'attracteur avec la fibre $\{\omega\} \times I$ ne dépend pas des ω_i, $i \geqslant 0$. Donc, le vrai attracteur est le produit cartésien de notre dessin et de l'ensemble de Cantor.

Pour rendre la différence entre les attracteurs osseux et non osseux plus claire nous avons également esquissé l'attracteur (encore une fois, le facteur de l'attracteur par l'espace Σ^2_+) de l'application construite par les applications f_0 et f_1 de la même manière que l'application F est construite par les applications f_0, f_1 et f_2.



(a) Les applications f_i

(b) Un attracteur osseux, construite par les applications f_0, f_1 et f_2

(c) Un attracteur non-osseux, construite par les applications f_0 et f_1

Figure 1.1 Les graphes des applications (1.1), un attracteur osseux et un attracteur non-osseux

Dans le Chapitre 2, nous allons prouver que cette application a un attracteur osseux[2]. Ensuite, nous allons suivre une stratégie proposée par Yu. S. Ilyashenko et A. Gorodetski pour obtenir un ouvert de difféomorphismes C^2 du tore $\mathbb{T}^3$ ayant un attracteur osseux.

Cette stratégie repose sur deux ingrédients importants.

- Les partitions de Markov des difféomorphismes d'Anosov du tore $\mathbb{T}^2$ nous permettent de passer des automorphismes de $\Sigma^k \times S^1$ à des difféomorphismes du tore $\mathbb{T}^3$ d'un genre particulier (« produits croisés », voir Subsection 2.1.4).
- Une stratégie élaborée par A. Gorodetski et Yu. S. Ilyashenko qui nous permet de passer des produits croisés à un ouvert dans l'espace des difféomorphismes C^2. Cette stratégie est basée sur le théorème de M. W. Hirsch, C. C. Pugh et M. Shub [7, Théorème 6.8] et ses améliorations obtenues par A. Gorodetski, Yu. S. Ilyashenko et A. Negut [6 and 10].

Dans le Chapitre 3 « Billiards », on discute les orbites périodiques des billards planaires. Les résultats de ce chapitre ont été obtenus en collaboration avec A. Glutsyuk, UMPA, ÉNS Lyon.

Un billard mathématique est un modèle pour décrire le mouvement d'une particule (une boule idéale de taille nulle) dans une table de billard (dont le bord n'est pas necessairement un polygone). La boule se déplace à vitesse constante à l'intérieur de la table, et se reflète sur son bord suivant la règle standard (l'angle d'incidence est égal à l'angle de réflexion).

Pourquoi est-il intéressant d'étudier de tels systèmes ? Il y a plusieurs raisons, parmi lesquelles on voici trois.

D'abord, les billards apparaissent comme des modèles mathématiques dans plusieurs problèmes physiques. Par exemple, si Ω est l'intérieur d'une chambre dont le sol, le plafond et les murs sont des miroirs, un rayon de lumière va suivre une trajectoire du billard Ω. Un autre modèle célèbre, qui nous ramène à un billard est un gaz idéal de Boltzmann. Effectivement, le mouvement de N boules qui se reflètent parfaitement elastiquement peut être décrit par une trajectoire de billard dans un domaine Ω de l'espace $\mathbb{R}^{3N}$.

Ensuite, il est plus simple d'étudier certaines propriétés (comme l'ergodicité ou les propriétés de mélange) pour une classe spécifique de systèmes plutôt, qu'en toute généralité.

Finalement, un flot de billard est un homologue naturel du flot géodésique, et dans certains cas ses trajectoires périodiques jouent le role des géodésiques fermées. En particulier, c'est le cas pour la théorie spéctrale de l'opérateur de Laplace $\Delta u = \sum_i \frac{\partial^2 u}{\partial x_i^2}$. J. J. Duistermaat et V. Guillemin [3] ont montré qu'il y a un lien entre le comportement des géodésiques fermées sur une variété riemanniene M sans bord et le comportement asymptotique des valeurs propres du problème de Dirichlet pour le laplacien. Plus tard, V. Ivrii a montré que pour le cas d'une variété *a bord*, l'ensemble des géodésiques fermées doit être remplacé par l'ensemble des trajectoires de billard périodiques.

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[2] Pour la définition formelle d'un attracteur osseux pour les applications de ce type voir Section 2.4 « Basic example ».

Ainsi, il s'avère qu'il existe un lien entre les valeurs propres du laplacien (c'est-à-dire, combien d'harmoniques de haute fréquence peut avoir un tambour d'une forme donnée a) et les trajectoires du billard correspondant. Puisqu'on peut lire et comprendre le texte principal de la présente thèse sans comprendre ce propos, nous allons formuler le théorème de V. Ivrii en petites charactères dans les prochains paragraphes.

Soit Ω un domaine dans $\mathbb{R}^n$ dont le bord est lisse par morceaux. Considérons le problème de Dirichlet pour l'operateur de Laplace dans ce domaine,

$$\Delta u = u, \quad u|_{\partial\Omega} = 0.$$

En 1911, H. Weyl a montré que le nombre $N(\lambda)$ de valeurs propres μ qui sont plus petites que λ^2, admet la formule asymptotique suivante :

$$N(\lambda) = c_0 \mathrm{Vol}_m(\Omega)\lambda^m + o(\lambda^m),$$

où $c_0 = c_0(m)$ est une constante connue.

Il a aussi conjecturé que

$$N(\lambda) = c_0 \mathrm{Vol}_m(\Omega)\lambda^m + c_1 \mathrm{Vol}_{m-1}(\partial\Omega)\lambda^{m-1} + o(\lambda^{m-1}),$$

où $c_1 = c_1(m)$, et Vol_{m-1} est le volume $(m-1)$-dimensionnel.

En 1975 J. J. Duistermaat et V. Guillemin [3] ont montré la conjecture de Weyl pour les variétés sans bord[3] satisfaisant la condition géométrique suivante : la mesure de l'ensemble de géodésiques fermées est nulle.

En 1980 V. Ivrii [13] a généralisé le résultat de J. J. Duistermaat et V. Guillemin au cas de variétés à bord. Il s'est révélé que dans ce cas ce sont les trajectoires fermées de billard qui jouent le rôle des géodésiques fermées. Plus précisément, V. Ivrii a démontré la conjecture de Weyl pour les domaines $\Omega \subset \mathbb{R}^m$ tels que l'ensemble des orbites périodiques du billard correspondant est de mesure nulle.

Puis, V. Ivri a proposé la conjecture suivante.

Conjecture 1.1 *(V. Ivrii, 1980) Pour tout domaine $\Omega \subset \mathbb{R}^m$ dont le bord est une surface C^∞-lisse, l'ensemble des trajectoires périodiques du billard correspondant est de mesure nulle.*

Dans le présent travail on ne discute que le cas d'un billard planaire, $m = 2$. Dans ce cas, la conjecture d'Ivrii peut être reformulée sous la forme suivante : est-il possible de fabriquer une table de billard telle qu'un joueur qui met la boule en un point choisi au hasard et qui la lance dans une direction choisie aussi au hasard, a une probabilité positive d'obtenir une trajectoire périodique ? Il se trouve que cette question est équivalente à la question suivante : peut-on fabriquer une table de billard telle que si on met la boule pres d'un point donné (avec une précision finie) et qu'on la lance dans une direction proche

[3] Nous avons formulé la conjecture de Weyl uniquement pour les domaines dans $\mathbb{R}^m$, mais en fait H. Weyl l'a formulée pour toute variéte riemannienne. Dans ce cas il faut remplacer les volumes dans la partie droite par certaines integrales des fonctions dépendant de la métrique.

d'une direction donnée (aussi, avec une précision finie), ceci qu'on la périodicité de la trajectoire ainsi obtenue ?

Quand V. Ivrii a enoncé cette conjecture dans le séminaire de Sinai, les participants lui ont dit que cette conjecture serait montré dans quelques jours, puis dans quelques semaines, puis dans quelques mois… La conjecture reste ouverte depuis déjà 30 ans !

Dans le Chapitre 3 « Billiards » nous allons démontrer un cas particulier de la conjecture d'Ivrii. Plus précisément, nous montrerons que pour tout domaine $\Omega \subset \mathbb{R}^2$ dont le bord est suffisamment lisse par morceaux, l'ensemble des trajectoires périodiques *quadrangulaire* est de mesure nulle. L'idée principale de la preuve est d'étudier la frontière de l'ensemble des trajectoires quadrangulaires périodiques. Il s'est trouvé qu'un point générique de la frontière corresponde à une trajectoire « dégénérée ». Nous considérons toutes les dègénérescences possibles, et nous montrons qu'il y a au plus un ensemble dénombrable de trajectoires dégénérées de chaque type sur le bord. Mais la frontière doit être de la cardinalité de $\mathbb{R}$, et cette contradiction montre le théorème.

Je voudrais remercier Yulij Sergeevich Ilyashenko pour son support permanent pendant mes études universitaires et doctorales, et Étienne Ghys pour les discussions et beaucoup d'aide dans la préparation de ce texte. Si vous le comprenez, c'est grâce à Étienne ; si vous ne le comprenez pas, c'est de ma faute.

Je voudrais aussi remercier mon co-auteur pour « Billiards » Alexey Glutsyuk, pour son excellente collaboration. Un merci spécial à Victor Kleptsyn, qui a eu une grande influence sur mon choix du directeur de thèse il y a six ans, et qui m'a aidé dans divers domaines entre-temps. Un merci gigantesque à ma femme Natalie Goncharuk pour sa patience et son assistance ; elle a été la première lectrice et correctrice de la plupart de ce texte.

2 Bony attractors

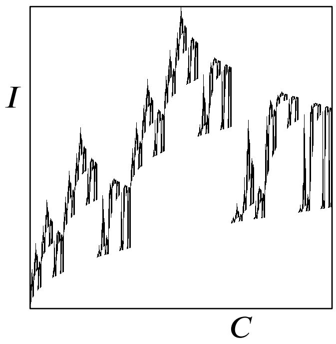


Figure 2.1 A sketch of a bony attractor

In this chapter we will construct a non-empty open set of maps $F\colon \mathbb{T}^3 \to \mathbb{T}^3$ such that each map from this set has a *bony attractor*, i. e. it has a unique attractor that consists of a graph of a function, and a dense set of arcs (bones).

For an informal description of the phenomena of a bony attractor see the "Introduction". An example of a bony attractor is sketched in Figure 2.1. The phase space of this system is $C \times C \times [0, 1]$, where C is a Cantor set. The horizontal coordinate corresponds to the first C, the vertical one corresponds to $[0, 1]$, and actually one should multiply this picture by another Cantor set C to obtain the attractor. For more details about this picture see either Chapter 1 "Introduction", or Section 2.4 "Basic example".

In the first section "Preliminaries" we will give the required definitions and introduce some useful notions. Next, in Section 2.2 "Definition of a bony attractor" we will introduce the formal notion of a bony attractor.

Section 2.3 "General strategy: from random systems to diffeomorphisms" describes a strategy proposed by A. Gorodetski and Yu. S. Ilyashenko. This strategy allows us to transfer interesting effects from the universe of random dynamical systems to the universe of standard dynamical systems.

In Section 2.4 "Basic example" we will construct one particular example of a random dynamical system that has a bony attractor. In the next sections ("Open set of step skew products", "Smooth example", "Mild skew products" and "Open set of smooth examples") we will follow the Gorodetski–Ilyashenko strategy. As a result, in the "Open set of smooth examples" we will prove the main theorem of this chapter, i. e. we will prove that there exists a non-empty open set of maps $F\colon \mathbb{T}^3 \to \mathbb{T}^3$ such that each map from this set has a bony attractor.

In the last section of this chapter we will discuss some possible directions of further research (open problems, new constructions etc.).

2.1 Preliminaries

In this section we will give accurate definitions for the concepts used in this chapter, and introduce some useful notation.

2.1.1 Stability

The following notions were introduced by Lyapunov in his work [17] (see also English version [18]).

Definition 2.1.1 Let F be a continuous mapping of a metric space X into itself. A fixed point $x \in X$, $Fx = x$ is called *Lyapunov stable* if for arbitrarily small neighborhood U of x there exists a smaller neighborhood $V \ni x$ such that any trajectory starting with a point of V never leaves U. In other words,

$$\forall \varepsilon > 0 \exists \delta > 0 \forall y \in B(x, \varepsilon) \forall n > 0 \; d(x, F^n y) < \varepsilon.$$

Consider the maps sketched in Figure 2.2. The north–south map has two fixed points, N and S. The north pole N is unstable, and the south pole S is stable. The maps sketched in Figures 2.2 (b)–(f) have a single fixed point $(0, 0)$. This point is Lyapunov stable for the maps from Figures 2.2 (b) and (c) and is Lyapunov unstable for the maps from Figures 2.2 (d)–(f).

There is a big difference between a center (see Figure 2.2 (b)) and a stable focus (see Figure 2.2 (c)). In the latter case the orbits not just stay in some small neighborhood of the fixed point, but also tend to the fixed point as time tends to infinity. This motivates the following definition.

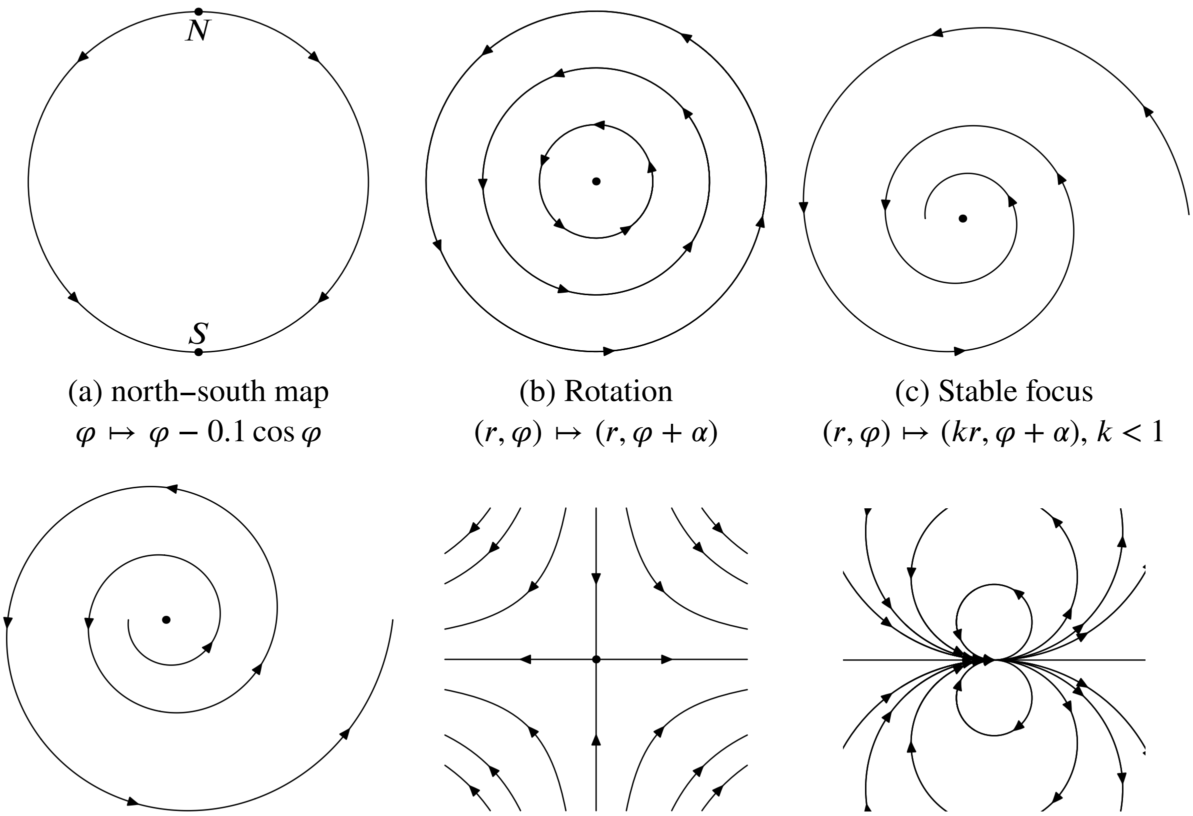
Definition 2.1.2 A Lyapunov stable fixed point x is called *asymptotically stable* if there exists $\varepsilon > 0$ such that for any $y \in B(x, \varepsilon)$ the sequence $F^n(y)$ tends to x as $n \to +\infty$.

Note that the condition of Lyapunov stability cannot be omitted. Indeed, consider the map sketched in Figure 2.2 (f). One can extend this map to the sphere $\bar{\mathbb{C}} = S^2$. Any trajectory of the extended map tends to the origin as time tends to infinity, but the origin is *not Lyapunov stable*.

We will need to speak about stable and unstable invariant subsets that contain more than one point. Recall the definition.

Definition 2.1.3 Let F be a continuous mapping of a metric space X into itself. An invariant subset $A \subset X$ (i. e. a subset such that $F(A) = A$) is said to be *Lyapunov stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any orbit that starts with a point from the δ-neighborhood of A does not leave the ε-neighborhood of A. In other words,

$$\forall \varepsilon > 0 \exists \delta > 0 \forall y \in X \forall n \in \mathbb{N} \left(d(A, y) < \delta \Rightarrow d(A, F^n y) < \varepsilon \right).$$



(a) north–south map
$\varphi \mapsto \varphi - 0.1\cos\varphi$

(b) Rotation
$(r, \varphi) \mapsto (r, \varphi + \alpha)$

(c) Stable focus
$(r, \varphi) \mapsto (kr, \varphi + \alpha)$, $k < 1$

(d) Unstable focus
$(r, \varphi) \mapsto (kr, \varphi + \alpha)$, $k > 1$

(e) Linear saddle point,
$(x, y) \mapsto (ax, y/a)$

(f) Time one map
for $\dot{z} = z^2$ on the
Riemann sphere

Figure 2.2 Examples of stable and unstable fixed points

2.1.2 Attractors

Let F be a dynamical system with discrete time, i. e. a map of a complete metric space X into itself, $F\colon X \to X$. Informally, an attractor of the map F is a closed set $A \subset X$ that attracts sufficiently many orbits of the map F such that any smaller closed set $A' \subset A$ attracts much less orbits of the map F. There are several formalizations of this concept. Let us formulate some of them.

2.1.2.1 Maximal attractor

First recall that a domain $U \subset X$ is called an *absorbing domain* of the mapping F if $F(U) \Subset U$, i. e. U includes the closure of the image $F(U)$, and this closure is a compact set.

Definition 2.1.4 Let F be a mapping of a complete metric space X into itself. Let U be an absorbing domain of F. The *maximal attractor* of the restriction $F|_U$ is the intersection of all the images of the domain U under the iterations[4] of F:

$$A_{max}(F) = \bigcap_{n \geqslant 0} F^n(U).$$

This attractor is called maximal because it attracts *all* points of the absorbing domain U.

Note that the notion of the maximal attractor depends on an absorbing domain $U \subset X$. For instance, consider the mapping given in polar coordinates by the formula

$$(r, \varphi) \mapsto (2r - r^2, \varphi - 0.1 \cos \varphi). \tag{2.1}$$

The restriction of this map to the unit circle is the north–south map sketched in Figure 2.2 (a). Some possible absorbing domains and the corresponding maximal attractors are listed in the table below.

Absorbing domain	**Maximal attractor**
Disk $\{(x, y) \mid r < 2\}$	Disk $\{(x, y) \mid r \leqslant 1\}$
Ring $\{(x, y) \mid 0.5 < r < 1.5\}$	Circle $\{(x, y) \mid r = 1\}$
Disk $B_1(S) = \left\{(x, y) \middle\vert x^2 + (y + 1)^2 < 1\right\}$	South pole $\{S\}$

Table 2.1 Absorbing domains and the corresponding maximal attractors for the north-south mapping (2.1)

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[4] In the sequel, if g is a map, then g^k stands for the k-th iterate of g.

How one can decide which of these maximal attractors is better? We will provide one of the possible answers in the next subsubsection.

2.1.2.2 Milnor likely limit set

Imagine that we use a numerical experiment to investigate the limit behaviour of system (2.1). We choose random point, calculate its images under many iterations of the mapping F, draw high-resolution black pixels at these images and look at the black set. For the north-south mapping (2.1), the images of any point tend to the south pole — excluding the case when this point is located exactly at the ray $\varphi = \pi/2$. But this event has probability zero, so it cannot happen in a real experiment. Hence, we will see only the black domain near the south pole.

Thus it would be natural to modify the definition of an attractor so that the circle $r = 1$ will not be an attractor anymore. John Milnor [19] suggested such definition. First, we recall the notion of the *ω-limit set*.

Definition 2.1.5 Let F be a map of a metric space X into itself. The set $\omega(x)$ of limit points of the sequence $F^n(x)$, $n > 0$, is called *ω-limit set* of the point $x \in X$.

Definition 2.1.6 Let X be a metric space with measure. A closed subset $A \subset X$ is called a *Milnor attractor* if it satisfies two conditions:

- the *realm of attraction* of A, consisting of all points $x \in X$ for which $\omega(x) \subset A$, must have strictly positive measure; and
- there is no strictly smaller closed set $A' \subset A$ so that the realm of attraction of A' coincides with the realm of attraction of A up to a set of measure zero.

Though a map $F\colon X \to X$ can have many attractors in the sense of this definition, it has the unique attractor that contains all other attractors.

Definition 2.1.7 The *likely limit set* $A_M(F)$ is the smallest closed subset of X with the property that $\omega(x) \subset A_M(F)$ for every point $x \in X$ outside of a set of measure zero.

Lemma 2.1.8 *(Milnor, [19, p. 4]) This likely limit set A_M is well defined and is a Milnor attractor for F. In fact, A_M is the unique maximal Milnor attractor, which contains all others.*

We will give two other equivalent definitions of the likely limit set.

Definition 2.1.9 Let X be a complete metric space with measure μ, $F\colon X \to X$ be continuous map that preserves the class of the measure μ. The *likely limit set* of the map

F is the smallest closed subset $A_M \subset X$, such that for μ-almost every point $x \in X$ the distance $d(A_M, F^n(x))$ tends to zero as n tends to $+\infty$.

Definition 2.1.10 Let X be a complete metric space with measure μ, $F\colon X \to X$ be continuous map that preserves the class of the measure μ. We will say that a point $x \in X$ belongs to the likely limit set of the map F if for any neighborhood $U \ni x$ the set of points y such that $F^n y \in U$ for infinitely many natural numbers n has a positive measure.

2.1.3 Bernoulli and Markov shifts

In this subsection we will introduce the notions of Bernoulli and Markov shifts. The phase space of a Bernoulli (Markov) shift is (a subset of) the space Σ^k of bi-infinite sequences of numbers $0, \dots, k-1$, and both maps are left shifts $(\sigma\omega)_i = \omega_{i+1}$. The main difference between these two notions is the measure we choose on Σ^k.

In the case of Bernoulli shift, ω_i are independent and identically distributed random variables. Formally, we take a tuple of probabilities $p_0, \dots, p_{k-1}$ such that $\sum p_i = 1$. Then we define the measure on the cylinders

$$C_a^b(v) := \left\{\omega \,\middle|\, \omega_a = v_a, \dots, \omega_b = v_b\right\}$$

by the formula

$$\mu(C_a^b(v)) := p_{v_a} \cdot \ldots \cdot p_{v_b},$$

and extend the measure to the sigma-algebra generated by these cylinders.

We also equip this space with "(λ_-, λ_+)-adic" metric

$$d(\omega, \widetilde{\omega}) = \max(\lambda_-^{-n_-(\omega,\widetilde{\omega})}, \lambda_+^{-n_+(\omega,\widetilde{\omega})}), \tag{2.2}$$

where $n_-(\omega, \widetilde{\omega})$ (resp., $n_+(\omega, \widetilde{\omega})$) is the least integer non-negative number n such that $\omega_{-n} \neq \widetilde{\omega}_{-n}$ (resp., $\omega_n \neq \widetilde{\omega}_n$). We will often consider (k, k)-adic metric.

Definition 2.1.11 The *Bernoulli shift* is the left shift on the space Σ^k,

$$\sigma\colon \Sigma^k \to \Sigma^k, \quad (\sigma\omega)_j = \omega_{j+1}.$$

In the case of Markov shift, we equip the space Σ^k with the measure corresponding to a time-homogeneous Markov chain. In other words,

- ω_n are identically distributed random variables, $P(\omega_n = i) = p_i$;
- ω_{n+1} depends only on ω_n;
- the probabilities $A_{ij} = P(\omega_{n+1} = j \mid \omega_n = i)$ are independent on n.

Formally, take a $k \times k$ matrix A and a tuple of numbers p_i satisfying the following conditions.

- Each transition probability is non-negative, $A_{ij} \geqslant 0$.
- Each probability p_i is postive, $p_i > 0$.
- For any symbol i, the total probability to pass from this symbol to another one equals one,
$$\sum_{j=0}^{k-1} A_{ij} = 1.$$
- The sum of probabilities p_i equals one,
$$\sum_{i=0}^{k-1} p_i = 1.$$
- The probability that a given symbol j appears at a given position can be computed as the sum of probabilities of transitions to this symbol from the symbol located in the previous position,
$$\sum_{i=0}^{k-1} p_i A_{ij} = p_j.$$

In other words, p_i is a left eigenvector of the matrix A with eigenvalue 1.

Let the measure of a cylinder $C_a^b(v)$ be given by
$$\mu_A(C_a^b(v)) = p_{v_a} A_{v_a v_{a+1}} \dots A_{v_{b-1} v_b}.$$

Due to the properties of the matrix A and the tuple p the measure μ_A can be extended to the sigma-algebra generated by the cylinders $C_a^b(v)$.

Given a $k \times k$ matrix A, consider the corresponding directed graph, i. e. the graph on vertices $0, \dots, k-1$ such that a vertex i is connected to a vertex j if and only if $A_{ij} \neq 0$. Recall that a square matrix A is called *irreducible* if the corresponding graph is strongly connected, i. e. for any two vertices i and j there exists a directed path from i to j. A square matrix A is called *aperiodic* if the greatest common divisor of the lengths of the directed cycles in the corresponding directed graph equals one.

Perron–Frobenius Theorem states that for any irreducible aperiodic matrix A such that $\sum_{j=0}^{k-1} A_{ij} = 1$ the following holds.

- There exists a unique left eigenvector p_i of the matrix A.
- The eigenvalue 1 is simple, and all other eigenvalues of A have absolute values less than one.

The first part implies that an irreducible aperiodic matrix A determines a unique Markov chain.

Consider the subset $\Sigma_A^k \subset \Sigma^k$ consisting of the *admissible sequences*, i.e. the sequences ω such that all the probabilities $A_{\omega_n\omega_{n+1}}$ are positive,

$$\Sigma_A^k = \left\{\omega \in \Sigma^k \,\middle|\, \forall n \in \mathbb{Z} : A_{\omega_n\omega_{n+1}} \neq 0\right\}. \tag{2.3}$$

Obviously, Σ_A^k is a compact subset of Σ^k, and $\mu_A\Sigma_A^k = \mu_A\Sigma^k = 1$.

Definition 2.1.12 Let a $(k \times k)$-matrix A and a k-tuple of numbers p satisfy the conditions listed above. The *Markov shift* is the restriction of the left shift to the space Σ_A^k equipped with Markov measure μ_A and some (λ_-, λ_+)-adic metric.

Clearly, a Bernoulli shift is a Markov shift, but not vice versa.

2.1.4 Skew products

2.1.4.1 The notion of a skew product

Recall that the *Cartesian product* of two maps $h\colon B \to B$ and $f\colon M \to M$ is the map $(h\times f)\colon B\times M \to B\times M, (b, m) \mapsto (h(b), f(m))$. In other words, a map $F\colon B\times M \to B\times M$ is a Cartesian product if it preserves the structure of the Cartesian product $B \times M$.

The first example of a system having a bony attractor will not be a Cartesian product but it will belong to a larger class of *skew products*. We will give three equivalent definitions of a skew product.

Definition 2.1.13 Let $h\colon B \to B$ be a continuous map of metric space B into itself, M be a metric space. A continuous map $F\colon B \times M \to B \times M$ is called a *skew product* over the map h with fiber M if it has the following form,

$$F\colon B \times M \to B \times M, \quad F\colon (b, x) \mapsto (h(b), f_b(x)).$$

The maps f_b are called *fiber maps* of the skew product F.

Definition 2.1.14 A continuous map $F\colon B\times M \to B\times M$ is a skew product over a map $h\colon B \to B$ if the following diagram commutes,

$$\begin{array}{ccc} B\times M & \xrightarrow{\ F\ } & B\times M \\ \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle \pi_B} \\ B & \xrightarrow{\ h\ } & B \end{array}$$

Here $\pi_B\colon B\times M \to B$ is the projection $(b, m) \mapsto b$.

Definition 2.1.15 A continuous map $F\colon B\times M \to B\times M$ is called *a skew product* if it preserves the vertical fibration $\{b\} \times M$.

2.1.4.2 The examples of skew products

The class of skew products plays important role in the dynamical systems theory. Many types of the limit behaviour of a dynamical system were first observed for skew products.

Example 2.1.16 Clearly, any Cartesian product of two maps is a skew product.

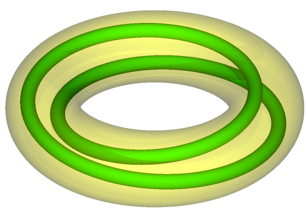

Figure 2.3 Solenoid map[5] for $k = 2$

Example 2.1.17 (Solenoid map) Another well-known skew product is the Smale—Williams solenoid mapping of the solid torus

$$S^1 \times D^2 = \{(z, w) \,|\, |z| = 1, |w| \leqslant 1\}$$

into itself defined by the formula

$$s\colon (z, w) \mapsto (z^k, 0.5z + \varepsilon w), \tag{2.4}$$

where ε is a small positive constant. We will discuss some properties of this map in Subsubsection 2.3.2.2.

Example 2.1.18 (Hairy attractor) Consider the map $F\colon S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ given by

$$F\colon (y, x) \mapsto (2y, h(y)x), \quad h(y) = 1 + 0.5\cos 2\pi y.$$

[5] © Ilya Schurov, the work is in public domain. The picture was drawn using gnuplot and pov-ray. The sources available at `http://en.wikipedia.org/wiki/File:Smale-Williams_Solenoid.png`

Clearly, this map is a skew product over the map $y \mapsto 2y$. One can show that F cannot be conjugated to a Cartesian product. We will neither prove nor use this fact.
Consider the compactification $\hat{F}\colon S^1\times\mathbb{RP}^1 \to S^1\times\mathbb{RP}^1$ of the map F. Since $\int_{S^1} \log h(y)\, dy < 0$, Birkhoff–Khinchin Theorem implies that almost any point of the phase space tends to the circle $S^1 \times \{0\}$, thus this circle includes the likely limit set of $\hat{F}$. One can show that actually the likely limit set coincides with the circle $S^1 \times \{0\}$.
On the other hand, there exists an uncountable set of fibers $\{\mathrm{pt}\} \times \mathbb{RP}^1$ (the 'hair') such that their points tend to the circle $S^1 \times \{\infty\}$. Therefore the likely limit set of the map $\hat{F}$ is not Lyapunov stable.

Example 2.1.19 (Intermingled basins) Consider the space of boundary-preserving skew products over the angle-doubling map of the circle,

$$F\colon S^1 \times [0,1] \to S^1 \times [0,1], \quad (y,x) \mapsto (2y, f_y(x)), \quad f_y(0) = 0, \quad f_y(1) = 1.$$

Itai Kan [14] found a non-empty open set in this space such that any system from this set has two attractors, $S^1\times\{0\}$ and $S^1\times\{1\}$, and their realms of attraction are highly intermingled: the closure of each realm of attraction is the whole phase space. Near each attractor the Itai Kan example looks like an example of a hairy attractor, i. e. $\int_{S^1} \log f'_y(i)\, dy < 0$ for $i = 0, 1$.
Later Yu. Ilyashenko, V. Kleptsyn, P. Saltykov [9 and 15] and independently C. Bonatti, L. Díaz and M. Viana [2] have shown that these properties survive under a small perturbation in the class of C^2 smooth boundary-preserving maps of the cylinder.

2.1.4.3 Step and mild skew products over a left shift

Though studying the skew products is easier than studying a map without any known structure, sometimes it is convenient to start with even smaller class of *step skew products* over a Markov (Bernoulli) shift.

Definition 2.1.20 A skew product F over a Markov shift $\sigma_A\colon \Sigma_A^k \to \Sigma_A^k$ is called a *step skew product* if the fiber maps f_ω depend only on the current symbol ω_0 of the sequence ω,

$$F\colon \Sigma_A^k \times M \to \Sigma_A^k \times M, \quad (\omega, x) \mapsto (\sigma\omega, f_{\omega_0}(x)), \quad f_i\colon M \to M.$$

The class of step skew products can be used as a playground to find new interesting types of the limit behaviour of dynamical systems.

A step skew product over a Bernoulli shift can be considered as a *random dynamical system* on M: every time we randomly choose which of the maps f_i should be applied at the moment.

General skew products over a Markov shift are called *mild* skew products. We will use this term to underline that a skew product is not required to be a step skew product.

2.1.5 Hausdorff dimension

Definition 2.1.21 Let X be a metric space. Consider an open covering U of the space X, i. e. finite or countable family of open balls Q_j of radii r_j such that the union of the balls Q_j coincides with the space X. Define *d-dimensional volume* $V_d(U)$ of the covering U by the formula

$$V_d(U) = \sum_j r_j^d.$$

The *Hausdorff dimension* of the space X is the infimum of the set of numbers d such that there exists an open covering of the space X of arbitrarily small d-dimensional volume,

$$\dim_H X = \inf\left\{d \,\middle|\, \forall \varepsilon > 0 \exists \text{ a covering } U \text{ of the metric space } X \text{ such that } V_d(U) < \varepsilon\right\}.$$

Recall that the Hausdorff dimension of a compact d-dimensional manifold M equals d. The same holds for a subset $A \subset M$ of positive Lebesgue measure. Probably the most famous example of a metric space having a non-integer Hausdorff dimension is the Cantor set,

$$\dim_H \left\{ x \in [0,1] \,\middle|\, \forall n \in \mathbb{N}\ \{3^n x\} \in \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \right\} = \log_3 2.$$

The following lemma estimates the distortion of the Hausdorff dimension of a set under a Hölder continuous mapping.

Lemma 2.1.22 *(Falconer, [4]) Let Z be a Riemannian manifold, A be some subset of Z. Let $\varphi\colon Z \to Z$ be a Hölder continuous map with Hölder exponent α. Then*

$$\dim_H \varphi(A) < \frac{\dim_H A}{\alpha}. \qquad (2.5)$$

In particular, the inequality $\dim_H A < \alpha \dim Z$ implies that $\mu(\varphi(A)) = 0$. Indeed, if the former inequality holds, then due to Falconer Lemma $\dim_H \varphi(A) < \dim Z$, hence $\mu(\varphi(A)) = 0$.

Note that the condition $\mu(A) = 0$ does not imply that $\mu(\varphi(A)) = 0$. For example, the image of the standard Cantor set under the Cantor function is the interval.

2.2 Definition of a bony attractor

Informally, a bony attractor consists of two parts: the graph of a continuous function, and rather large family of vertical segments (bones).

Definition 2.2.1 Let X be a compact manifold with or without boundary. We will say that a continuous map $G\colon X \to X$ has a *bony attractor* if there exists a G-invariant fibration with smooth compact one-dimensional fibers such that the following holds.

1. Each fiber of the invariant fibration either does not intersect the likely limit set A_M (see Definition 2.1.9), or intersects A_M on a single point, or intersects A_M on a topological segment (a *bone*).
2. The union of the bones is dense in the likely limit set, and the set of bones has the cardinality of $\mathbb{R}$.
3. Let $Y \subset X$ be the saturation of the likely limit set by the fibers. Then the Hausdorff dimension $\dim_H A_M$ of the likely limit set is less than the Hausdorff dimension of Y,

$$\dim_H A_M < \dim_H Y. \tag{2.6}$$

We will say that the *graph part* Γ of the attractor is the relative complement of the union of the bones in the likely limit set.

The second condition means that the set of bones is rather large. The last condition means that the set of bones is not very large.

Definition 2.2.2 We will say that a continuous map $G\colon X \to X$ has a *bony attractor without holes* if it has a bony attractor, and two additional conditions hold.

1. The graph part Γ is dense in the likely limit set, $\operatorname{Cl}\Gamma = A_M$.
2. The likely limit set is asymptotically stable.

The aim of this chapter is to prove the following result.

Theorem 2.2.3 *There exists a non-empty open set in the space of C^2-diffeomorphisms of the three-torus such that each map from this set has a bony attractor without holes.*

In fact, the domain $U = \mathbb{T}^2 \times [0,1] \subset \mathbb{T}^2 \times S^1$ is an absorbing domain, see Subsection 2.1.2, for all the maps from this open set, and we will first prove the conditions of Definition 2.2.1 and the first condition of Definition 2.2.2 for the *maximal* attractor in this domain instead of the likely limit set. Then, we will prove that the maximal attractor for this domain coincides with the likely limit set.

The proof will follow the general strategy due to Yu. Ilyashenko and A. Gorodetski. In the next section we will describe the Gorodetski–Ilyashenko strategy, then we will follow this strategy.

2.3 General strategy: from random systems to diffeomorphisms

Though it is interesting to find a *single example* of a dynamical system with unusual behaviour, it is much more interesting to find a *non-empty open set* of dynamical systems with such behaviour.

One can simplify the task of constructing the examples of exotic behaviour by replacing the classical dynamical system (i. e. the action of the cyclic group $\mathbb{Z}$) by an action of a free finitely generated semigroup. Indeed, given a manifold, we have much more freedom with actions of the free semigroup than with actions of the cyclic group, hence we can construct interesting examples with less effort.

A. Gorodetski and Yu. Ilyashenko [6] found a very strong relation between the dynamics generated by a generic action of a free semigroup on a compact manifold, on the one hand, and a generic classical dynamical system, on the other hand. This relation leads to the following heuristic principle formulated in the same article: *all phenomena observed generically in the dynamics of a free semigroup can be also found among generic diffeomorphisms*[6].

The Gorodetski–Ilyashenko strategy was used in, e.g., [8, 9, 11, 12, 15 and 22].

In Subsections 2.3.1–2.3.3 we will describe the Gorodetski–Ilyashenko strategy, step by step, and discuss some difficulties that one has to overcome on this path. In Subsection 2.3.5 we will briefly list the main steps of the strategy.

2.3.1 Step skew products

Given a free semigroup $\mathscr{F}$ generated by k smooth maps of a compact manifold M into itself, $f_i\colon M \to M$, $i = 0, \ldots, k-1$, one can consider the step skew product F over the Bernoulli shift (see Subsubsection 2.1.4.3) generated by f_i. It is easy to see that the orbits of the semigroup $\mathscr{F}$ coincide with the projections of the positive semitrajectories of the map F onto the fiber along the base. Therefore, one can reformulate the properties of the action of $\mathscr{F}$ in terms of the step skew product F.

We will skip this first step, and start with studying step skew products. In Section 2.4 we will construct a *single* step skew product having a bony attractor, then in Section 2.5 we will show that all the required properties survive under a small perturbation *in the class of step skew products*.

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[6] More precisely, generic in some non-empty open set of diffeomorphisms.

2.3.2 Smooth realization

One can say that $\Sigma^k \times M$ is not a manifold, so there is nothing surprising in the fact that a dynamical system on $\Sigma^k \times M$ can have an unusual asymptotic behaviour. This subsection deals with different ways to find a smooth realization of the map F. First of all, we need a smooth realization of the Bernoulli shift. There are two well-known maps that have a maximal hyperbolic set Λ such that the restriction of the map to Λ is (semi-)conjugated to a Bernoulli shift, and one well-known map semi-conjugated to a Markov shift. To obtain a smooth realization of the skew product, one should replace the shift σ by one of these maps. Now we will describe these maps.

2.3.2.1 Smale's horseshoe

This is the simplest smooth realization of the Bernoulli shift. Take k "horizontal" rectangles D_i and k "vertical" rectangles D'_i,

$$D_i = [0,1] \times \left[\frac{2i-1}{2k+1}, \frac{2i}{2k+1}\right]; \quad D'_i = \left[\frac{2i-1}{2k+1}, \frac{2i}{2k+1}\right] \times [0,1].$$

The Smale's horseshoe map h maps each "horizontal" rectangle D_i onto the corresponding "vertical" rectangle D'_i shrinking in the horizontal direction and expanding in the vertical direction,

$$h|_{D_i}(x,y) = \left(\frac{2i+x-1}{2k+1}, (2k+1)y - (2i-1)\right).$$

One can easily extend this map to a homeomorphism of the two-dimensional sphere. The map h (and, hence, its extension) has the hyperbolic invariant set Λ such that the restriction $h|_\Lambda$ is conjugated to the Bernoulli shift $\sigma\colon \Sigma^k \to \Sigma^k$. The set Λ consists of the points (x, y) such that both x and y have only odd digits in base $2k + 1$ representation.

One can easily obtain a smooth realization of a given step skew product using Smale's horseshoe. It is sufficient to take any skew product G over h such that $g_b = f_i$ for $b \in D_i$. Then the restriction $G|_{\Lambda\times M}$ will be conjugated to the original step skew product F.

The main drawback of this realization is that the set Λ is not Lyapunov stable for h, hence the set $\Lambda \times M$ is not Lyapunov stable for G.

2.3.2.2 Solenoid map

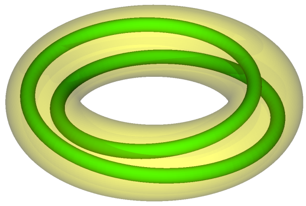

Figure 2.4 Solenoid map[7] for $k = 2$.

Recall (see Subsubsection 2.1.4.2) that the Smale–Williams mapping of the solid torus

$$S^1 \times D^2 = \{(z, w) \mid |z| = 1, |w| \leqslant 1\}$$

into itself is given by the formula

$$s\colon (z, w) \mapsto (z^k, 0.5z + \varepsilon w), \tag{2.7}$$

where ε is a small positive constant.

The image of the solid torus under the map s is homeomorphic to another solid torus, wrapped k times inside the initial one (see Figure 2.4 for $k = 2$). Now it is easy to see that the maximal attractor of s intersects each disk $\{z\} \times D^2$ on a Cantor set.

The restriction of s to its maximal attractor Λ is a quotient map of the Bernoulli shift $\sigma\colon \Sigma^k \to \Sigma^k$, i.e. there exists a continuous map $\Phi\colon \Sigma^k \to \Lambda$ such that the following diagram commutes.

$$\begin{array}{ccc} \Sigma^k & \xrightarrow{\ \sigma\ } & \Sigma^k \\ \Big\downarrow \Phi & & \Big\downarrow \Phi \\ \Lambda & \xrightarrow{\ s\ } & \Lambda \end{array}$$

The conjugation Φ is called *the fate map*. Split the circle $|z| = 1$ into k equal arcs d_i, and split the initial solid torus into k parts

$$A_i = d_i \times D^2, \quad d_i = \left\{ z \in S^1 \,\middle|\, i \leqslant k \arg z < i + 1 \right\}$$

For any sequence $\omega \in \Sigma^k$, its image $\Phi(\omega)$ is the only point x of the solid torus such that for any integer number j the point $s^j(x)$ belongs to the closure of A_{ω_j}. Let $\Sigma_0^k \subset \Sigma^k$ be the set of sequences $\omega \in \Sigma^k$ such that ω has no right tail of symbols '$k - 1$'. Then the fate map Φ is continuous on Σ^k and the restriction of the fate map to the set Σ_0^k is a bijective map. Moreover, the forward image of the standard Bernoulli measure on Σ^k is the SRB-measure on Λ.

The solenoid Λ is the maximal attractor of the mapping s. Hence, unlike the Smale horseshoe, the set $\Lambda \times M$ is Lyapunov stable for G. It makes the example obtained by this realization more interesting.

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7 © Ilya Schurov, the work is in public domain. The picture was drawn using gnuplot and pov-ray. The sources available at `http://en.wikipedia.org/wiki/File:Smale-Williams_Solenoid.png`

However, in this case we cannot just take $g_b(x) = f_i$ for $b \in A_i$ because the skew product G defined in this way will not be continuous.

One of the ways to overcome this difficulty is to take $g_b(x) = f_i$ for $b \in A_i \smallsetminus (d_i' \times D^2)$, where d_i' are small arcs near the ends of d_i, and glue them into a continuous skew product G using the fiber maps over the points $b \in d_i' \times D^2$. In this case one should somehow avoid the points from $d_i' \times D^2$ in the proofs. It seems to be easy, but actually sometimes one needs to avoid the points whose trajectories intersect $d_i' \times D^2$ as well, and the set of such points has full measure.

The other way to overcome this difficulty is to take the solenoid map that wraps the solid torus $2k$ times. In this case we can take $g_b(x) = f_i$ for $b \in A_{2i}$, and extend G to a smooth skew product. It seems that in this case one has to avoid more sequences (about half of the phase space), but the projection of the image of each part A_{2i} under the solenoid map onto z is the whole circle, $\pi_z \circ s(A_{2i}) = S^1$.

Actually, there are two other small technical difficulties with this smooth realization. First, the Bernoulli shift is not *conjugated* to the solenoid map, but *semi-conjugated*. Second, the solenoid map contracts in the stable direction stronger than it extends in the unstable direction. Therefore, if we want the fate map to be Lipschitz, we have to equip Σ^k with (λ_-, λ_+)-adic metric, $\lambda_- \neq \lambda_+$ (see Subsection 2.1.3). Both of these problems just add some technical details to the proofs.

2.3.2.3 Anosov diffeomorphism

We will not discuss now the general concept of Anosov diffeomorphism. Instead, we will define a linear Anosov diffeomorphism of the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let A be an integer area-preserving 2×2 matrix, $A \in SL_2(\mathbb{Z})$. Then the action of the matrix A on the plane $\mathbb{R}^2$ descends to a diffeomorphism of the two-torus $\mathbb{T}^2$. We will use the same letter A for this diffeomorphism.

Definition 2.3.1 The action of an integer area-preserving 2×2 matrix $A \in SL_2(\mathbb{Z})$ on the two-torus is called a *linear Anosov diffeomorphism* if A has two distinct real eigenvalues.

It turns out that this Anosov diffeomorphism is a quotient map of a Markov shift on the space $\Sigma_B^k \subset \Sigma^k$ of admissible words (see (2.3)). To prove this statement, one should construct a Markov partition of the torus into parallelograms A_i, and consider the corresponding fate map.

This smooth realization of the left shift is in some sense the best one, but needs even more technical work than the previous one. The main advantage of this realization over the horseshoe map and the solenoid map is that the maximal hyperbolic set Λ coincides with the whole phase space. Usually this allows us to construct a *diffeomorphism* having the same properties as the initial step skew product.

The main new difficulty with this smooth realization is that linear Anosov diffeomorphism is a quotient map of a *Markov shift*, not a Bernoulli shift. This means that the symbols ω_i are not independent anymore. For details see Subsection 2.1.3.

2.3.3 Perturbations

In the previous subsection we have described how one can obtain a *single* diffeomorphism that has the same properties as a given class of skew products over a Markov shift. In our case 'has the same properties' will mean 'has a bony attractor'.

Now it is natural to ask whether the properties of the constructed diffeomorphism survive under a small perturbation. Before studying the perturbations in the class of all diffeomorphisms, one should prove that the properties survive under a small perturbation in the class of *skew products* over the chosen smooth realization of the left shift σ (σ can be either Bernoulli or Markov shift, depending on the chosen smooth realization).

The space of skew products is just a tiny part (e.g., it has infinite codimension) of the space of all diffeomorphisms. So, a generic perturbation of a skew product is not a skew product anymore. However, Hirsch, Pugh and Shub [7] proved that if the initial skew product is *partially hyperbolic* with vertical central fibers, then a sufficiently small perturbation of this diffeomorphism is a skew product *with respect to another structure of the trivial fibration*. In other words, there exists a coordinate change in the phase space that conjugates the perturbed diffeomorphism to a skew product close to the original one.

Though every particular fiber of the new fibration is smooth and "almost vertical" in C^r metric, Hirsch–Pugh–Shub Theorem gives us only continuous dependence of the fibers on the point in the base. Hence, if we use this theorem to conjugate the perturbed map with a skew product, we will obtain a skew product whose fiber maps depend continuously on the point in the base, and the conjugation itself *is not smooth*. So, the conjugation can map a set of measure zero to a set of the full measure (this effect is called "Fubini nightmare").

A. Gorodetski [5] proved that actually the new fibers depend *Hölder-continuously* on the point of the base. Later this theorem was generalized by Yu. Ilyashenko and A. Negut [10]. We recall their theorem in Subsection 2.3.4. In particular, Gorodetski–Ilyashenko–Negut Theorem states the following. Given a skew product over one of the smooth realizations described above[8], the Hölder exponent tends to 1 as the magnitude of the perturbation tends to zero.

This theorem together with the Falconer Lemma (see Lemma 2.1.22 at page 20) allows us to transfer the statements about Hausdorff dimension from the skew products to the diffeomorphisms.

2.3.4 Gorodetski–Ilyashenko–Negut Theorem

This subsection follows the Introduction of the article [10].

Let F be a skew product over a hyperbolic diffeomorphism h. Suppose that the restriction of dh to the stable direction contracts all vectors not less than in λ_- and not more than in λ times, and the restriction of dh^{-1} to the unstable direction contracts not less than in μ_- and not more than in μ times,

$$\lambda_- \|v\| \leqslant \|dh(v)\| \leqslant \lambda\|v\|, \text{ for } v \in E^s; \tag{2.8}$$

$$\mu_- \|v\| \leqslant \|dh^{-1}(v)\| \leqslant \mu\|v\|, \text{ for } v \in E^u. \tag{2.9}$$

Suppose that the stable and unstable fibrations are trivial.

Definition 2.3.2 We will say that a skew product F over a hyperbolic diffeomorphism h satisfies the *modified dominated splitting condition*, if the following two inequalities hold,

$$\max(\lambda, \mu) + \left\| \frac{\partial f_b^{\pm 1}}{\partial b} \right\|_{C^0(X)} < \max(\lambda^{-1}, \mu^{-1}).$$

$$\left\| \frac{\partial f_b^{\pm 1}}{\partial x} \right\|_{C^0(X)} < \max(\lambda^{-1}, \mu^{-1})$$

Theorem 2.3.3 *(Ilyashenko–Negut) Let F be a skew product over a diffeomorphism h. Suppose that h has a hyperbolic invariant set Λ. Suppose that the diffeomorphism h satisfies conditions (2.8) and (2.9), and the skew product F satisfies the modified dominated splitting condition. Then for sufficiently small $\delta > 0$ and any smooth map G which is δ-close to F in C^1 metric the following holds.*

- *There exists a G-invariant set $Y \subset X$ and a continuous map $p\colon Y \to \Lambda$ such that $p \circ G = h \circ p$. Moreover, the map*

  $$H\colon Y \to \Lambda \times M, \quad H(b, x) = (p(b, x), x)$$

  is a homeomorphism.

[8] Actually, there are some assumptions on the initial skew product. We will formulate them later.

- *The fibers $W_b = p^{-1}(b)$ are Lipschitz close to vertical (constant) fibers, and Hölder continuous in b. This means that W_b is the graph of a Lipschitz map $\beta_b \colon M \to B$ such that*

$$d_{C^0}(\beta_b, b) \leqslant O(\delta), \quad Lip\beta_b \leqslant O(\delta),$$

$$d_{C^0}(\beta_b, \beta_{b'}) \leqslant d(b, b')^{\alpha - O(\delta)} O(\delta^{-\alpha}),$$

where

$$\alpha = \min\left(\frac{\log\lambda}{\log\lambda_-}, \frac{\log\mu}{\log\mu_-}\right).$$

Moreover, the map H^{-1} is also Hölder continuous, with the same α.

2.3.5 Short plan of the strategy

This subsection briefly recalls the main steps of the Ilyashenko–Gorodetski strategy.

- Construct an action of the free semigroup having interesting properties such that these properties survive under a small perturbation.
- Use the explicit construction from Subsection 2.3.1 to obtain a step skew product with interesting dynamical properties, such that the required properties survive under a small perturbation in the class of step skew products.
- Use one of the smooth realizations of the Bernoulli shift (see Subsection 2.3.2) to construct an example of a smooth map that has similar properties. On this step one has to pass from the space of step skew products to the space of mild skew products.
- Prove that the interesting properties of the system survive under small perturbations in the space of Hölder skew products with smooth fiber maps. One can assume that the Hölder exponent is close to 1.
- Use Gorodetski–Ilyashenko–Negut Theorem (see Subsection 2.3.3) to prove that the properties survive under a small perturbation in the class of *smooth maps*. On this step one has to overcome some difficulties related to "Fubini nightmare" effect, and probably reformulate the required properties for the general case of (partially hyperbolic) diffeomorphisms.

2.4 Basic example

In this section we will give a single example of step skew product having a bony attractor without holes. In the next subsection we will provide this example and a sketch of its attractor. Formally, a step skew product cannot have a bony attractor in sense of

Definition 2.2.1 since the phase space of a step skew product is not a manifold. It turns out that it is easy to modify the definition of a bony attractor for the class of (step) skew products over any compact metric space with probability measure (see Subsection 2.4.2). In Subsection 2.4.3 we will introduce some useful notation and conventions. The rest of this section (Subsections 2.4.4–2.4.7) will be devoted to the proof of the fact that our example has a bony attractor without holes.

2.4.1 Example

Consider the following three maps (see Figure 2.5),

$$f_0(x) = 0.6x, \quad f_1(x) = 1 - 0.6(1 - x), \quad f_2(x) = \frac{1}{2\pi}\arctan(10x - 5) + \frac{1}{2}. \tag{2.10}$$

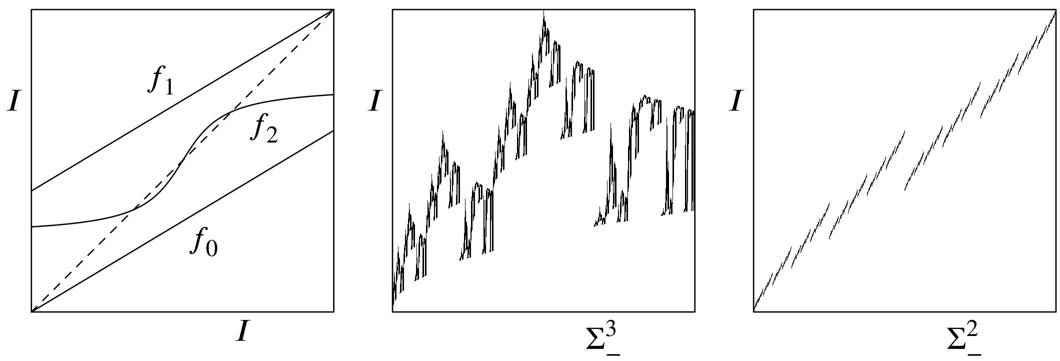
The maps f_0 and f_1 are linear contractions to the endpoints of I, and the map f_2 has a repelling fixed point 0.5.

Consider the corresponding step skew product over the Bernoulli shift,

$$F_0 \colon \Sigma^3 \times I \to \Sigma^3 \times I, \quad (\omega, x) \mapsto (\sigma\omega, f_{\omega_0}(x)). \tag{2.11}$$

We will equip Σ^3 with (⅓, ⅓, ⅓) Bernoulli measure and (3, 3)-adic metric.

The graphs of the functions f_i are drawn in Figure 2.5 (a). The quotient space of the maximal attractor of F_0 by the space of right tails $\omega_0\omega_1\ldots\omega_n\ldots$ of sequences $\omega \in \Sigma^3$ is sketched in Figure 2.5 (b). To make the difference between a bony attractor and a non-bony attractor more clear we have also computed the maximal attractor of the skew product over $\sigma\colon \Sigma^2 \to \Sigma^2$ with fiber maps f_0 and f_1, and sketched the quotient space of this attractor by the space of right tails $\omega_0\omega_1\ldots\omega_n\ldots$ of sequences $\omega \in \Sigma^2$ (see Figure 2.5 (b)).



(a) The fiber maps

(b) A bony attractor, for f_0, f_1 and f_2

(c) A thin attractor without bones, for f_0 and f_1

Figure 2.5 The fiber maps and the attractors

In this subsection we will prove the following theorem.

Theorem 2.4.1 *The skew product F_0 has a bony attractor without holes in sense of Definition 2.4.2 (see next subsection).*

Later we will prove more general Theorem 2.5.2, but we will prove this particular case separately in order to illustrate some key ideas in the simplest case.

2.4.2 Definition

Formally, the phase space $\Sigma^k \times I$, $I = [0, 1]$ is not a smooth manifold, so Definition 2.2.1 (thus Definition 2.2.2 as well) does not work for step skew products. So, we will replace the invariant fibration by the vertical fibration with fibers $\{\omega\} \times I$, $\omega \in \Sigma^k$. As we told in the "Definition of a bony attractor" at page 21, we will first prove that all conditions hold for the maximal attractor, then show that it coincides with the likely limit set. That is why we will replace the likely limit set by the maximal attractor in all conditions of the following definition but the last one.

Definition 2.4.2 We will say that a skew product $F\colon \Sigma^k \times I \to \Sigma^k \times I$ has a *bony attractor without holes* if the following conditions hold.

1. The maximal attractor intersects each fiber either on a single point, or on a segment (a *bone*).
2. The union of the bones is dense in the maximal attractor, and the set of bones has the cardinality of $\mathbb{R}$.
3. The Hausdorff dimension of the maximal attractor is less than the Hausdorff dimension of the phase space,
$$\dim_H A_{max} < \dim_H(\Sigma^k \times I).$$
4. The relative complement to the union of the bones in the maximal attractor is dense in the maximal attractor.
5. The likely limit set coincides with the maximal attractor,
$$A_M = A_{max}, \quad A_{max} := \bigcap_{i \geqslant 0} F^i(\Sigma^k \times I).$$

The *graph part* Γ of the maximal attractor is the relative complement of the union of the bones in the maximal attractor.

All the conditions except for the last one are completely analogous to the conditions of Definitions 2.2.1 and 2.2.2, and the last condition is slightly stronger than the last condition of Definition 2.2.2. Actually, the first condition is trivial in the case of a skew product (see Lemma 2.4.3 below).

2.4.3 Notation, conventions and first observations

In this subsection we will introduce some useful notation. Though this section is devoted to step skew products (more precisely, the step skew product given by (2.10) and (2.11)), we will introduce notation in a more general settings.

Let B be a compact metric space equipped with a measure μ. Let $F\colon B\times I \to B\times I$ be a skew product over a homeomorphism $h\colon B \to B$ with strictly increasing fiber maps $f_b\colon I \to I$.

There are many equivalent metrics on the Cartesian product $B\times I$ that define the same Cartesian product topology. In the sequel we will assume that $B\times I$ is equipped by the metric

$$d_{B\times I}((b,x),(b',x')) = \max(d_B(b,b'), d_I(x,x'))$$

because a ball in this metric is simply the Cartesian product of two balls with equal radii, one in the base and one in the fiber.

For any $b\in B$ and $n\in\mathbb{N}$ the image $F^n(B\times I)$ intersects the vertical fiber $\{b\}\times I$ on the segment $\{b\}\times I_{b,n}$, where

$$I_{b,n} = f_{h^{-1}(b)}\circ\ldots\circ f_{h^{-n}(b)}(I). \tag{2.12}$$

Clearly, $B\times I = F^0(B\times I)\supset F^1(B\times I)\supset\ldots\supset F^n(B\times I)\supset\ldots$ are compact subsets of $B\times I$, hence the maximal attractor $A_{max}(F) = \mathrm{Cl}(\bigcap_n F^n(B\times I))$ coincides with the intersection $\bigcap_n F^n(B\times I)$. Thus for any $b\in B$ the maximal attractor A_{max} intersects the fiber $\{b\}\times I$ on the set $\{b\}\times I_b$, where

$$I_b = \bigcap_{n\geqslant 0} I_{b,n} = \bigcap_{n\geqslant 0} f_{h^{-1}(b)}\circ\ldots\circ f_{h^{-n}(b)}(I). \tag{2.13}$$

Let $\sigma_n^{\pm}(b)$ (resp., $\sigma^{\pm}(b)$) be the endpoints of the segment $I_{b,n}$ (resp., I_b),

$$I_{b,n} =: [\sigma_n^-(b), \sigma_n^+(b)]; \quad I_b =: [\sigma^-(b), \sigma^+(b)]. \tag{2.14}$$

It is not convenient to write compositions like $f_{h^{-1}(b)}\circ\ldots\circ f_{h^{-n}(b)}$ every time, so we will introduce the following notation. For any integer n and any point b of the base B, let $f_{b,n}\colon I\to I$ be the fiber component of the restriction of F^n to the fiber $\{b\}\times I$, i. e.

$$
\begin{aligned}
f_{b,n} &= f_{h^{n-1}(b)} \circ \ldots \circ f_{h(b)} \circ f_b, && n > 0;\\
f_{b,0} &= id, && n = 0;\\
f_{b,n} &= (f_{h^n(b),-n})^{-1} = f^{-1}_{h^n(b)} \circ \ldots \circ f^{-1}_{h^{-2}(b)} \circ f^{-1}_{h^{-1}(b)}, && n < 0.
\end{aligned}
$$

In the special case of a step skew product $F\colon \Sigma^k_A \times I \to \Sigma^k_A \times I$ over a Markov shift, for any admissible finite word u denote by $f_u\colon I \to I$ the composition $f_{u_{|u|}} \circ \ldots \circ f_{u_1}$,

$$f_u = f_{u_{|u|}} \circ f_{u_{|u|-1}} \circ \ldots \circ f_{u_2} \circ f_{u_1}, \text{for a finite word } u \text{ and step skew product } F.$$

Now we can rewrite Formulas (2.12), (2.13) and (2.14) in the following way,

$$
\begin{aligned}
I_{b,n} &= f_{h^{-n}(b),n}(I); & \sigma^-_n(b) &= f_{h^{-n}(b),n}(0); & \sigma^+_n(b) &= f_{h^{-n}(b),n}(1);\\
I_b &= \bigcap_{n\geqslant 0} f_{h^{-n}(b),n}(I); & \sigma^-(b) &= \lim_{n\to+\infty} \sigma^-_n(b); & \sigma^+(b) &= \lim_{n\to+\infty} \sigma^-_n(b).
\end{aligned}
$$

Let us show that the functions $\sigma^\pm\colon B \to I$ are semi-continuous.

Lemma 2.4.3 *Let $F\colon B\times I \to B\times I$ be a skew product with strictly increasing fiber maps. Then the maximal attractor of F intersects each fiber $\{b\}\times I$ on the segment $[\sigma^-(b), \sigma^+(b)]$, where the function $\sigma^-\colon B \to I$ (resp., $\sigma^+\colon B \to I$) is lower semi-continuous (resp., upper semi-continuous).*

Proof The map F is continuous, hence the functions σ^-_n and σ^+_n are continuous as well. Therefore, the function $\sigma^-(b) = \sup_n \sigma^-_n(b)$ is lower semi-continuous, and the function $\sigma^+(b) = \inf_n \sigma^+_n(b)$ is upper semi-continuous. ∎

Denote by Ω the set of the sequences $\omega \in \Sigma^k$ such that the maximal attractor A_{max} intersects the fiber $\{\omega\} \times I$ on a single point (i. e. $\sigma^-(\omega) = \sigma^+(\omega)$). Then the graph part of the attractor (see Definition 2.4.2) is the intersection $A_{max} \cap (\Omega \times I)$, and the bones are the segments $[\sigma^-(\omega), \sigma^+(\omega)]$ for $\omega \notin \Omega$.

2.4.4 Existence of bones

Consider the segment $\tilde{I} = [0.4, 0.6]$. Note that

$$
\begin{aligned}
f_2(\tilde{I}) &= [f_2(0.4), f_2(0.6)]\\
&= \left[\frac{1}{2\pi}\arctan(-1) + 0.5, \frac{1}{2\pi}\arctan(1) + 0.5\right] = [0.375, 0.625] \supset \tilde{I}.
\end{aligned}
$$

Consider a sequence ω such that $\omega_{-n} = 2$ for any $n > m$, i. e.

$$\omega = \ldots 22\ldots 2\omega_{-m}\omega_{-m+1}\ldots\omega_{-1}\omega_0\omega_1\ldots \tag{2.15}$$

Then for any $n > m$,

$$I_{\omega,n} = f_{\omega_{-1}} \circ f_{\omega_{-2}} \circ \ldots \circ f_{\omega_{-m}} \circ f_2^{n-m}(I) =$$
$$= f_{\sigma^{-m}\omega,m} \circ f_2^{n-m}(I) \supset f_{\sigma^{-m}\omega,m} \circ f_2^{n-m}(\tilde{I}) \supset f_{\sigma^{-m}\omega,m}(\tilde{I}).$$

Therefore, the segment $I_{\omega,n}$ includes the non-trivial segment $f_{\sigma^{-m}\omega,m}(\widetilde{I})$. Recall that I_ω is the intersection of the segments $I_{\omega,n}$. Hence, I_ω includes the segment $f_{\sigma^{-m}\omega,m}(\tilde{I})$ as well, thus $\omega \notin \Omega$.

Denote by E the set of sequences of the form (2.15). In the previous paragraph we have shown that $E \cap \Omega = \varnothing$. Clearly, $|E| = |\mathbb{R}|$ and E is dense in Σ^3. Hence, $|\Sigma^3 \setminus \Omega| = |\mathbb{R}|$ and $\Sigma^3 \setminus \Omega$ is dense in Σ^3. Recall (see Subsection 2.4.3) that the function σ^- (resp., σ^+) is lower semi-continuous (resp., upper semi-continuous), and $\sigma^-(\omega) = \sigma^+(\omega)$ for $\omega \in \Omega$. Therefore, the density of $\Sigma^3 \setminus \Omega$ in Σ^3 implies the density of the union of the bones in the maximal attractor. Thus F_0 satisfies the second condition of Definition 2.4.2.

2.4.5 Hausdorff dimension and measure

The only property of the fiber maps we will use in this subsection is that the images of the maps f_0^2 and f_1^2 have empty intersection. Therefore, for any point $x \in I$ either $x \notin \operatorname{Im} f_0^2$ or $x \notin \operatorname{Im} f_1^2$.

Consider a finite word $v = v_{-n}\ldots v_{-1}$. Recall that $C(v)$ is the set of infinite words $\omega \in \Sigma^3$ such that $\omega|_{[-n,-1]} = v$. The fiber map depends only on the current symbol of the sequence, hence the segment $I_{\omega,n}$ is the same for all $\omega \in C(v)$. Denote by I_v this segment,

$$I_v = I_{\omega,n} \text{ for any sequence } \omega \in C(v).$$

Each interval I_v can be covered by $\lceil 3^n |I_v| \rceil$ segments of length 3^{-n}. Each cylinder $C(v)$ can be covered by 3^{n+1} balls of radii 3^{-n}. Therefore, the Cartesian product $C(v) \times I_v$ can be covered by $3^{n+1} \lceil 3^n |I_v| \rceil$ balls of radii 3^{-n}.

The union of the Cartesian products $C(v) \times I_v$ for all words v of length n coincides with $F^n(\Sigma^3 \times I)$, hence this union includes the maximal attractor. On the other hand, this union can be covered by

$$3^{n+1} \sum_{|v|=n} \lceil 3^n |I_v| \rceil \leqslant 3^{n+1} \sum_{|v|=n} (3^n |I_v| + 1) = 3^{2n+1} \sum_{|v|=n} |I_v| + 3^{2n+1} \tag{2.16}$$

balls of radii 3^{-n}.

Let us estimate the sum $\Sigma_v |I_v|$. Let $D_{x,n}$ be the set of the words v such that $x \in I_v$,

$$D_{x,n} = \{v \,|\, |v| = n, x \in I_v\}.$$

Applying Fubini Theorem to the set $\{(v,x) \,|\, |v| = n, x \in I_v\}$, we obtain

$$\sum_{|v|=n} |I_v| = \int_I |D_{x,n}|\, dx \leqslant \max_{x\in I} |D_{x,n}|.$$

Therefore, it is sufficient to estimate the right hand side of this inequality.

Let us prove that $|D_{x,n+2}| \leqslant 8\,|D_{x,n}|$ for any $x \in I$. Note that for any word $v \in D_{x,n+2}$ the word $u = v_{-n} \dots v_{-1}$ must be an element of $D_{x,n}$, i. e. $v = v_{-n-2}v_{-n-1}u$, $u \in D_{x,n}$. Indeed,

$$I_v = (f_{v_{-1}} \circ \dots \circ f_{v_{-n}}) \circ (f_{v_{-n-1}} \circ f_{v_{-n-2}})(I) \subset f_{v_{-1}} \circ \dots \circ f_{v_{-n}}(I) = I_u,$$

hence $x \in I_u$, and $u \in D_{x,n}$. Therefore, $|D_{x,n+2}| \leqslant 9\,|D_{x,n}|$. In order to prove that $|D_{x,n+2}| \leqslant 8\,|D_{x,n}|$, it is sufficient to show that for any word $u \in D_{x,n}$ there exists a couple of symbols v_{-n-2} and v_{-n-1} such that $v_{-n-2}v_{-n-1}u \notin D_{x,n+2}$.

Consider the point $x' = f^{-1}_{u_{-n}} \circ \dots \circ f^{-1}_{u_{-1}}(x)$. As we noted in the beginning of this subsection, either $x' \notin f_0^2(I)$, or $x' \notin f_1^2(I)$, therefore either $00u \notin D_{x,n+2}$, or $11u \notin D_{x,n+2}$. Hence, $|D_{x,n+2}| \leqslant 8\,|D_{x,n}|$.

Thus $|D_{x,n}| \leqslant \mathrm{const}(\sqrt{8})^n$, and

$$\sum_{|v|=n} |I_v| \leqslant \int_I \mathrm{const}(\sqrt{8})^n = \mathrm{const}(\sqrt{8})^n.$$

Now let us estimate the sum (2.16).

$$3^{2n+1} \sum_{|v|=n} |I_v| + 3^{2n+1} \leqslant \mathrm{const} \cdot 3^{2n+1}(\sqrt{8})^n + 3^{2n+1} < \mathrm{const} \cdot (9\sqrt{8})^n.$$

Hence, the Hausdorff dimension of the maximal attractor is at most $\log_3 9\sqrt{8} < 3$, thus the measure of A_{max} is zero[9]. Hence, F_0 satisfies the third condition of Definition 2.4.2.

2.4.6 Density of the graph

Now let us prove that the bones belong to the closure of the graph, i. e. the maximal attractor coincides with the closure of the graph Γ of the restriction $\sigma^+|_\Omega$. Let (ω, x) be

[9] Recall that we equip the phase space $\Sigma^3 \times I$ with the Cartesian product of the Bernoulli measure on Σ^3 and the Lebesgue measure on I.

a point of the maximal attractor, $C(v) \times U$ be its standard neighborhood, i. e. $C(v)$ is a cylinder corresponding to the word $v = \omega_{-n} \dots | \omega_0 \dots \omega_n$, and U is a neighborhood of x.

We need to find a point of the graph Γ in the product $C(v) \times U$. Actually, we will find a point of Γ in the smaller set $C(v) \times \{x\}$. The graph Γ is invariant under F, hence it is sufficient to find a point of Γ in $F^{-n}(C(v) \times \{x\}) = C(\sigma^{-n} v) \times \left\{ f_{\omega,-n}(x) \right\}$. Denote $x' = f_{\omega,-n}(x)$. Then $F^{-n}(C(v) \times \{x\}) = C(\sigma^{-n} v) \times \left\{ x' \right\}$. So we are looking for a sequence η such that $\eta \in C(\sigma^{-n} w)$ and $I_\eta = \left\{ x' \right\}$. Note that the former property depends only on the right tail of η, while the latter property depends only on the left tail of η. Hence, we can choose those tails independently. For the right tail η^+ we can just take any sequence starting with $\sigma^{-n} v$.

Note that the images of the maps f_0 and f_1 cover I. Hence, for any point $y \in I$, either $f_0^{-1}(y)$, or $f_1^{-1}(y)$ is defined. Therefore, we can choose a sequence $\eta^- = \dots \eta_{-k} \dots \eta_{-1}$ of zeroes and ones such that for any k the point $f_{\eta^-,-k}(x')$ belongs to the image of $f_{\eta_{-k-1}}$, thus $x' \in I_{\eta^-}$. On the other hand, the maps f_0 and f_1 contract on I with coefficient 0.6, hence the length of $I_{\eta^-,m}$ is equal to 0.6^m. Finally, 0.6^m tends to zero as m tends to infinity, thus $|I_{\eta^-}| = 0$, and $I_{\eta^-} = \left\{ x' \right\}$.

So, the map F_0 satisfies the fourth condition of Definition 2.4.2.

2.4.7 Coincidence of attractors

Let us prove that the maximal attractor of F_0 coincides with the likely limit set. Note that a point (η, x') of the graph Γ belongs to the ω-limit set of a point (ω, x) if the sequence η belongs to the ω-limit set of the sequence ω. Indeed, if the sequences $\sigma^n \omega$ and η coincide on the segment $[-N, N]$, then $f_{\sigma^n \omega}(x) \in I_{\eta,N}$. On the other hand, x' is the only common point of the intervals $I_{\eta,N}$.

The likely limit set (see Definition 2.1.9 at page 14) of the Bernoulli shift is Σ^3, hence the likely limit set of the skew product F_0 includes the graph Γ. Therefore, the likely limit set includes the closure of the graph, i. e. the maximal attractor. On the other hand, the maximal attractor includes the likely limit set for any dynamical system. Thus, $A_{max} = A_M$.

Finally, the map F_0 has a bony attractor without holes in the sense of Definition 2.4.2.

2.5 Open set of step skew products

In the previous section we have constructed a *single* dynamical system with a bony attractor. Now it is natural to ask whether this property survives under a small perturbation in some ambient space. This section deals with the perturbations in the space of step skew products. At the same time we will replace a Bernoulli shift by a Markov shift.

Theorem 2.5.1 *Consider a Markov shift with transition matrix A of size $k \times k$, $k \geqslant 2$, such that $A_{ij} \neq 0$ for any i, j. Then there exists a non-empty open set in the space of step skew products over the Markov shift σ_A with the fiber $I = [0, 1]$ such that each skew product from this set has a bony attractor without holes.*

Let $J = J(f_0, \ldots, f_{k-1})$ be the convex hull of the set of fixed points of the fiber maps. Then the strip $\Sigma_A^k \times J$ includes the maximal attractor $A_{max}(F)$. Indeed, all the points above this strip go downwards, and all the points below this strip go upwards.

The following theorem provides simple sufficient conditions for Theorem 2.5.1.

Theorem 2.5.2 *Let $f_0, \ldots, f_{k-1} : I \to I$ be strictly increasing maps such that*

1. *there exists a finite set of elements f_{u^j} of the semi-group generated by the maps f_i such that each map f_{u^j} contracts on I, and the images of the segment J under these maps cover the segment J;*
2. *there exists a finite composition of the maps f_i such that this composition has a repelling fixed point.*

Then the corresponding step skew product $F : \Sigma_A^k \times I \to \Sigma_A^k \times I$ has a bony attractor without holes.

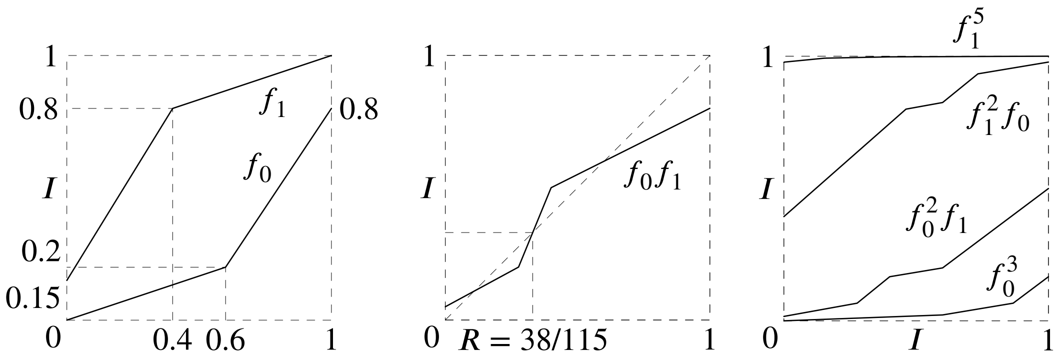
The proof of this theorem is very similar to the proof of Theorem 2.4.1. The composition from the second condition plays the role of the map f_2, and the maps f_{u^j} play the role of the maps f_0 and f_1.

The next subsection reduces Theorem 2.5.1 to Theorem 2.5.2, and the rest of this section is devoted to the proof of Theorem 2.5.2.

2.5.1 An open set of examples

In this subsection we will construct a non-empty open set of step skew products such that each system from this set satisfies the assumptions of Theorem 2.5.2. Thus we will reduce Theorem 2.5.1 to Theorem 2.5.2.

For any $k \geqslant 3$ one can find a non-empty open set of step skew products such that each skew product from this set is similar to the basic example, see Section 2.4. Namely, let f_0 and f_1 be two maps contracting to the points $x_0 < x_1$, and $0.5 < f_0', f_1' < 1$. Let us choose maps $f_2, \ldots, f_{k-1}$, such that $f_i(I) \subset (x_0, x_1)$, and one of these maps has a repelling fixed point. Then Condition 2 holds. Moreover, in this case $J = [x_0, x_1]$, and the inequality $0.5 < f_0', f_1'$ implies Condition 1. It is easy to see that the set of tuples $(f_0, \ldots, f_{k-1})$ described in this paragraph is not empty and open in C^1 topology.



(a) The fiber maps

(b) The composition $f_0 \circ f_1$ has a repelling fixed point R

(c) Contracting compositions whose images cover I

Figure 2.6 A step skew product over $\sigma\colon \Sigma^2 \to \Sigma^2$ that satisfies the conditions of Theorem 2.5.2.

It is harder to find an example for $k = 2$. Let f_0 be the piecewise-linear map with the vertices at $(0, 0)$, $(0.6, 0.2)$, $(1, 0.8)$ and f_1 be the piecewise-linear map with the vertices at $(0, 0.15)$, $(0.4, 0.8)$, $(1, 1)$ (see Figure 2.6 (a)). It is easy to see that $\frac{38}{115}$ is a repelling fixed point of the composition $f_0 f_1$ (see Figure 2.6 (a)). The hard step is to find the compositions f_{w^j}. I have written a computer program (thanks `C++`, `STL` and `Boost`), and this program found that f_0^3, $f_0^2 f_1$, $f_1^2 f_0$ and f_1^5 (see Figure 2.6 (c)) satisfy the first condition of Theorem 2.5.2. Note that once the compositions are found, one can check that they really satisfy the first condition of Theorem 2.5.2 without computer.

The only problem is that the constructed example is not smooth, but it is easy to overcome this problem by replacing the vertices by sufficiently small arcs. Sufficiently small neighborhood of this system will satisfy assumptions as well.

2.5.2 Existence of bones

Let $f_v = f_{v_1} \dots f_{v_n}$ be a composition having a repelling fixed point R. Then for sufficiently small segment $\tilde{I} \ni R$, the image of $\tilde{I}$ under f_v includes $\tilde{I}$. Then for any sequence of the form

$$\omega = \dots vv \dots v\omega_{-m} \dots \omega_{-1} | \omega_0 \dots$$

and any $n = m + l|v|$ we have

$$I_{\omega,n} = f_{\omega_{-1}} \circ \dots \circ f_{\omega_{-m}} \circ f_v^l(I) \supset f_{\omega_{-1}} \circ \dots \circ f_{\omega_{-m}}(\tilde{I}).$$

Hence, the intersection $I_\omega = \bigcap_n I_{\omega,n}$ includes the segment $f_{\omega_{-1}} \circ \ldots \circ f_{\omega_{-m}}(\tilde{I})$ as well. Thus $\omega \notin \Omega$. Obviously, the set of sequences ω of this form is dense in Σ_A^k, and has the cardinality of $\mathbb{R}$.

So, the second condition of Definition 2.4.2 holds.

2.5.3 Hausdorff dimension and measure

The proof mostly repeats the proof for the basic example. Consider the maps f_{u^j}. The images of J under these maps cover J, hence at least two of these maps contract to different points. Without loss of generality, we can assume that the maps f_{u^0} and f_{u^1} contract to different points. Therefore, the images of I under some powers of these maps have empty intersection, $f_{u^0}^m(I) \cap f_{u^1}^m(I) = \varnothing$.

Lemma 2.5.3 *Consider a step skew product over a Markov shift $\sigma \colon \Sigma_A^k \to \Sigma_A^k$ such that the transition matrix A has no zeroes. Let $f_0, \ldots, f_{k-1} \colon I \to I$ be the fiber maps of this skew product. Suppose that there exists a finite number of compositions f_{v^j} such that the intersection of images of I under these compositions is empty,*

$$\bigcap_j f_{v^j}(I) = \varnothing.$$

Then the maximal attractor has Hausdorff dimension less than 3*, and its measure is zero.*

Proof Denote by ε the least of the probabilities in the matrix A. Denote by N the length of the longest word among v^j. Then *for any point $x \in I$ there exists a composition f_v of length N such that $x \notin f_v(I)$.*

Denote by $D_{x,n}$ the set of the words $w = w_{-n} \ldots w_{-1}$ of length n such that $x \in f_w(I)$,

$$D_{x,n} = \left\{ w = w_{-n} \ldots w_{-1} \,\middle|\, x \in f_w(I) \right\}. \tag{2.17}$$

For each word $w \in D_{x,n+N}$, the word $u = w_{-n} \ldots w_{-1}$ must be an element of $D_{x,n}$. Hence, $\# D_{x,n+N} \leqslant k^N \# D_{x,n}$. Moreover, for any word $u \in D_{x,n}$ there exists a word v of length N such that $vu \notin D_{x,n+N}$. Therefore, $\# D_{x,n+N} \leqslant (k^N - 1) \# D_{x,n}$ and $\mu C(D_{x,n+N}) \leqslant (1 - \varepsilon^N) \mu C(D_{x,n})$. Here $C(D_{x,n})$ is the union of all cylinders corresponding to the words in $D_{x,n}$,

$$C(D_{x,n}) = \bigcup_{w \in D_{x,n}} C(w).$$

Let D_x be the set of the infinite words $\omega \in \Sigma_A^k$ such that $x \in I_\omega$. Obviously, D_x is the intersection of all the cylinders $C(D_{x,n})$,

$$D_x := \left\{\omega \in \Sigma^k \,\middle|\, (\omega, x) \in A_{max}\right\} = \bigcap_n C(D_{x,n}).$$

Hence, the measure of D_x equals zero for all x. Therefore, the measure of A_{max} equals zero as well.

Now let us show that the Hausdorff dimension of A_{max} is less than 3. As in the basic example, we have $\# D_{x,n} \leqslant \text{const} \cdot (k^N - 1)^{n/N}$. Therefore,

$$\sum_{|w|=n} |I_w| = \int_I \# D_{x,n}\, dx \leqslant \text{const} \cdot (k^N - 1)^{n/N},$$

and the maximal attractor can be covered by at most

$$k^{n+1} \sum_{|w|=n} \lceil k^n |I_w| \rceil \leqslant k^{2n+1} \sum_{|w|=n} |I_w| + k^{2n+1}$$
$$\leqslant \text{const} \cdot k^{2n+1}(k^N - 1)^{n/N} + k^{2n+1} \leqslant \text{const} \cdot k^{2n}(k^N - 1)^{n/N}$$

balls of radii k^{-n}. Therefore, the Hausdorff dimension of the maximal attractor is at most

$$\dim_H A_{max} \leqslant \log_k k^2(k^N - 1)^{1/N} < 3.$$

∎

So, the third condition of Definition 2.4.2 holds.

2.5.4 Absence of holes

The proof of the last two conditions of Definition 2.4.2 for a step skew product F satisfying the assumptions of Theorem 2.5.2 repeats the proof for the basic example F_0 with minor modifications.

In the proof from Subsection 2.4.6 it is sufficient to replace the maps f_0 and f_1 by the maps f_{uj}, and the number 0.6 by the maximum of the derivatives of the maps f_{uj}.

The proof in Subsection 2.4.7 needs even less modifications — it is sufficient to replace Σ^3 by Σ^k_A.

2.6 Smooth example

In this section we will construct a single smooth dynamical system having a bony attractor without holes. Our system will be a skew product over the Smale–Williams solenoid map (2.4) for $k = 6$ and $\varepsilon = 0.2$:

$$s\colon (z, w) \mapsto (z^6, 0.5z + 0.2w).$$

The solenoid map is not bijective on the whole solid torus, but the map s is bijective on its maximal attractor $A_{max}(s)$, and the restriction of the solenoid map to the maximal attractor is semi-conjugated to the Bernoulli shift $\sigma\colon \Sigma^6 \to \Sigma^6$. For the details about the fate map Φ that semi-conjugates s to the Bernoulli shift see Subsection 2.3.2.

In our example the fiber maps depend only on the future part of the fate, i. e. only on z coordinate of the point in the base,

$$F_S\colon ((z, w), x) \mapsto (s(z, w), f_z(x)).$$

On the subsets A_0, A_2 and A_4 the fiber maps f_z coincide with f_0, f_1 and f_2, respectively. Then we extend the map to the solid torus using linear interpolation on the subsets A_1, A_3 and A_5.

Theorem 2.6.1 *The restriction of the map F_S to the set $A_{max}(s)\times I$ has a bony attractor without holes.*

The proof of this theorem mostly repeats the proof of Theorem 2.5.2. The only difficult part is to adopt the definition of $D_{x,n}$ and prove the inequality $\#D_{x,n+N} \leqslant (6^N - 1)\#D_{x,n}$. It is slightly more difficult because the preimage $f_{\omega,-n}$ now depends on the whole sequence ω instead of only the symbols ω_{-n}, …, ω_{-1}. However, it turns out that the set of possible preimages $f_{\omega,-n}$ for ω belonging to a ball or radius λ^{-n} is uniformly small. This uniform estimate allows us to complete the proof.

We will not provide the detailed proof of Theorem 2.6.1. In the next section we will prove a more general Theorem 2.7.3.

2.7 Mild skew products

In this section we will construct a non-empty open set of mild skew products over the Bernoulli shift having bony attractor without holes. In order to simplify the proof in Section 2.8, we need to consider not only the skew products such that the fiber maps f_ω are Lipschitz on ω, but also the skew products such that the fiber maps f_ω are only Hölder on ω.

Fix some constants $C > 0$ and $\alpha < 1$. Consider the space of Hölder skew products with smooth fiber maps,

$$g_\omega \in C^1(I), \tag{2.18}$$

$$\|g_\omega - g_\eta\|_{C^0} \leqslant C\,d(\omega, \eta)^\alpha \tag{2.19}$$

The distance between two maps G and $\widetilde{G}$ in this space is the maximum of the C^1 distances between their fiber maps,

$$d(G, \widetilde{G}) = \max_{\omega \in \Sigma^k} \|g_\omega - \widetilde{g}_\omega\|_{C^1}.$$

Theorem 2.7.1 *There exists a non-empty open set in the space of Hölder skew products such that each map from this set has a bony attractor without holes.*

Definition 2.7.2 We will say that a skew product F over the Bernoulli (Markov) shift σ has *step set* $S \subset \{0, \ldots, k-1\}$ if $\omega_0 = \eta_0 \in S$ implies $f_\omega = f_\eta$. The maps f_ω for $\omega_0 \in S$ are called *step fiber maps* of F.

We will use slightly modified versions of the fiber maps from the basic example. Namely, let the functions $\tilde{f}_0$, $\tilde{f}_1$ and $\tilde{f}_2$ be given by

$$\tilde{f}_0 \colon x \mapsto 0.9x + 0.001,$$
$$\tilde{f}_1 \colon x \mapsto 0.999 - 0.9(1-x),$$
$$\tilde{f}_2 \colon x \mapsto \frac{1}{2\pi} \arctan(10(x - 0.5)) + 0.5.$$

The following theorem provides us a list of conditions such that for any Lipschitz skew product F satisfying this condition there exists a small neighborhood of F in the space of Hölder continuous skew products that satisfies Theorem 2.7.1. In Subsection 2.7.1 we will prove some technical lemmas. Then in Subsections 2.7.2—2.7.5 we will show that a sufficiently small perturbation of the map F has a bony attractor without holes.

Theorem 2.7.3 *Suppose that a skew product F over a Markov shift with an irreducible aperiodic transition matrix A satisfies the following conditions.*

1. *For any symbol v_0 from the alphabet $\{0, \ldots, k-1\}$, and $i = 0, 1, 2$ there exists a symbol u_0 such that $u_0 v_0$ is an admissible word, u_0 belongs to the step set of F, and the corresponding step fiber map f_{u_0} is $\tilde{f}_i$.*
2. *Both fiber maps f_ω and the inverse maps f_ω^{-1} are bi-Lipschitz with constant L_f,*

$$\sup_\omega \max \left(\|f'_\omega\|_{C^0}, \|(f_\omega^{-1})'\|_{C^0} \right) < L_f. \qquad (2.20)$$

3. *The fiber maps f_ω and the inverse maps f_ω^{-1} are Lipschitz on ω with constant L_b,*

$$\|f_\omega - f_\eta\|_{C^0} < L_b d(\omega, \eta), \quad \|f_\omega^{-1} - f_\eta^{-1}\|_{C^0} < L_b d(\omega, \eta). \qquad (2.21)$$

4. *The map F is partially hyperbolic with vertical central fibration. Moreover,*

$$L_f < \lambda \text{ and } L_f < \lambda^\alpha.$$

5. *There exists an interval* $J = [a, b] \subset I$ *such that*
   - *if* $f_\omega(x) \in I \setminus J$, *then* $f'_\omega(x) < \text{const} < 1$;
   - *the open intervals*

   $$\left(\tilde f_0(a) + \frac{L_b}{\lambda - L_f}, \tilde f_0(b) - \frac{L_b}{\lambda - L_f}\right) \text{ and } \left(\tilde f_1(a) + \frac{L_b}{\lambda - L_f}, \tilde f_1(b) - \frac{L_b}{\lambda - L_f}\right)$$

   cover J.

Then for sufficiently small δ *any Hölder skew product* G *with Hölder exponent* α *which is* δ*-close to* F *in* C^1 *metric,*

$$d(F, G) = \max_{\omega \in \Sigma^k} \|f_\omega - g_\omega\|_{C^1} < \delta, \tag{2.22}$$

has a bony attractor without holes.

We can assume that the perturbation is so small that the fiber maps $g_\omega^{\pm 1}$ are Lipschitz with constant L_f as well,

$$\sup_\omega \max\left(\|g'_\omega\|_{C^0}, \|(g_\omega^{-1})'\|_{C^0}\right) < L_f.$$

Notation 2.7.4 Denote by S_j the set of symbols $i \in S$ such that the corresponding step fiber map is $\tilde f_j$. Then the first condition of Theorem 2.7.3 can be reformulated in the following way. For any symbol v_0 and any $j = 0, 1, 2$ there exists $u_0 \in S_j$ such that the word $u_0 v_0$ is admissible.

2.7.1 Technical lemmas

The following lemma generalizes Lemma 3.1 from [6]. The original lemma deals with the case $s = m$. Gorodetski and Ilyashenko also studied the case $s > m/2$ in Lemma 4.1 of the same article.

Lemma 2.7.5 *For any* L_f, α, λ *and* C *there exists a number* $K = K(L_f, \alpha, \lambda, C)$ *such that the following holds. Let* F *be a skew product over a Markov shift that satisfies conditions 2, 3 and 4 of Theorem 2.7.3, let* S *be the step set of the map* F*. Let* G *be a Hölder continuous skew product* δ*-close to* F *in* C^1 *metric. Let* $m > s \geqslant 0$ *be two integer numbers. Let* ω *and* η *be two bi-infinite words such that*

- $\omega_i = \eta_i$ *for* $|i| < m$;

- *$\omega_i \in S$ for $m - s \leqslant i < m$.*

Then

$$|g_{\omega,m} - g_{\eta,m}| \leqslant K\delta^{\beta} + \frac{L_b}{\lambda - L_f} \cdot \left(\frac{L_f}{\lambda}\right)^s,$$

where $\beta = 1 - \log_{\lambda^{\alpha}} L_f$.

Proof Denote by δ_i the C^0-norm of the difference $g_{\omega,i} - g_{\eta,i}$,

$$\delta_i = \|g_{\omega,i} - g_{\eta,i}\|_{C^0}.$$

Since fiber maps g_{ω} are Lipschitz with constant L_f,

$$\delta_{i+1} \leqslant L_f \delta_i + \|g_{\sigma^i\omega} - g_{\sigma^i\eta}\|_{C^0}. \tag{2.23}$$

Indeed, for any $x \in I$ we have

$$\begin{aligned} |g_{\omega,i+1}(x) - g_{\eta,i+1}(x)| = |g_{\sigma^i\omega}(g_{\omega,i}(x)) - g_{\sigma^i\eta}(g_{\eta,i}(x))| \leqslant \\ \leqslant |g_{\sigma^i\omega}(g_{\omega,i}(x)) - g_{\sigma^i\omega}(g_{\eta,i}(x))| + |g_{\sigma^i\omega}(g_{\eta,i}(x)) - g_{\sigma^i\eta}(g_{\eta,i}(x))| \leqslant \\ \leqslant L_f\delta_i + \|g_{\sigma^i\omega} - g_{\sigma^i\eta}\|_{C^0}. \end{aligned}$$

Step 0. Preparations. In order to use (2.23), we need to estimate the second summand. For each $i = 0, \dots, m-1$, we will use one of the following estimations.

1. Due to (2.19),

$$|g_{\sigma^i\omega} - g_{\sigma^i\eta}| \leqslant C \cdot d(\sigma^i\omega, \sigma^i\eta)^{\alpha} \leqslant C\lambda^{(-m+i)\alpha}. \tag{2.24}$$

2. Due to (2.21) and (2.22),

$$|g_{\sigma^i\omega} - g_{\sigma^i\eta}| \leqslant |g_{\sigma^i\omega} - f_{\sigma^i\omega}| + |f_{\sigma^i\omega} - f_{\sigma^i\eta}| + |f_{\sigma^i\eta} - g_{\sigma^i\eta}| \leqslant 2\delta + L_b\lambda^{-m+i}. \tag{2.25}$$

3. Moreover, for $i \geqslant m - s$ we have $f_{\sigma^i\omega} = f_{\sigma^i\eta}$, hence

$$|g_{\sigma^i\omega} - g_{\sigma^i\eta}| \leqslant 2\delta, \quad \text{for } i \geqslant m - s. \tag{2.26}$$

Let us choose l such that

$$C\lambda^{(-l-1)\alpha} \leqslant 2\delta < C\lambda^{-l\alpha}. \tag{2.27}$$

We will use the first estimate for $i < m - l$, the second one for $m - l \leqslant i < m - s$, and the last one for $i \geqslant m - s$. These estimates will lead to linear recurrent inequalities for δ_i. The key idea is that the solutions of the corresponding recurrent equations are sums of some increasing geometric progressions, hence they are of the same order as the last summand.

The only technical detail is that some of the ranges $[0, m-l-1]$, $[m-l, m-s-1]$ and $[m-s, m-1]$ can be empty. However, it is sufficient to prove the lemma for sufficiently large m. Indeed, if $m_1 < m_2$, then the upper estimate for δ_{m_1-i} for $m = m_1$ is less than the upper estimate for δ_{m_2-i} for $m = m_s$. In particular, the upper estimate for δ_{m_1} for $m = m_1$ is less than the upper estimate for δ_{m_2} for $m = m_2$. Hence, we can assume that $m > l$.

Step 1. $0 \leqslant i < m - l$. Due to (2.23) and (2.24),

$$\delta_{i+1} \leqslant L_f \delta_i + C\lambda^{(-m+i)\alpha}.$$

Recall that $\delta_0 = 0$, hence by induction one can easily prove that

$$\delta_i \leqslant C\lambda^{-m\alpha} \frac{\lambda^{i\alpha} - L_f^i}{\lambda^\alpha - L_f}.$$

In particular, if $m \geqslant l$, then

$$\delta_{m-l} \leqslant C\lambda^{-m\alpha} \frac{\lambda^{(m-l)\alpha} - L_f^{m-l}}{\lambda^\alpha - L_f} < C\frac{\lambda^{-l\alpha}}{\lambda^\alpha - L_f} \leqslant \frac{2\delta\lambda^\alpha}{\lambda^\alpha - L_f}. \tag{2.28}$$

Step 2. $m - l \leqslant i$. Due to (2.23) and (2.25),

$$\delta_{i+1} \leqslant L_f \delta_i + 2\delta + L_b \lambda^{-m+i}.$$

Moreover, for $i \geqslant m - s$,

$$\delta_{i+1} \leqslant \delta_i L_f + 2\delta.$$

Let us divide each inequality by L_f^{i+1}, and sum up the results for $i = m - l, \dots, m - 1$,

$$\frac{\delta_m}{L_f^m} \leqslant \frac{\delta_{m-l}}{L_f^{m-l}} + \frac{2\delta}{L_f} \sum_{i=m-l}^{m-1} \frac{1}{L_f^i} + \frac{L_b}{\lambda^m L_f} \sum_{m-l \leqslant i < m-s} \left(\frac{\lambda}{L_f}\right)^i, \tag{2.29}$$

where the last sum is zero if $l \leqslant s$.

Let us estimate the sums of finite geometric progressions by the sums of the corresponding infinitely decreasing geometric progressions,

$$\sum_{i=m-l}^{m-s} \frac{1}{L_f^i} < \sum_{i=m-l}^{\infty} \frac{1}{L_f^i} = \frac{1}{L_f^{m-l-1}(L_f - 1)}.$$

$$\sum_{m-l\leqslant i<m-s}\left(\frac{\lambda}{L_f}\right)^i < \sum_{i=s-m+1}^{\infty}\left(\frac{L_f}{\lambda}\right)^i = \frac{L_f^{s-m+1}}{\lambda^{s-m}(\lambda - L_f)}$$

Let us substitute these estimates into (2.29) and multiply both sides by L_f^m,

$$\delta_m \leqslant \delta_{m-l}L_f^l + \frac{2\delta L_f^l}{L_f - 1} + \frac{L_b L_f^s}{\lambda^s(\lambda - L_f)} \tag{2.30}$$

Due to the right part of the inequality (2.27),

$$L_f^l = (\lambda^{\alpha l})^{\frac{\log L_f}{\alpha \log \lambda}} < \left(\frac{C}{2\delta}\right)^{\frac{\log L_f}{\alpha \log \lambda}}.$$

Let us substitute this estimate and (2.28) into (2.30),

$$\begin{aligned}\delta_m &< \left(\frac{2\delta\lambda^\alpha}{\lambda^\alpha - L_f} + \frac{2\delta}{L_f - 1}\right)\cdot\left(\frac{C}{2\delta}\right)^{\frac{\log L_f}{\alpha \log \lambda}} + \frac{L_b}{\lambda - L_f}\left(\frac{L_f}{\lambda}\right)^s \\ &= K(L_f, \alpha, \lambda, C)\delta^\beta + \frac{L_b}{\lambda - L_f}\left(\frac{L_f}{\lambda}\right)^s,\end{aligned}$$

where

$$K(L_f, \alpha, \lambda, C) = \frac{2L_f(\lambda^\alpha - 1)}{(\lambda^\alpha - L_f)(L_f - 1)}\cdot\left(\frac{C}{2}\right)^{\frac{\log L_f}{\alpha \log \lambda}}.$$

∎

Note that in the case of a mild skew product the segment $I_{\omega,n}$ depends on all symbols of the sequence ω, not just $\omega_{-n}, \ldots, \omega_{-1}$. So, we will need to change the definitions of the segment I_v and of the set $D_{x,n}$.

Definition 2.7.6 Given a word $v = v_{-n}\ldots v_{-1}|v_0\ldots v_m$, $n, m>0$, denote by I_v the minimal segment that includes all the segments $I_{\omega,n}$ for $\omega \in C(v)$,

$$I_v = \left[\inf_{\omega\in C(v)} f_{\sigma^{-n}\omega,n}(0), \sup_{\omega\in C(v)} f_{\sigma^{-n}\omega,n}(1).\right]$$

Denote by $D_{x,n}$ the set of the words $v = v_{-n}\ldots v_n$ such that $x \in I_v$.

Recall that Lemma 2.5.3 followed from the fact that for any word $v \in D_{x,n}$ there exists a word u of the fixed length N such that $uv \notin D_{x,n+N}$. The following lemma is analogue of this fact in the case of a mild skew product.

Lemma 2.7.7 *There exists a natural number $N = N(\lambda, L_f, L_b)$ such that for any C and any α, for sufficiently small positive δ the following holds. Let G be a skew product that satisfies all the assumptions of Theorem 2.7.3. Then for any admissible finite word $v = v_{-m} \dots |v_0 \dots v_{m+N}$ and any point $x \in I$, there exists a word u of length N such that the word uv is admissible, and $x \notin I_{uv}$.*

Proof Choose a small number δ such that $K(L_f, \alpha, \lambda, C)\delta^\beta < 0.01$ and for any two sequences ω and η such that $\omega_0, \dots, \omega_9 \in S_0$ and $\eta_0, \dots, \eta_9 \in S_1$ the 0.01-neighborhoods of the images of $g_{\omega,10}$ and $g_{\eta,10}$ have empty intersection. Next, let us choose N such that for any sequence that ends with $N - 10$ elements of the step set the second term in the estimate from Lemma 2.7.5 is less that 0.01,

$$\frac{L_b}{\lambda - L_f} \cdot \left(\frac{L_f}{\lambda}\right)^{N-10} < 0.01.$$

Let us prove the lemma for the chosen number N. First, let us choose a word $\widetilde{u}$ that consists of $N - 10$ elements of the step set, and the word $\widetilde{u}v$ is admissible. Then, due to Lemma 2.7.5, $|f_{\omega,-N+10}(x) - f_{\eta,-N+10}(x)| < 0.02$ for any two sequences $\omega, \eta \in C(\widetilde{u}v)$. Therefore, the set of possible values of $f_{\omega,-N+10}(x)$ is either strictly above the set of possible images of $g_{\omega,10}$ for $\omega_0, \dots, \omega_9 \in S_0$, or strictly below the set of possible images of $g_{\omega,10}$ for $\omega_0, \dots, \omega_9 \in S_1$. Without loss of generality we can assume that the set of possible values of $f_{\omega,-N+10}(x)$ is strictly above the set of possible images of $g_{\omega,10}$. Then one can prepend 10 elements of S_0 to the word $\widetilde{u}$, and obtain the word u that satisfies the assertion of the lemma. ∎

Lemma 2.7.8 *Let x be a point of the segment I and n be a natural number. Suppose that $g_{\omega,-n}(x) \in J$. Then $g_{\eta,-n-1}(x)$ belongs to J either for any $\eta \in C^\infty_{-n}(\omega)$ such that $\eta_{-n-1} \in S_0$, or for any $\eta \in C^\infty_{-n}(\omega)$ such that $\eta_{-n-1} \in S_1$.*

Proof Recall that $J = [a, b]$. Due to Assumption 5 of Theorem 2.7.3, for sufficiently small $\delta > 0$ the segments

$$J_0 = [\tilde{f}_0(a + \delta) + K\delta^\beta + \frac{L_b}{\lambda - L_f}, \tilde{f}_0(b - \delta) - K\delta^\beta - \frac{L_b}{\lambda - L_f}],$$

$$J_1 = [\tilde{f}_1(a + \delta) + K\delta^\beta + \frac{L_b}{\lambda - L_f}, \tilde{f}_1(b - \delta) - K\delta^\beta - \frac{L_b}{\lambda - L_f}],$$

cover J.

Therefore, either $g_{\omega,-n}(x) \in J_0$ or $g_{\omega,-n}(x) \in J_1$. Without loss of generality we can assume that $g_{\omega,-n}(x) \in J_0$. Let η be a sequence such that η coincides with ω on the ray $[-n, +\infty)$ and $\eta_{-n-1} \in S_0$. Let us prove that $g_{\eta,-n-1}(x) \in J$. Since $\eta \in C^\infty_{-n}(\omega)$, due to

Lemma 2.7.5 for $s = 0$ the distance between the preimages $g_{\omega,-n}(x)$ and $g_{\eta,-n}(x)$ is at most $K\delta^\beta + \frac{L_b}{\lambda - L_f}$. Therefore,

$$g_{\eta,-n}(x) \in [\tilde{f}_0(a+\delta), \tilde{f}_0(b-\delta)] \subset [g_{\sigma^{-n-1}\eta}(a), g_{\sigma^{-n-1}\eta}(b)].$$

The latter inclusion holds because $\|g^{-1}_{\sigma^{-n-1}\eta} - \tilde{f}_0^{-1}\|_{C^0} < \delta$.

Finally, $g_{\eta,-n}(x) \in [g_{\sigma^{-n-1}\eta}(a), g_{\sigma^{-n-1}\eta}(b)]$, therefore $g_{\eta,-n-1}(x) \in [a,b] = J$. ∎

Lemma 2.7.9 *Suppose that for any $x \in I$ the cardinality of the set $D_{x,n}$ grows exponentially slower than the cardinality of the set A_n of all admissible words $v_{-n} \dots v_n$,*

$$\frac{\left|D_{x,n}\right|}{\left|A_n\right|} < (1-\varepsilon)^n.$$

Then the Hausdorff dimension of the maximal attractor is less than the Hausdorff dimension of the phase space. More precisely,

$$\dim_H A_{max} < \dim_H(\Sigma_A^k \times I) - \min(1, \log_\lambda \frac{1}{1-\varepsilon}).$$

Proof Due to the definitions of the set $D_{x,n}$ and of the segment I_v, the image of the map F^n is included into the union of the Cartesian products $C^n_{-n}(v) \times I_v$ for all words $v \in D_{x,n}$. Hence, the image $F^n(X)$ can be covered by at most

$$N(n) = \sum_{v \in A_n} \lceil \lambda^n |I_v| \rceil \leqslant \lambda^n \sum_{v \in A_n} |I_v| + \left|A_n\right|$$

balls of radii λ^{-n}. Estimate the first summand.

$$\lambda^n \sum_{v \in A_n} |I_v| = \lambda^n \int_I \left|D_{x,n}\right| dx \leqslant \lambda^n (1-\varepsilon)^n \left|A_n\right|.$$

The growth rate of the sum $N(n)$ is the maximum of the rates of the summands,

$$\begin{aligned} \dim_H(A_{max}) &\leqslant \lim_{n\to\infty} \frac{\log_\lambda N(n)}{n} \\ &\leqslant \max\left(\lim_{n\to\infty} \frac{1}{n} \log_\lambda(\lambda^n (1-\varepsilon)^n \left|A_n\right|), \lim_{n\to\infty} \frac{\log_\lambda \left|A_n\right|}{n} \right) \\ &= \max\left(\dim_H(\Sigma_A^k) + 1 + \log_\lambda(1-\varepsilon), \dim_H(\Sigma_A^k)\right) \\ &= \dim_H(\Sigma_A^k \times I) - \min(1, \log_\lambda \frac{1}{1-\varepsilon}). \end{aligned}$$

∎

2.7.2 Bony attractor

As in the previous cases (see Subsections 2.4.4 and 2.5.2), consider a segment $\widetilde{I}$ such that $\tilde{f}_2(\widetilde{I}) \supset \widetilde{I}$. Then for sufficiently small values of δ and any sequence ω such that $\omega_0 \in S_2$, the image $g_\omega(\widetilde{I})$ includes the segment $\widetilde{I}$. Therefore, for any sequence ω such that $\omega_{-k} \in S_2$ for $k > n$, the set I_ω includes the non-trivial interval $f_{\sigma^{-n}\omega,n}(\widetilde{I})$. Due to Assumption 1 of Theorem 2.7.3, the set of sequences of this form is dense in Σ_A^k and has the cardinality of $\mathbb{R}$.

2.7.3 Hausdorff dimension and measure

Let N be the number from Lemma 2.7.7. Let δ be so small that we can apply this lemma.

Let $A_n^{i,j}$ be the set of admissible words $v = v_{-n} \dots v_n$ such that $v_{-n} = i$ and $v_n = j$. Denote by $D_{x,n}^{i,j}$ the intersection $D_{x,n} \cap A_n^{i,j}$. Clearly,

$$\left|A_n\right| = \sum_{i,j} \left|A_n^{i,j}\right|, \quad \left|D_{x,n}\right| = \sum_{i,j} \left|D_{x,n}^{i,j}\right|.$$

Due to Lemma 2.7.9, it is sufficient to show that the sequence $\left|D_{x,n}\right|$ grows exponentially slower than the sequence $\left|A_n\right|$. Obviously, this would follow from the fact that some other linear combination of the numbers $\left|D_{x,n}^{i,j}\right|$ with positive coefficients grows exponentially slower than the linear combination of the numbers $\left|A_n^{i,j}\right|$ with the same coefficients.

Consider two matrix sequences, $(\mathscr{A}_n)_{ij} = \left|A_n^{i,j}\right|$ and $(\mathscr{D}_{x,n})_{ij} = \left|D_{x,n}^{i,j}\right|$. Let A_{top} be the topological transition matrix,

$$(A_{top})_{ij} = \begin{cases} 0, & \text{if } A_{ij} = 0; \\ 1, & \text{else.} \end{cases}$$

It is easy to check that $\mathscr{A}_n = A_{top}^{2n+1}$.

Let us estimate the elements of the matrix $\mathscr{D}_{x,n+N}$ in terms of the matrices $\mathscr{D}_{x,n}$ and A_{top}. For any word $v \in D_{x,n+N}$ the word $v_{-n} \dots v_n$ belongs to the set $D_{x,n}$. Therefore,

$$(\mathscr{D}_{x,n})_{ij} \leqslant (A_{top}^N \mathscr{D}_{x,n} A_{top}^N)_{ij}.$$

Denote by $\mathscr{B}_{x,n}$ the difference between these two matrices,

$$\mathscr{B}_{x,n} := A_{top}^N \mathscr{D}_{x,n} A_{top}^N - \mathscr{D}_{x,n}.$$

The element $(\mathscr{B}_{x,n})_{ij}$ is the number of the admissible words $v = v_{-n-N} \dots v_{n+N}$ such that $v_{-n} \dots v_n \in D_{x,n}$, $v \notin D_{x,n+N}$, $v_{-n-N} = i$ and $v_{n+N} = j$. Due to Lemma 2.7.7, the total number of the admissible words $v = v_{-n-N} \dots v_{n+N} \notin D_{x,n+N}$ such that $v_{-n} \dots v_n \in D_{x,n}$ is at least $\left|D_{x,n}\right|$,

$$\sum_{i,j} (\mathscr{B}_{x,n})_{ij} \geqslant \sum_{i,j} (\mathscr{D}_{x,n})_{ij}. \tag{2.31}$$

Due to Perron–Frobenius Theorem, matrix A_{top} has exactly one left eigenvector a with positive coordinates and exactly one right eigenvector b with positive coordinates,

$$a A_{top} = \lambda_P a, \quad A_{top} b = \lambda_P b.$$

Let us study the asymptotic behaviour of the sequences $a \mathscr{A}_n b$ and $a \mathscr{D}_{x,n} b$. The first sequence is just a geometric progression,

$$a \mathscr{A}_n b = a A_{top}^{2n+1} b = \lambda_P^{2n+1} a b.$$

Let us estimate the second sequence,

$$\begin{aligned} a \mathscr{D}_{x,n+N} b &= a A_{top}^N \mathscr{D}_{x,n} A_{top}^N b - a \mathscr{B}_{x,n} b \\ &= \lambda_P^{2N} a \mathscr{D}_{x,n} b - a \mathscr{B}_{x,n} b. \end{aligned}$$

Due to (2.31),

$$a \mathscr{B}_{x,n} b \geqslant \varepsilon a \mathscr{D}_{x,n} b,$$

where ε depends only on a and b, hence ε depends only on the matrix A.
Therefore,

$$a \mathscr{D}_{x,n+N} b \leqslant \lambda_P^{2N} a \mathscr{D}_{x,n} b - a \mathscr{B}_{x,n} b \leqslant (\lambda_P^{2N} - \varepsilon) a \mathscr{D}_{x,n} b.$$

Thus the sequence $a \mathscr{D}_{x,n} b$ grows exponentially slower than the sequence $a \mathscr{A}_n b$. Hence, the sequence $\left|D_{x,n}\right|$ grows exponentially slower than the number of admissible words of length $2n+1$. Now Lemma 2.7.9 implies that the Hausdorff dimension of the maximal attractor is less than the Hausdorff dimension of the phase space.

Note that the estimate on the Hausdorff dimension of the maximal attractor depends only on the matrix A_{top} and the number N (which depends on L_f, L_b, λ and k). In particular, the estimate *does not depend on* α.

2.7.4 Density of the graph

Take a point $p = (\omega, x)$ of the maximal attractor, and its standard neighborhood $U = C^N_{-N}(\omega) \times (x - \widetilde{\varepsilon}, x + \widetilde{\varepsilon})$.

First, let us prove that for some $n > N$ the preimage $G^{-n}(\{\omega\} \times (x - \widetilde{\varepsilon}, x + \widetilde{\varepsilon}))$, $n > N$ intersects the strip $\Sigma^k \times J$. Indeed, otherwise due to Assumption 5 of Theorem 2.7.3 the lengths of these preimages would exponentially grow as $n \to \infty$ which is impossible. Let $m > N$ be a number such that the preimage $G^{-n}(\{\omega\} \times (x - \widetilde{\varepsilon}, x + \widetilde{\varepsilon}))$ intersects the strip $\Sigma^k \times J$. Let $y \in (x - \widetilde{\varepsilon}, x + \widetilde{\varepsilon})$ be a point such that $g_{\omega,-m}(y) \in J$.

Applying Lemma 2.7.8 infinitely many times, one can easily show that there exists an infinite sequence $\eta = \dots\eta_{-m-1}$ such that $\eta_i \in S_0 \cup S_1$ for any i and $y \in I_{\eta\omega_{-m}\dots|\omega_0\dots}$.

Recall that the maps $\tilde{f}_0$ and $\tilde{f}_1$ uniformly contract, hence $I_{\eta\omega_{-m}\dots|\omega_0\dots} = \{y\}$, therefore the point $(\eta\omega_{-n}\dots|\omega_0\dots,\ y)$ belongs to the graph Γ.

Finally, in any neighborhood of a point $p \in A_{max}$ there exists a point of the graph Γ, hence Γ is dense in A_{max}.

2.7.5 Coincidence of attractors

Let us prove that $A_{max} = A_M$.

Indeed, the likely limit set of a skew product intersects a fiber $\{\omega\} \times I$ whenever ω belongs to the likely limit set of the map in the base. The likely limit set of a Markov shift is the whole phase space, hence the likely limit set of the skew product G intersects each fiber $\{\omega\} \times I$.

On the other hand, the maximal attractor includes the likely limit set, therefore if (ω, x) is a point of the graph Γ, then $(\omega, x) \in A_M$. But the graph Γ is dense in the maximal attractor, therefore $A_M = A_{max}$.

2.8 Open set of smooth examples

In this section we will prove Theorem 2.2.3.

In Subsection 2.8.1 we will choose a linear Anosov diffeomorphism T and construct a skew product $\mathscr{F}$ over T with fiber $I = [0, 1]$ such that the corresponding skew product F over a Markov shift satisfies all the assumptions of Theorem 2.7.3. In Subsection 2.8.2 we will use Gorodetski–Ilyashenko–Negut Theorem to show that any smooth map C^2-close to $\mathscr{F}$ has a bony attractor without holes.

Finally, in Subsection 2.8.3 we will attach another strip $\mathbb{T}^2 \times I'$ and extend $\mathscr{F}$ up to a skew product on the three-torus so that the restrictions of the fiber maps to the attached strip will uniformly expand. Then the original strip $\mathbb{T}^2 \times I$ will become an absorbing domain, and the results of Subsection 2.8.2 will imply that any diffeomorphism of the three-torus sufficiently close to the extended $\mathcal{F}$ in C^2 metric has a bony attractor without holes.

2.8.1 Smooth example with fiber [0, 1]

Our example will be a skew product over the linear Anosov diffeomorphism given by the matrix

$$T_q = \begin{pmatrix} q & q+1 \\ q-1 & q \end{pmatrix}, \quad q = 1000.$$

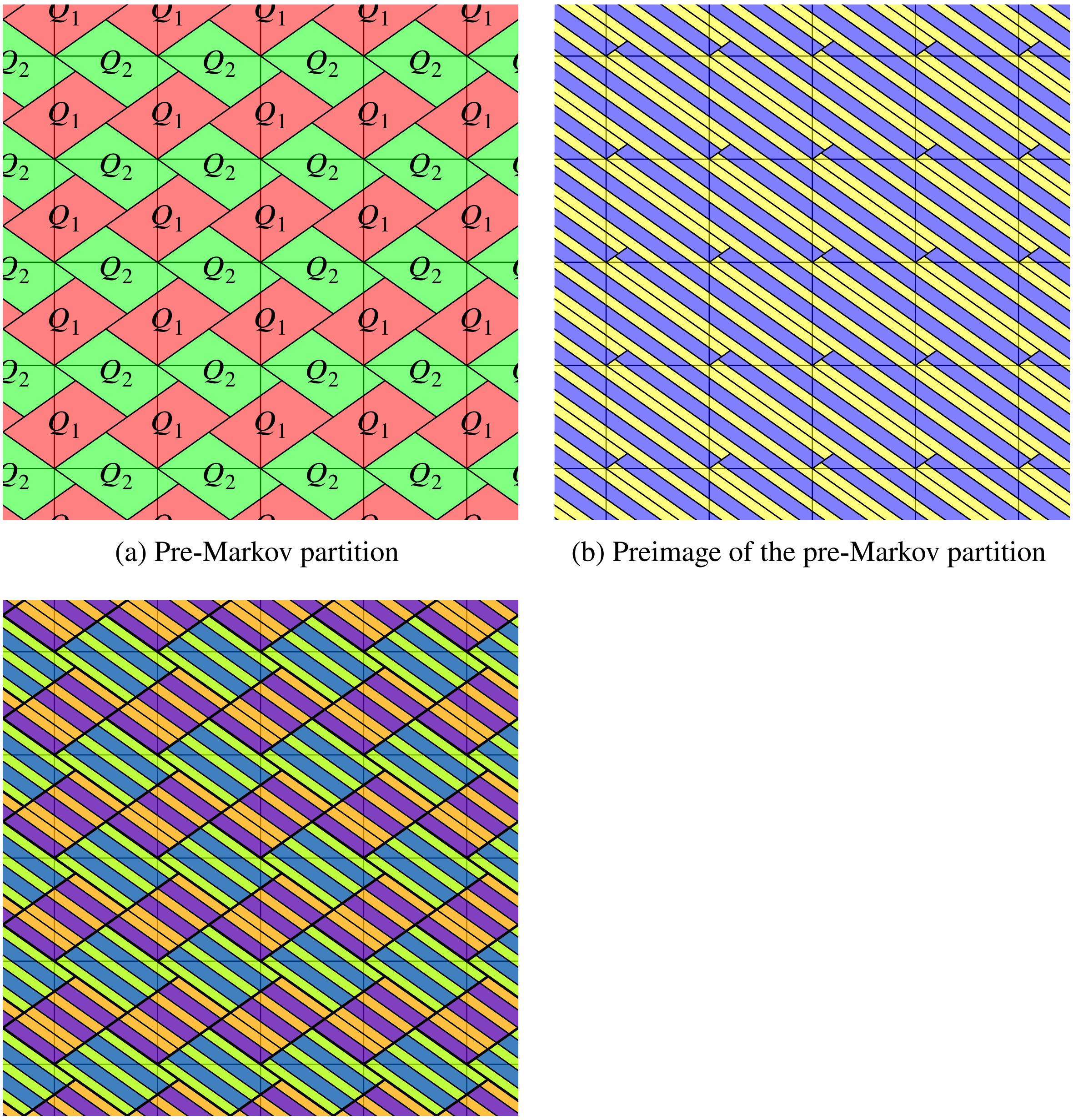
One can easily check that the eigenvalues of this matrix are $\lambda = q + \sqrt{q^2 - 1}$ and $\lambda^{-1} = q - \sqrt{q^2 - 1}$.

For any $q > 10$ we will construct a skew product over T_q with fiber I that satisfies all assumptions of Theorem 2.7.3 but Assumption 5. In order to satisfy this assumption, we will need to take $q \geqslant 1000$. Probably the picture is the same for any $q > 10$, but we cannot prove it yet.

Let us describe the Markov partition of the torus that we will use in our proof. This construction was introduced in [1]. For a more detailed description see [16]. First, let us split the torus into two parallelograms Q_1 and Q_2 with sides parallel to the eigenvectors of T_q as shown in Figure 2.7 (a). This is a *pre-markov* partition. Then take the preimage of this partition under T_q (see Figure 2.7 (b)), and draw both the original partition and its preimage under T_q on the same picture (see Figure 2.7 (c)). One can show that the intersections of the parallelograms of the initial pre-Markov partition with their preimages under T_q form a Markov partition for T_q.
Formally, the parallelograms of the Markov partition are closures of the connected components of the intersections $Q_{ij} = (\operatorname{int} Q_i) \cap T_q^{-1}(\operatorname{int} Q_j)$. We will say that a parallelogram Q of the Markov partition *has type* (i, j) if $Q \subset Q_{ij}$. One can show that the following holds.

- There are $(T_q)_{ij}$ parallelograms of type (i, j).
- Let Q be a parallelogram of type (i, j), and Q' be a parallelogram of type (i', j'). Then the intersection $T_q(\operatorname{int} Q) \cap \operatorname{int} Q'$ is not empty if and only if $j = i'$.

Therefore, the topological transition matrix for the Markov chain corresponding to this Markov partition is



(a) Pre-Markov partition

(b) Preimage of the pre-Markov partition

(c) Markov partition

Figure 2.7 Pre-Markov and Markov partition for $q = 3$

$$\begin{pmatrix} 1_{q\times q} & 1_{q\times(q+1)} & 0_{q\times(q-1)} & 0_{q\times q} \\ 0_{(q+1)\times q} & 0_{(q+1)\times(q+1)} & 1_{(q+1)\times(q-1)} & 1_{(q+1)\times q} \\ 1_{(q-1)\times q} & 1_{(q-1)\times(q+1)} & 0_{(q-1)\times(q-1)} & 0_{(q-1)\times q} \\ 0_{q\times q} & 0_{q\times(q+1)} & 1_{q\times(q-1)} & 1_{q\times q} \end{pmatrix}, \qquad (2.32)$$

where $0_{a\times b}$ (resp., $1_{a\times b}$) is the $a \times b$ matrix consisting of zeroes (resp., ones). The correspondence between the rows (columns) of this matrix and the types of parallelograms is shown in Table 2.2.

Rows (columns) range	**Type of parallelograms**
1 to q (first q)	$(1, 1)$
$q + 1$ to $2q + 1$ (next $q + 1$)	$(1, 2)$
$2q + 2$ to $3q$ (next $q - 1$)	$(2, 1)$
$3q + 1$ to $4q$ (last q)	$(2, 2)$

Table 2.2 The correspondence between rows (columns) of the topological transition matrix (2.32) and parallelograms of the Markov partition

Let us choose six parallelograms R_{ij} of the Markov partition, $i = 0, 1, 2$, $j = 1, 2$ such that

- $R_{ij} \subset Q_j$, i. e. R_{ij} is either of type $(1, j)$, or of type $(2, j)$;
- the distance between $R_{i_1 j_1}$ and $R_{i_2 j_2}$ is at least 0.1 provided that $i_1 \neq i_2$.

This is possible for, e. g., $q > 100$. Choose the fiber maps $\tilde{f}_b$ to be $\tilde{f}_i$ for $b \in R_{i1} \cup R_{i2}$, and extend this skew product up to a skew product over T_q with Lipschitz constants $L_b, L_f < 10$ such that all fiber maps are linear combinations of $\tilde{f}_i$ with positive coefficients, and the sum of the coefficients equals one.

Let us show that the constructed skew product satisfies all assumptions of Theorem 2.7.3 for sufficiently large values of q (say, for $q > 1000$). First of all, the square of the transition matrix has no zero elements, hence the transition matrix is irreducible. Next, $A_{11} \neq 0$, hence A is an aperiodic matrix. Now let us pass to enumerated conditions of Theorem 2.7.3.

- *For any symbol v_0 from the alphabet $\{0, \ldots, k - 1\}$, and $i = 0, 1, 2$ there exists a symbol u_0 such that $u_0 v_0$ is an admissible word, u_0 belongs to the step set of F, and the corresponding step fiber map f_{u_0} is $\tilde{f}_i$.*

  Let $(\tilde{i}, \tilde{j})$ be the type of the parallelogram corresponding to v_0. Then it is sufficient to take u_0 to be the symbol corresponding to the parallelogram $R_{i\tilde{i}}$.

- *Both fiber maps f_ω and the inverse maps f_ω^{-1} are bi-Lipschitz with constant L_f.*
  This property holds due to the choice of fiber maps.
- *The fiber maps f_ω and the inverse maps f_ω^{-1} are Lipschitz on ω with constant L_b.*
  This property holds due to the choice of fiber maps as well.
- *The map F is partially hyperbolic with vertical central fibration. Moreover,*

$$L_f < \lambda \text{ and } L_f < \lambda^\alpha.$$

  The first inequality holds since $L_f < 10 < \lambda$. The second inequality holds for α sufficiently close to one.
- *There exists an interval $J = [a, b] \subset I$ such that*
  - *if $f_\omega(x) \in I \setminus J$, then $f'_\omega(x) <$ const<1;*
  - *the open intervals*

$$\left(\tilde{f}_0(a) + \frac{L_b}{\lambda - L_f}, \tilde{f}_0(b) - \frac{L_b}{\lambda - L_f}\right) \text{ and } \left(\tilde{f}_1(a) + \frac{L_b}{\lambda - L_f}, \tilde{f}_1(b) - \frac{L_b}{\lambda - L_f}\right)$$

    cover J.

  Take $J = [0.2, 0.8]$. The first property holds for the fiber maps $\tilde{f}_i$, hence it holds for all fiber maps as well.

  Recall that $L_b, L_f < 10$ and $\lambda > 2q - 1$. Thus for $q > 1000$ the fraction $\frac{L_b}{\lambda - L_f}$ is less than $\frac{1}{190}$. Substituting this estimate, we obtain that the second property holds as well.

2.8.2 Perturbations

Consider a smooth map $\mathscr{G}\colon \mathbb{T}^2 \to \mathbb{T}^2$ such that $d_{C^1}(\mathscr{F}, \mathscr{G}) < \delta$, where a small positive number δ will be chosen later. Due to Ilyashenko–Negut Theorem, this map is conjugated to a Hölder continuous skew product $\widetilde{\mathscr{G}}$ over T_q with smooth fiber maps. Due to Hirsch–Pugh–Shub Theorem, the fiber maps of this skew product are C^1-close to the fiber maps of the map $\mathscr{F}$. Let $G\colon \Sigma_A^k \times I \to \Sigma_A^k \times I$ be the skew product over a Markov shift $\sigma_A : \Sigma_A^k \to \Sigma_A^k$ semi-conjugated to the diffeomorphism $\mathscr{G}$.

Note that Conditions 1 and 2 of Definition 2.2.1 and Condition 1 of Definition 2.2.2 are purely topological, hence they survive under the continuous conjugation H from Ilyashenko–Negut Theorem.

Let us show that Condition 3 of Definition 2.2.1 holds for sufficiently small $\delta > 0$. Let $d = d(L_f, L_b, T)$ be the estimate for the Hausdorff dimension of the maximal attractor obtained in Subsection 2.7.3. Let us choose a positive δ such that the conjugation H from Ilyashenko–Negut Theorem is Hölder continuous with exponent $\alpha > d/3$. Then the Hausdorff dimension of the maximal attractor of $\mathscr{G}$ is less than $d/\alpha < 3$.

The only remaining part is the coincidence of the maximal attractor and the likely limit set. We will need the following lemma.

Lemma 2.8.1 *Consider a skew product $\mathcal{F}$ over a linear Anosov diffeomorphism T of the two-torus,*

$$X = \mathbb{T}^2 \times M, \quad \mathcal{F} : X \to X, \quad (b, m) \mapsto (Tb, \tilde{f}_b(m)),$$

where M is a compact manifold. Suppose that the map $\mathcal{F}$ is partially hyperbolic with vertical central fibration $\{b\} \times M$. Then there exists $\delta > 0$ such that for any smooth map $\mathcal{G}$, $d_{C^2}(\mathcal{F}, \mathcal{G}) < \delta$ the likely limit set of $\mathcal{G}$ with respect to the Lebesgue measure m_3 on X intersects each central fiber of the map $\mathcal{G}$.

A very similar result was proved (though not formulated as an isolated statement) in [2, p. 215]. I would like to thank V. Kleptsyn who pointed me to this book. The following proof essentially repeats the last paragraph of the proof of Proposition 11.1 in this book, providing much more details.

Proof Choose δ such that for any smooth map $\mathcal{G}$, $d_{C^2}(\mathcal{F}, \mathcal{G}) < \delta$ the dominated splitting condition holds, and the strongly unstable bundle of the original map $\mathcal{F}$ belongs to the strongly unstable cone of the perturbed map $\mathcal{G}$.

Suppose that there exists a C^2-smooth perturbation $\mathcal{G}$, $d_{C^2}(\mathcal{F}, \mathcal{G}) < \delta$ and a fiber of the central fibration of $\mathcal{G}$ such that the likely limit set $A_M(\mathcal{G})$ does not intersect this fiber. Then the likely limit set of the map $\mathcal{G}$ does not intersect the saturation $\widetilde{U}$ of a small neighborhood of this fiber by the fibers of the central fibration.

Choose an open set $U \Subset \widetilde{U}$, and consider the set V of points $p \in \mathbb{T}^2 \times M$ such that $\mathcal{G}^n(p) \notin U$ for all $n \geqslant 0$,

$$V = \left\{ p \in X \,\middle|\, \forall n \geqslant 0 \, \mathcal{G}^n(p) \notin U \right\} = \bigcap_{n \geqslant 0} \mathcal{G}^{-n}(X \setminus U).$$

The set V is the intersection of a family of closed sets, hence V is a closed set as well. The union of all preimages $\mathcal{G}^{-n}(V)$ includes the set of points p such that the ω-limit set of p does not intersect $\widetilde{U}$,

$$\bigcup_{n \geqslant 0} \mathcal{G}^{-n}(V) \supset \left\{ p \in X \,\middle|\, (\text{the } \omega\text{-limit set of } p) \cap \widetilde{U} = \varnothing \right\}.$$

Indeed, if p does not belong to the union $\bigcup_{n \geqslant 0} \mathcal{G}^{-n}(V)$, then $\mathcal{G}^n(p) \in U$ for infinitely many $n \in \mathbb{N}$. Thus the orbit of the point p has a limit point in $\operatorname{Cl} U \subset \widetilde{U}$, hence the ω-limit set of the point p intersects $\widetilde{U}$.

Recall that the likely limit set does not intersect $\widetilde{U}$, hence the set of points p such that the ω-limit set of p does not intersect $\widetilde{U}$ has a full Lebesgue measure. Therefore, the union of all preimages $\mathscr{G}^{-n}(V)$ has a full Lebesgue measure as well. Thus, the set V has a positive Lebesgue measure.

Due to Fubini Theorem, there exists a point $m_0 \in M$ and an unstable leaf $\widetilde{\gamma} \colon \mathbb{R} \to \mathbb{T}^2$ of the linear Anosov diffeomorphism T such that the intersection $V \cap (\widetilde{\gamma}(\mathbb{R}) \times \{m_0\})$ has a positive one-dimensional Lebesgue measure. Denote by γ the curve $\gamma(t) = (\widetilde{\gamma}(t), m_0)$. Then the set $\{t \in \mathbb{R} \mid \gamma(t) \in V\}$ has a positive Lebesgue measure. Without loss of generality, we can assume that zero is a Lebesgue point of this set,

$$\lim_{\varepsilon \to 0} \frac{\operatorname{meas}(\gamma^{-1}(V) \cap (-\varepsilon, \varepsilon))}{2\varepsilon} = 1.$$

Fix a small positive number $\varepsilon > 0$, and choose an interval $(-\varepsilon', \varepsilon')$ so that

$$\operatorname{meas}(\gamma^{-1}(V) \cap (-\varepsilon', \varepsilon')) > 2\varepsilon'(1 - \varepsilon).$$

Let $n(\varepsilon')$ be the smallest natural number such that the image $\gamma_{\varepsilon'}$ of the curve $\gamma|_{(-\varepsilon', \varepsilon')}$ under the map $\mathscr{G}^{n(\varepsilon')}$ is longer than one. Denjoy Distortion Lemma implies that there exists $C = C(\mathscr{F}, \delta)$ such that

$$\max_{t \in (-\varepsilon', \varepsilon')} \left\| (\mathscr{G}^{n(\varepsilon')})'(\gamma(t)) \right\| < C(\mathscr{F}, \delta) \min_{t \in (-\varepsilon', \varepsilon')} \left\| (\mathscr{G}^{n(\varepsilon')})'(\gamma(t)) \right\|$$

Therefore,

$$\operatorname{meas}_1(\gamma_{\varepsilon'}(-\varepsilon', \varepsilon') \cap V) > (1 - C\varepsilon) \operatorname{meas}_1(\gamma_{\varepsilon'}(-\varepsilon', \varepsilon')). \tag{2.33}$$

Consider the family of the curves $\gamma_{\varepsilon'} = \mathscr{G}^{n(\varepsilon')}(\gamma(-\varepsilon', \varepsilon'))$ for small positive numbers ε'. Due to Arzelà–Ascoli Theorem, this family has a limit point in the space of C^1-smooth curves. Denote by γ_0 the limit curve parametrized by arc length. Inequality (2.33) implies that the intersection $\operatorname{Im} \gamma_0 \cap V$ has a full measure in $\operatorname{Im} \gamma_0$. Recall that V is a closed set, hence $\operatorname{Im} \gamma_0 \subset V$.

Due to Hirsch–Pugh–Shub and Gorodetski Theorems, there exists a projection $\pi \colon X \to \mathbb{T}^2$ such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\mathscr{G}} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{T}^2 & \xrightarrow{T} & \mathbb{T}^2 \end{array}$$

The image $\pi \circ \gamma_0$ of the curve γ_0 under the projection π is a smooth curve in $\mathbb{T}^2$ such that the images of this curve under the iterations of T do not intersect the open set πU which is impossible. This contradiction proves the lemma. ∎

Now it is easy to prove that the maximal attractor coincides with the likely limit set. Indeed, the likely limit set of the perturbed map $\mathcal{G}$ intersects each fiber of the central fibration of $\mathcal{G}$. On the other hand, the maximal attractor includes the likely limit set. Hence, the likely limit set includes the graph part Γ of the maximal attractor. The graph part Γ is dense in the maximal attractor, therefore the likely limit set coincides with the maximal attractor.

Finally, the map $\mathcal{G}$ has a bony attractor without holes.

2.8.3 From segment to a circle

So, we have constructed a skew product $\mathcal{F}$ over the linear Anosov diffeomorphism with *segment* as a fiber such that any smooth map $\mathcal{G}$ sufficiently close to $\mathcal{F}$ has a bony attractor without holes. Consider a strip $Y = \mathbb{T}^2 \times I$. Let us glue together our phase space X and this strip into three-torus and extend the map $\mathcal{F}$ to the whole torus using linear expanding maps on the attached strip Y. Then $\mathcal{F}(X) \subset X$, hence $\mathcal{G}(X) \subset X$ for $\mathcal{G}$ sufficiently close to $\mathcal{F}$, thus the maximal attractor of the initial system $\mathcal{G}|_X$ is a locally maximal attractor of the new system, hence the new system has bony attractor without holes as well.

2.9 Further research

2.9.1 Skew products with fiber [0, 1]

In Section 2.7 we constructed a non-empty open set of Hölder skew products such that any system from this set has a bony attractor without holes.

One can ask which properties from Definition 2.4.2 hold for a typical skew product over a Markov shift. Actually, there are two series of questions.

- What properties has a typical step skew product over a Markov shift?
- What properties has a typical mild skew product over a Markov shift?

For mild skew products, it is natural to consider only Hölder continuous skew products with smooth fiber maps that satisfy the inequality $L_f < \lambda^\alpha$ (see Theorem 2.7.3).

V. Kleptsyn and D. Volk [22] described some properties of a typical (step, Hölder continuous) skew product.

Theorem 2.9.1 *(V. Kleptsyn, D. Volk, [22]) Let $F\colon \Sigma_A^k \times I \to \Sigma_A^k \times I$ be a Hölder skew product over a Markov shift σ_A. Suppose that F satisfies the following conditions (cf. theorem 2.7.3).*

- *The fiber maps $x \mapsto f_\omega(x)$ and their inverse maps are C^1 smooth on x with Lipschitz constant L.*
- *The maps $\Sigma_A^k \to C^1(I)$, $\omega \mapsto f_\omega$ and $\omega \mapsto f_\omega^{-1}$ are Hölder continuous on ω with Hölder exponent α.*
- *$\lambda^\alpha > L$.*

Then there exists a finite collection of strips $U_i = \left\{(\omega, x) \,\middle|\, g_i^-(\omega) \leqslant x \leqslant g_i^+(\omega)\right\}$, $g_i^\pm : \Sigma_A^k \to I$, such that the following holds.

- *Each strip U_i is absorbing either for F or for F^{-1}.*
- *The realms of attraction[10] of the strips U_i (with respect to F or F^{-1}) cover Σ_A^k except for a closed subset whose projection to Σ_A^k has measure zero.*
- *The maximal attractor[10] of each strip U_i is the union of the graph of a continuous function $\varphi_i\colon \Omega_i \to I$, $\Omega_i \subset \Sigma_A^k$, $\mu_A(\Omega_i) = 1$, and a subset of $(\Sigma_A^k \setminus \Omega_i) \times I$.*
- *Let m be any invariant ergodic measure on $\Sigma_A^k \times I$ such that $m(U \times I) = \mu_A(U)$ for any μ_A-measurable set $U \subset \Sigma_A^k$. Then there exists a strip U_i from the collection $\left\{U_i\right\}$ whose maximal attractor includes $\operatorname{supp} m$, and m is the SRB-measure for U_i.*
- *The Lyapunov exponent in the vertical direction of each invariant ergodic measure is not equal to zero.*

So, this theorem basically states that the phase space of a typical skew product over a Markov shift with fiber I can be splitted into a finite collection of domains U_i such that for any domain U_i either the restriction $F|_{U_i}$, or the restriction $F^{-1}|_{U_i}$ has an attractor 'not too worse' than a bony attractor.

For example, this theorem implies that the likely limit set of a typical skew product over a Markov shift with fiber I has measure zero. Though this theorem provides us a lot of information on the limit behaviour of a typical skew product over a Markov shift, there are still some open questions.

Question 2.9.2 Is the likely limit set of a typical (step, Hölder continuous) skew product over a Markov shift Lyapunov stable?

Question 2.9.3 Does the likely limit set of a typical (step, Hölder continuous) skew product over a Markov shift intersect each vertical fiber on a finite union of segments and points?

---

[10] With respect to F if U_i is an absorbing domain for F, and with respect to F^{-1} otherwise.

2.9.2 Multi-dimensional bones

Another possible direction of further research is to provide a reasonable definition of a bony attractor with multi-dimensional bones, and study the properties of such attractors. In particular, it would be interesting to test whether systems with bony attractors are counter-examples to some well-known open problems. I will just list some possible questions.

- Does there exist an open set of diffeomorphisms having a bony attractor with multi-dimensional bones?
- Does there exist an open set of diffeomorphisms having a bony attractor *with holes* (e. g., the likely limit set is asymptotically unstable)?
- Does there exist an open set of diffeomorphisms having a thick attractor (i. e., $0 < \mu(A_M) < \mu(X)$, where X is the phase space)?

3 Billiards

This chapter is a joint work with A. Glutsyuk.

3.1 Introduction

3.1.1 Main results

Given a domain $\Omega \subset \mathbb{R}^m$ with (piecewise) smooth boundary, consider the *billiard dynamical system* which describes the trajectories of a particle (a billiard ball) moving inside this domain. The ball moves along straight lines inside Ω and reflects against the boundary of Ω by the standard reflection law.

Formally, the phase space of this system is the set of pairs (x, v), where $x \in \partial\Omega$ is a point of reflection and v, $\|v\| = 1$ is velocity of the ball at this point (a unit vector directed towards the interior of the domain Ω). The billiard map sends a pair (x, v) to the pair (x', v'), where x' is the first point along the ray $\{x + tv \mid t \in (0, +\infty)\}$ that belongs to the border $\partial\Omega$ and v' is the speed of the ball after reflection.

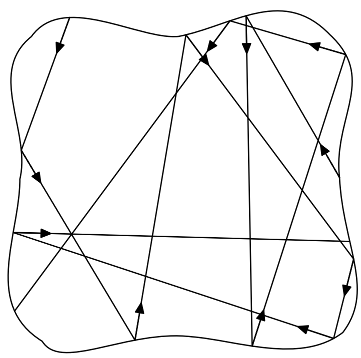


Figure 3.1 A couple of billiard trajectories

This chapter is devoted to a particular case of the following long-standing problem.

Conjecture 3.1.1 *(V. Ivrii, 1978) Given a domain in the Euclidean space with sufficiently smooth boundary, the set of periodic orbits of the corresponding billiard has measure zero.*

More precisely, we study the set of pairs (x, v) such that the orbit of (x, v) under the billiard map is finite.

Clearly, it is sufficient to prove that for any k the set Per_k of k-gonal orbits has measure zero. For $k = 2$, this statement is trivial. For triangular trajectories in a planar billiard (i. e., a billiard in $\Omega \subset \mathbb{R}^2$), this statement was proved by M. Rychlik [21]. Later Rychlik's result was generalized by Ya. Vorobets [23] for higher dimensional billiards.

We will show that the set of quadrilateral periodic orbits of a planar billiard has measure zero.

Theorem 3.1.2 *There exists a natural number r such that for any planar billiard with piecewise C^r smooth boundary, the set* Per_4 *has measure* 0.

In what follows, μ denotes the Lebesgue measure on the billiard phase space (i. e., the set of pairs (x, v) described above).

Obviously, Theorem 3.1.2 is implied by the two following theorems.

Theorem 3.1.3 *Suppose that for some k and m and for any r there exists a billiard in* $\mathbb{R}^m$ *with piecewise* C^r *smooth boundary such that* $\mu\,\mathrm{Per}_k > 0$. *Then there exists a billiard in* $\mathbb{R}^m$ *with piecewise-analytic boundary such that the set* Per_k *has an inner point in the space of all orbits.*

Theorem 3.1.4 *For any planar billiard with piecewise-analytic boundary, the set* Per_4 *has no inner points.*

3.1.2 From Weyl to Ivrii

Though Conjecture 3.1.1 is a pure billiard theory question, it appeared as a geometrical condition in the following PDE problem.

Consider the Dirichlet problem for the Laplace operator Δ in some domain $\Omega \subset \mathbb{R}^m$. The Laplace operator Δ is a negatively-determined self-adjoint operator, therefore its eigenvalues with the Dirichlet boundary condition $u|_{\partial\Omega} = 0$ are negative real numbers $0 \geqslant -\lambda_1^2 \geqslant -\lambda_2^2 \geqslant \ldots \geqslant -\lambda_k^2 \geqslant \ldots$ Denote by $N(\lambda)$ the number of the eigenvalues $-\lambda_i^2$ such that $\lambda_i^2 < \lambda^2$, that is,

$$N(\lambda) = k \Leftrightarrow \lambda_k < \lambda \leqslant \lambda_{k+1}. \tag{3.1}$$

Question 3.1.5 What is the asymptotic behaviour of the function $N(\lambda)$?

H. Weyl [24] proved that $N(\lambda)$ is asymptotically proportional to λ^m, where m is the dimension of Ω.

Theorem 3.1.6 *(H. Weyl, 1911) Let* $\Omega \subset \mathbb{R}^m$ *be a domain such that* $\mathrm{mes}(\Omega) < \infty$ *and* $\mathrm{mes}(\partial\Omega) = 0$. *Then*

$$N(\lambda) = c_0\,\mathrm{mes}(\Omega)\lambda^m + o(\lambda^m),$$

where $c_0 = (2\pi)^{-m} w_m$ *and* w_m *stands for the volume of m-dimensional unit ball.*

After proving this theorem, Weyl obtained more precise asymptotic for $N(\lambda)$ for $\Omega = [a_1, b_1] \times \ldots \times [a_m, b_m]$. It turns out that in this case

$$N(\lambda) = c_0 \operatorname{mes}(\Omega)\lambda^m - c_1 \operatorname{mes}'(\partial\Omega)\lambda^{m-1} + o(\lambda^{m-1}), \qquad (3.2)$$

where $c_1 = \frac{1}{4}(2\pi)^{m-1}\omega_{m-1}$, and mes′ is the $(m-1)$-dimensional measure. Weyl conjectured that the same formula holds for any domain $\Omega \subset \mathbb{R}^m$ with sufficiently (piecewise) smooth boundary.

Many mathematicians, including R. Courant, B. Levitan, V.Avakumovič, L. Hörmander, J. J. Duistermaat, V. Guillemin, R. Seeley and V. Ivrii contributed to the proof of this conjecture. The best result was achieved by V. Ivrii (see [13]), who proved Weyl conjecture for domains satisfying an additional geometric condition.

Theorem 3.1.7 *(V. Ivrii, 1980) Let Ω be a domain in $\mathbb{R}^m$ with infinitely smooth boundary. Suppose that in the corresponding billiard the set of the periodic orbits has measure zero. Then for Ω, the asymptotic formula (3.2) holds.*

This geometric condition is analogous to the condition that appears in the same problem for Riemannian manifolds without border. In the latter case we should require the set of closed geodesics to have zero measure.

In 1980 V. Ya. Ivrii gave a talk in Ya. G. Sinai seminar (Moscow State University) — one of the best seminars on billiards, and he conjectured (see Conjecture 3.1.1) that this geometric condition holds for any domain in the Euclidean space with sufficiently smooth boundary. He was told that this conjecture will be proven in a couple of days… in a week… in a month… in a year…

The conjecture still stands!

As we have noted above, the case of triangular orbits was studied by M. Rychlik [21] and Ya. Vorobets [23]. We study the case of quadrilateral trajectories in planar billiards.

3.2 Reduction to the analytic case

In this section we will prove Theorem 3.1.3.

Suppose that for any r there exists a billiard with piecewise C^r smooth boundary such that the measure of Per_k is positive.

We will denote by $A_1, A_2, \ldots$ the vertices of a trajectory of the billiard map. If a trajectory is k-periodic, then it is natural to count these vertices modulo k, i. e. $A_0 = A_k$, $A_1 = A_{k+1}$ etc. We will be interested only in k-gons that are non-degenerate in the following sense.

Definition 3.2.1 A k-tuple of points $A_1, \ldots, A_k \in \mathbb{R}^m$ is called a *non-degenerate k-gon* if

- consequent vertices do not coincide, i. e. $A_i \neq A_{i+1}$ for $i = 1, \ldots, k$;

- none of the angles is equal to π, i. e. $\angle A_{i-1}A_iA_{i+1} \neq \pi$ for $i = 1, \dots, k$.

Otherwise this k-tuple is called a *degenerate k-gon*.

Let us explain why it is natural to require a periodic billiard orbit to be a non-degenerate k-gon. If $A_i = A_{i+1}$, then the reflection law at the vertices A_i and A_{i+1} makes no sense. If $\angle A_{i-1}A_iA_{i+1} = \pi$, then the billiard map is not smooth at $(A_{i-1}, \frac{\overrightarrow{A_{i-1}A_i}}{A_{i-1}A_i})$; moreover, in this case there exists a ray arbitrarily close to $A_{i-1}A_i$ that does not intersect the border $\partial\Omega$ near A_i.

Remark 3.2.2 *If $A_{i-1} \neq A_i$ and $\angle A_{i-1}A_iA_{i+1} \neq \pi$ and the border $\partial\Omega$ is C^r-smooth at the points A_{i-1} and A_i then the billiard map is C^{r-1}-smooth at the point $(A_{i-1}, \frac{\overrightarrow{A_{i-1}A_i}}{A_{i-1}A_i})$.*

The space of all non-degenerate k-gons is an open set in $\mathbb{R}^{mk}$.

Consider a billiard table $\Omega \subset \mathbb{R}^m$, and take a periodic non-degenerate orbit $A_1 \dots A_k$. The tangent space to the set of k-gons with vertices in $\partial\Omega$ at the point $A_1 \dots A_k$ is the Cartesian product of tangent hyperplanes $T_{A_i}\partial\Omega$. Due to the reflection law, the hyperplane $T_{A_i}\partial\Omega$ is the exterior bisector of the angle $A_{i-1}A_iA_{i+1}$ for $i = 1, \dots, k$, hence the space $\bigoplus_{i=1}^k T_{A_i}\partial\Omega$ is the same for all domains $\Omega \subset \mathbb{R}^m$ such that $A_1 \dots A_k$ is a periodic trajectory for the corresponding billiard.

Thus we obtain a $k(m-1)$-dimensional distribution in the space of all non-degenerate k-gons in $\mathbb{R}^m$. Denote by $\mathscr{F}$ this distribution. There is a strong connection between billiard tables with 'large' set of k-gonal orbits and non-trivial $2(m-1)$-dimensional integral surfaces of this distribution.

We will need the following definition.

Definition 3.2.3 We will say that an r-jet of a $2(m-1)$-dimensional surface in $\mathbb{R}^{mk}$ (i. e., an r-jet of a map $\varphi\colon (\mathbb{R}^{2(m-1)}, 0) \to \mathbb{R}^{mk}$) is an *integral* r-jet for the distribution $\mathscr{F}$ if the $(r-1)$-jet of the map $d\varphi$ satisfies the $(r-1)$-jet of the equations that define $\mathscr{F}$.
We will say that an integral r-jet (integral surface, germ of an integral surface) φ is *non-trivial* if for any $i = 1, \dots, k$ the composition of φ with the projection $\pi_i\colon (A_1, \dots, A_k) \mapsto A_i$ has rank $m-1$.

Clearly, one can define an integral r-jet of any dimension in the same way but we will need only $2(m-1)$-dimensional integral surfaces and integral jets, so we fixed the dimension in the definition above.

Lemma 3.2.4 *If there exists a domain $\Omega \subset \mathbb{R}^m$ with C^r-smooth boundary such that $\mu\,\mathrm{Per}_k > 0$, then there exists a non-trivial integral $(r-1)$-jet of the distribution $\mathscr{F}$. If there exists a C^r-smooth (resp., analytic) non-trivial integral surface of $\mathscr{F}$, then there exists a* *domain* $\Omega \subset \mathbb{R}^m$ *with piecewise* C^r*-smooth (resp., piecewise analytic) boundary such that the set* Per_k *has an inner point.*

The non-triviality condition is needed because $\mathscr{F}$ has some trivial $2(m-1)$-dimensional integral surfaces. For example, for $m = 2$ and $k = 4$ for any $X, Y \in \mathbb{R}^2$ and $s > XY$ the family

$$\left\{(X, A_2, Y, A_4) \,\middle|\, XA_2 + A_2Y = XA_4 + A_4Y = s\right\}$$

is a two-dimensional integral surface of $\mathscr{F}$ that does not correspond to any billiard table.

Proof Let us prove the first part of the lemma. Let us fix a domain Ω with piecewise C^r-smooth boundary such that $\mu\,\mathrm{Per}_k > 0$. Recall that the phase space of the billiard map has dimension $2(m-1)$. Therefore, the set $U \subset \mathbb{R}^{mk}$ of non-degenerate billiard trajectories $A_1 \dots A_k$ (including non-periodic trajectories) is a C^{r-1}-smooth $2(m-1)$-dimensional manifold as well. Consider the map that sends each pair (A_1, v) to the corresponding billiard trajectory $A_1 \dots A_k$ of length k. Clearly, the restriction of this map to the pre-image of U is a smooth map from an open dense set in the phase space of the billiard map to U. Let $\mathrm{Per}'_k \subset U$ be the image of the set Per_k under this map. Then the $2(m-1)$-dimensional Lebesgue measure of Per'_k is positive. For any element $A_1 \dots A_k \in \mathrm{Per}'_k$ the tangent space for U at the point $A_1 \dots A_k$ is a plane of the distribution $\mathscr{F}$. Therefore, for any Lebesgue point $A_1 \dots A_k$ of the set Per'_k the $(r-1)$-jet of U at $A_1 \dots A_k$ is an integral jet of $\mathscr{F}$.

It remains to show that the rank of the composition of this $(r-1)$-jet with each projection $\pi_i : (A_1, \dots, A_k) \mapsto A_i$ is equal to $m-1$. Since the space of possible directions has dimension $m-1$, it is sufficient to prove that the rank of the composition of this $(r-1)$-jet with each projection $\pi'_i : (A_1, \dots, A_k) \mapsto (A_i, \frac{\overrightarrow{A_iA_{i+1}}}{A_iA_{i+1}})$ is $2(m-1)$.

Fix an $i \in \{1, \dots, k\}$ and let us prove that the composition of this $(r-1)$-jet with π'_i has rank $2m-2$. For $i = 1$ this is clear because U is parametrized by A_1 and $\frac{\overrightarrow{A_1A_2}}{A_1A_2}$. Let $i > 1$. Denote by $\mathbf{B}$ the billiard map. Recall that $A_1 \dots A_k$ is a non-degenerate k-gon, hence both the billiard map $\mathbf{B}$ and the inverse billiard map $\mathbf{B}^{-1}$ are smooth at each pair $(A_i, \frac{\overrightarrow{A_iA_{i+1}}}{A_iA_{i+1}})$. Thus the rank of the map $\pi'_i = \mathbf{B}^{i-1} \circ \pi'_1$ is equal to $2m-2$. Therefore, the rank of the map π_i equals $m-1$. So, we have proven the first part of the lemma.

Now we will prove the second part of the lemma. In this proof (and only in this proof) we will use the notation C^ω for real analytic functions and allow r to be either a natural number or ω. This will allow us to consider the case of an analytic integral surface together with the case of C^r integral surface.

Let U be a non-trivial $2(m-1)$-dimensional C^r smooth integral surface of the distribution $\mathscr{F}$. Due to the non-triviality condition, the images of U under the projections π_i are

C^r submanifolds of dimension $m-1$. Obviously, every $A_1 \dots A_k$ is an inner point of the set Per_k for any billiard Ω with piecewise C^r-smooth boundary $\partial\Omega$ such that for every i the germ of $\partial\Omega$ at A_i is the image of the germ of U under projection π_i. This completes the proof. ∎

Let us apply Lemma 3.2.4 to complete the proof of Theorem 3.1.3.

Due to the first part of the lemma, for any r there exists a non-trivial integral r-jet of $\mathcal{F}$. The main theorem of Chapter XI [20, p. 342] implies that for r large enough and any integral $2(m-1)$-dimensional r-jet there exists a germ of $2(m-1)$-dimensional analytic integral surface of $\mathcal{F}$ having the same r-jet. Therefore, $\mathcal{F}$ has a non-trivial analytic integral surface U of dimension $2(m-1)$. The second part of Lemma 3.2.4 completes the proof.

3.3 Analytic case

3.3.1 Strategy of the proof

Recall that our aim is to prove that there does not exist a planar billiard Ω with piecewise analytic boundary such that the set Per_4 has an inner point.

Clearly, the property of being an inner point of the set Per_k is local, i. e. this property depends only on the germs of the boundary $\partial\Omega$ at the vertices of the trajectory. This motivates the following definition.

Definition 3.3.1 A *k-reflective billiard germ* is a k-tuple of germs of analytic curves $\gamma_i\colon (\mathbb{R},0) \to (\mathbb{R}^2, A_i)$ such that

- $A_i \neq A_{i+1}$ for $i = 1, \dots, k-1$, $A_k \neq A_1$ (otherwise the reflection law makes no sense);
- the reflection law with respect to the curves γ_i holds at the vertices A_i, $i = 1, \dots, k$;
- $A_1 A_2 \dots A_k$ is an inner point of the set Per_k of k-gonal billiard orbits.

Clearly, the following statement implies Theorem 3.1.4.

Theorem 3.3.2 *There does not exist an analytic* 4*-reflective billiard germ.*

We will prove this theorem instead of Theorem 3.1.4. In this subsection we will only give an idea of the proof, and the rest of this section is devoted to the detailed proof.

Assume the converse. Then there exists a 4-reflective analytic billiard germ (a, b, c, d). Let $ABCD$ be the corresponding periodic trajectory.

We can extend the mirrors and the families of periodic trajectories analytically. Our strategy will consist in extending the mirrors and the family of periodic trajectories sufficiently far to obtain a contradiction.

Namely, Lemma 3.3.11 lists the possible obstructions to analytic extension of a family of 4-periodic trajectories with fixed base vertex $A_i \in \gamma_i$. Then Proposition 3.3.15, Lemma 3.3.20, Proposition 3.3.21 and Proposition 3.3.30 show that each of these cases holds for at most countable set of base vertices in γ_i. On the other hand, the curve γ_i is uncountable. This contradiction will complete the proof.

3.3.2 First observations for k-gonal trajectories

As we noted above, we will study analytic extensions of the initial germs. Clearly, these extensions can intersect existing billiard trajectories, so we need to modify the definition of a billiard trajectory.

Definition 3.3.3 Let $\gamma_1, \gamma_2, \dots, \gamma_k \colon \mathbb{R} \to \mathbb{R}^2$ be analytic curves. A k-tuple of points $A_1 A_2 \dots A_k$ is called *a billiard trajectory* for the k-tuple of mirrors $\gamma_1, \dots, \gamma_k$ if $A_i \in \gamma_i$ and the reflection law holds.

We will need to apply this definition for the case when some of the vertices A_i are singular points of the respective mirrors γ_i. Thus we introduce the following convention.

Convention 3.3.4 Let $\gamma(t_0)$ be a singular point of a smooth curve γ. We will say that l is the *tangent line* to the curve γ at the point $\gamma(t_0)$ if l is the limit

$$l = \lim_{t \to t_0} T_{\gamma(t)} \gamma.$$

In particular, we say that there exists the tangent line at a cusp singular point.

Remark 3.3.5 *In a family of k-periodic billiard trajectories, the vertices of the polygon $A_1 \dots A_k$ move in the directions of the exterior bisectors of the angles of this polygon, therefore its perimeter is a constant. We may and will assume that this constant is equal to one, $A_1A_2 + \dots + A_{k-1}A_k + A_kA_1 = 1$.*

One of the possible obstructions to the analytic extension of a family of periodic trajectories is *degeneracy* of the limit trajectory. Recall the definition of a non-degenerate k-gon (we just replace m by 2 in definition 3.2.1).

Definition 3.3.6 A k-tuple of points $A_1, \dots, A_k \in \mathbb{R}^2$ is called a *non-degenerate k-gon* if

- consequent vertices do not coincide, i. e. $A_i \neq A_{i+1}$ for $i = 1, \dots, k$;
- none of the angles is equal to π, i. e. $\angle A_{i-1}A_iA_{i+1} \neq \pi$ for $i = 1, \dots, k$.

Otherwise this k-tuple is called a *degenerate k-gon*.

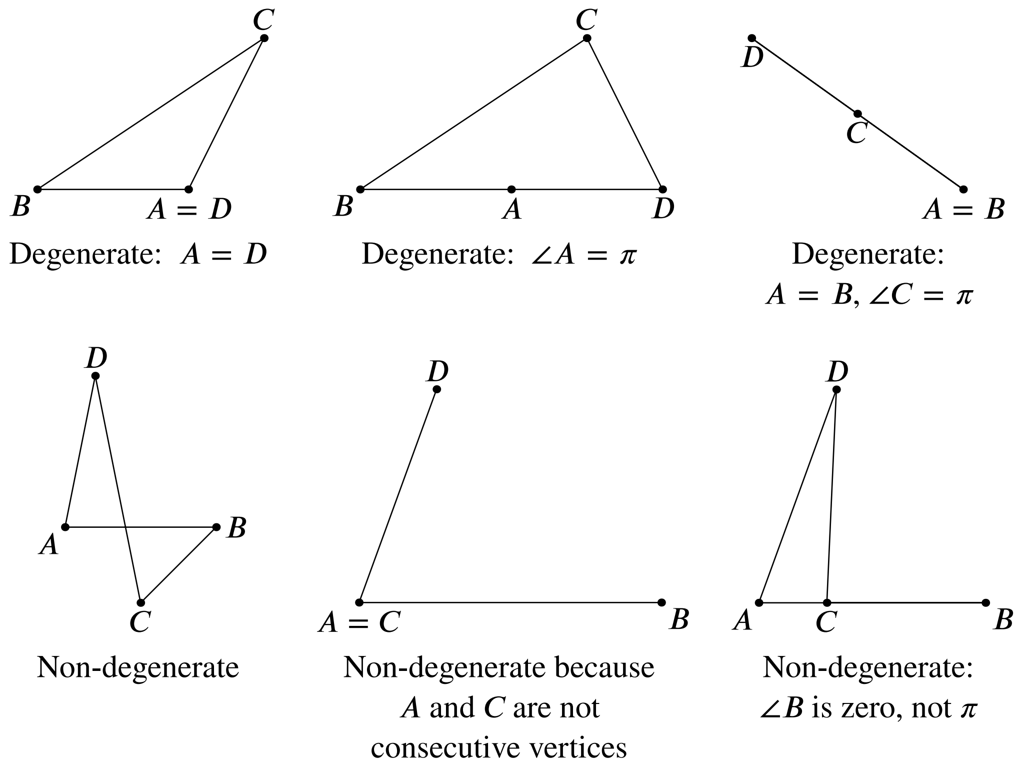


Figure 3.2 Degenerate and non-degenerate quadrilaterals $ABCD$

A k-gon such that $A_i = A_{i+1}$ for some i is an obstruction to the extension because the reflection law at A_i makes no sense for such polygons. A k-gon such that $\angle A_{i-1}A_iA_{i+1} = \pi$ is an obstruction to the extension because if, say, the line $A_{i-1}A_{i+1}$ and the mirror γ_i have 2-point contact at A_i, then there exists a ray arbitrarily close to $A_{i-1}A_i$ that does not intersect γ_i near A_i.

Some degenerate and non-degenerate quadrilaterals are shown in Figure 3.2.

3.3.3 Start of the proof of Theorem 3.3.2

Assume the converse. Then there exists an analytic 4-reflective billiard germ (a, b, c, d). Let us replace these germs by their maximal analytic extensions.

More precisely, given a germ $\gamma\colon (\mathbb{R}, 0) \to \mathbb{R}^2$ we will replace it by a curve $\tilde{\gamma}$ that contains the maximal analytic extension (as a map $\mathbb{R} \to \mathbb{R}^2$) of any analytic reparametrization of γ. Let us prove that such curve $\tilde{\gamma}$ exists. This fact should be known for ages but I have not found any reference.

Note that all singular points of an analytic curve (except for the endpoints if they exist) are cusps. Thus it is sufficient to extend γ beyond all cusps. Consider the unit speed parametrization $\bar{\gamma}$ of γ,

$$\bar{\gamma}(s) = \gamma(t(s)), \quad s(t) = \int\limits_0^t \|\dot{\gamma}(\tau)\|\, d\tau,$$

and replace $\bar{\gamma}$ by its maximal analytic extension (i. e., the extension till the next singular point).

Then we extend $\bar{\gamma}$ through all cusps. The resulting curve $\hat{\gamma}$ will have singularities at these cusps, and $\|\dot{\hat{\gamma}}\| = 1$ at any regular point. Therefore $\hat{\gamma}$ contains any analytic extension of any reparametrization of the initial germ γ, and we only need to find an analytic reparametrization $\tilde{\gamma}$ of the curve $\hat{\gamma}$. Note that for any s_i corresponding to a cusp there exists N_i such that the curve $\tilde{\gamma}_i\colon t \mapsto \hat{\gamma}(s_i + t^{N_i})$ is analytic at the origin. Thus we can change the analytic structure near each point s_i so that $\hat{\gamma}$ will become an analytic map from this new abstract analytic one-dimensional manifold to the plane. Indeed, it is sufficient to use $\sqrt[N_i]{s - s_i}$ as a new chart near s_i. Any contractible abstract analytic one-dimensional manifold is analytically equivalent to the real line, hence there exists an analytic curve $\tilde{\gamma}\colon \mathbb{R} \to \mathbb{R}^2$, $\tilde{\gamma}(\tau) = \hat{\gamma}(s(\tau))$, $s \in C(\mathbb{R})$. Clearly, $\tilde{\gamma}$ is a maximal analytic extension of γ.

So, we have proved the existence of a maximal analytic extension.

We will approach the border of the set Per_4 along the *angle families* $A = \mathrm{const}$. Formally, fix some initial 4-reflective trajectory $A^0B^0C^0D^0$. Let us fix $A = A^0$ and start increasing the angle $\alpha = \angle BAD$. Due to the 4-reflectivity of the initial billiard germ, we will obtain a small 1-parametric family $AB_\alpha C_\alpha D_\alpha$ of quadrilateral billiard trajectories. Consider the analytic extension of this family to the maximal possible interval $(\alpha_-, \alpha_+) \subset (0, \pi)$, i. e. we do not try to extend the family beyond $\alpha = 0$ and $\alpha = \pi$.

Clearly, the curves $\alpha \mapsto B_\alpha$, $\alpha \mapsto C_\alpha$ and $\alpha \mapsto D_\alpha$ are analytic reparametrizations of parts of the curves b, c and d, respectively.

Remark 3.3.7 *The vertices B_α, C_α and D_α can be singular points of the respective curves for some values of $\alpha \in (\alpha_-, \alpha_+)$.*

Notation 3.3.8 Denote by β_α, γ_α and δ_α the angles $\angle AB_\alpha C_\alpha$, $\angle B_\alpha C_\alpha D_\alpha$ and $\angle C_\alpha D_\alpha A$, respectively. Denote by B_+, C_+, D_+, β_+, γ_+ and δ_+ the limits (if they exist) of B_α, C_α, D_α, β_α, γ_α and δ_α as $\alpha \to \alpha_+$, respectively.

The 4-reflectivity is an analytic condition, hence all trajectories $AB_\alpha C_\alpha D_\alpha$ are 4-reflective. Formally, consider the fourth power of the billiard map, that is, the map of four successive reflections against the border. Since the initial trajectory is 4-reflective, this map is the identity map in some neighbourhood of the initial pair $(A, \frac{\overrightarrow{AB}}{AB})$. On the other hand, this map is analytic. Thus its analytic extension along the family of trajectories $AB_\alpha C_\alpha D_\alpha$ is the identity map, hence all trajectories $AB_\alpha C_\alpha D_\alpha$ are 4-reflective.

It is not convenient to consider the cases $B_+ \in b$ and $B_+ \notin b$, so we attach possible values of B_+ to the curve b itself.

Convention 3.3.9 If an analytic curve has a limit either in the forward direction, or in the reverse direction, we will attach these limits to the curve and consider them to be singular points of the resulting curve.

The following notion will be used in some proofs to consider the similar cases together.

Definition 3.3.10 Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be analytic curves. We say that a point X is a *marked point* if it is either a singular point of one of these curves γ_i (including self-intersection points and the limits attached to γ_i due to the previous convention) or an intersection point of two different curves.

We would like to underline that "two different curves" in this definition means that even if for some $i \neq j$ the curves γ_i and γ_j coincide, we do not mark all the points of γ_i. Thus, the set of marked points is at most countable.

The following lemma provides us the list of possible obstructions to the analytic extension of an angle family.

Lemma 3.3.11 *For any initial quadrilateral one of the following cases holds.*

1. *At least one of the limits* $B_+ = \lim\limits_{\alpha \to \alpha_+} B_\alpha$, $C_+ = \lim\limits_{\alpha \to \alpha_+} C_\alpha$ *and* $D_+ = \lim\limits_{\alpha \to \alpha_+} D_\alpha$ *does not exist.*
2. $AB_+C_+D_+$ *is a degenerate quadrilateral (see Definition 3.3.6).*
3. *At least two of the points* B_+, C_+ *and* D_+ *are singular points of the corresponding mirrors.*

Proof Assume the converse, then for some initial quadrilateral

- the limits B_+, C_+ and D_+ exist;
- the quadrilateral $AB_+C_+D_+$ is non-degenerate;
- at most one of the points B_+, C_+, D_+ is a singular point of the corresponding curve.

Without loss of generality we can assume that B_+ is a regular point of b, and either C_+ or D_+ is a regular point of c or d, respectively. Then we can easily extend the family B_α to some bigger interval. Note that the rays $B_\alpha C_\alpha$ and AD_α depend only on A, α and B_α. Indeed, the line $B_\alpha C_\alpha$ is the image of the line AB_α under the symmetry with respect to the tangent line to b at B_α, and the ray AD_α is the ray starting from A in the known direction.

Consider two cases.

Case I. C_+ is a regular point of c, then C_α can be extended to a bigger interval as the intersection point of the ray $B_\alpha C_\alpha$ and the curve c. Hence, we can define the ray $C_\alpha D_\alpha$ for

α close enough to α_+ (including the values of α greater than α_+). Therefore we can define D_α as the intersection point of the rays $C_\alpha D_\alpha$ and AD_α. Due to the inequality $\delta_+ \neq \pi$, for α sufficiently close to α_+ this intersection point exists, is unique and analytically depends on α. Finally, we can extend the family $AB_\alpha C_\alpha D_\alpha$ to a bigger interval, which contradicts the assumption that (α_-, α_+) is the maximal interval. Therefore, this case is impossible.

Case II. D_+ is a regular point of d, then D_α can be extended to a bigger interval as the intersection point of the ray AD_α and the curve d. Hence, we can define the ray $D_\alpha C_\alpha$ for α close enough to α_+ (including the values of α greater than α_+). Let us define C_α as the intersection point of the rays $D_\alpha C_\alpha$ and $B_\alpha C_\alpha$. Due to the inequality $\gamma_+ \neq \pi$, this intersection point exists, is unique and analytically depends on α. Finally, we can extend the family $AB_\alpha C_\alpha D_\alpha$ to a bigger interval, which contradicts the assumption that (α_-, α_+) is the maximal interval. Therefore, this case is also impossible.

Finally, both cases are impossible. This completes the proof of the lemma. ∎

It is convenient to choose which vertex to fix. In order to avoid renaming of the mirrors in the middle of the proof, we will now rename the mirrors so that the following convention holds.

Convention 3.3.12 (Naming convention) We say that a 4-reflective billiard germ (a, b, c, d) with marked mirror a satisfies *the naming convention* if

1. neither a nor c is a line;
2. if one of the mirrors is an ellipse then either b or d is either an ellipse or a line.

Note that it is possible to rename the mirrors so that the naming convention will hold unless at least two of the mirrors are straight lines. Indeed, if one of the mirrors is a line, let us rename the mirrors so that b is a line, and the naming convention will be satisfied; otherwise, none of the mirrors is a straight line, thus the first condition holds automatically, and it is easy to satisfy the second condition.

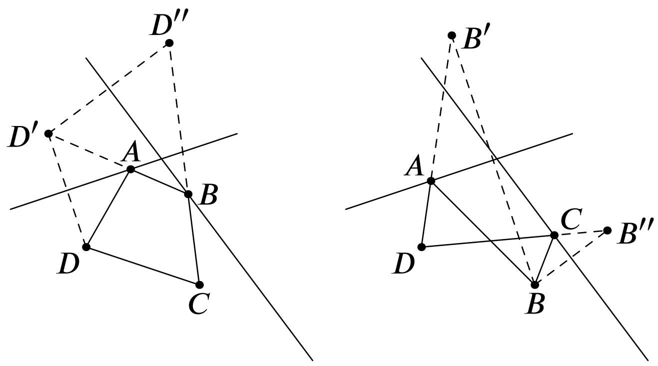
Lemma 3.3.13 *At most one of the mirrors a, b, c, d is a straight line.*

The following elegant proof was given by V. Kleptsyn.

Proof Assume that at least two of the curves a, b, c and d are lines. Let us consider two cases.

Case I. The curves a and b are straight lines (see Fig. 3.3(a)). Let us fix a point $D \in d$ and consider a small angle family $A_\delta B_\delta C_\delta D$. Denote by D' the image of the point D under the reflection with respect to the line a. Denote by D'' the image of the point D' under the reflection with respect to the line b. Then for any $C \in c$,

$$DC + CD'' = DC + D'B + BC = DC + DA + AB + BC = 1.$$



(a) The mirrors a and b are straight lines (b) The mirrors a and c are straight lines

Figure 3.3 Two mirrors are straight lines

Thus c is an ellipse with foci D and D'' *for any* $D \in d$. Therefore all points of the curve d are the foci of the same ellipse which is impossible. Therefore this case is impossible.

Case II. The curves a and c are lines (see Fig. 3.3 (b)). Let us fix a point $B \in b$ and consider a small angle family $A_\beta BC_\beta D_\beta$. Denote by B' and B'' the images of the point B under the reflection with respect to the lines a and c, respectively. Then for any $D \in d$,

$$B'D + B''D = BA + AD + BC + CD = 1.$$

Thus d is an ellipse with foci B' and B'' for every $B \in b$ which is impossible. Therefore this case is also impossible.

Finally, at most one of the curves a, b, c and d is a line. ∎

Later we will say "for a generic point $A \in a$" instead of "for a generic point $A \in a$ for any angle family corresponding to this point". In this chapter we use rather strong notion of genericity.

Convention 3.3.14 We say that some property holds *for a generic point $A \in a$*, if it holds for all but at most countable set of points $A \in a$.

The next subsections deal with the cases from Lemma 3.3.11 one by one and show that these cases hold for at most countable set of points $A \in a$. Hence there exists a point of the mirror a that satisfies none of these cases, but this contradicts Lemma 3.3.11. This contradiction will complete the proof.

3.3.4 Existence of the limits

In this Section we will prove the following proposition.

Proposition 3.3.15 *Suppose that the naming convention holds. Then for a generic point $A \in a$ the limits B_+, C_+ and D_+ exist, $B_+ \neq A$ and $D_+ \neq A$.*

In Lemma 3.3.16 we will prove that the limits B_+ and D_+ exist and do not coincide with A, and in Lemma 3.3.19 we will show that the limit C_+ exists as well.

Lemma 3.3.16 *Suppose that the naming convention holds. Then for a generic point $A \in a$ the limits B_+ and D_+ exist, $B_+ \neq A$ and $D_+ \neq A$.*

Proof First, let us prove that for a generic point $A \in a$, the limits B_+ and D_+ exist. Due to the symmetry between B and D it is sufficient to show that the limit B_+ exists. Assume the converse. Then the limit B_+ does not exist for uncountably many points $A \in a$. Take a point $A \in a$ such that the limit B_+ does not exist, see Figure 3.4. Note that the *line* AB_α depends only on A and α. Therefore this line tends to some limit position l as $\alpha \to \alpha_+$,

$$l = \lim_{\alpha \to \alpha_+} (\text{line } AB_\alpha).$$

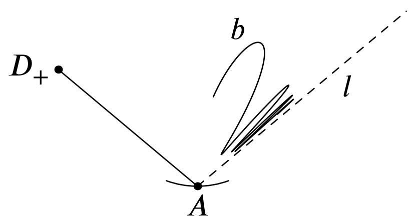


Figure 3.4 Oscillating curve b

Recall that the perimeter of the quadrilateral $AB_\alpha C_\alpha D_\alpha$ is one, hence B_α belongs to the unit disk centered at A. Therefore, $\text{dist}(B_\alpha, l)$ tends to zero as α tends to α_+.

The mirror a is not a line, hence the intersection $a \cap l$ is at most countable. Consider a point $A' \in a \setminus l$ such that the limit $B_+(A')$ does not exist as well. Consider the corresponding angle family $A'B'_\alpha C'_\alpha D'_\alpha$. Let l' be the limit position of the line $A'B'_\alpha$. Therefore, B'_α tends to the line l'.

Thus the curve b tends to both lines l and l', hence b tends to the intersection point $l \cap l'$, therefore both limits $B_+(A)$ and $B_+(A')$. Thus for a generic point $A \in a$ the limit B_+ exists.

Now, let us prove that $B_+ \neq A$ and $D_+ \neq A$. Again, we will only prove that $B_+ \neq A$. Assume the converse, i. e. for uncountably many points $A \in a$ the limit B_+ coincides with A.

Recall that we have attached the limits of the mirror b (if they exist) to the curve b itself, hence $B_+ \in b$. Therefore, the equality $A = B_+$ is possible only if $A \in b$. Note that if $a \neq b$, then the intersection $a \cap b$ is at most countable, thus $A \neq B_+$ for a generic point $A \in a$. Therefore, $a = b$.

Consider the set

$$V = \left\{ A \in a \,\middle|\, \exists B_+(A), A = B_+, A \text{ is neither a marked point nor an inflection point of } a \right\}.$$

The set of marked points is at most countable, as well as the set of inflection points of a. Therefore, the set V is uncountable.

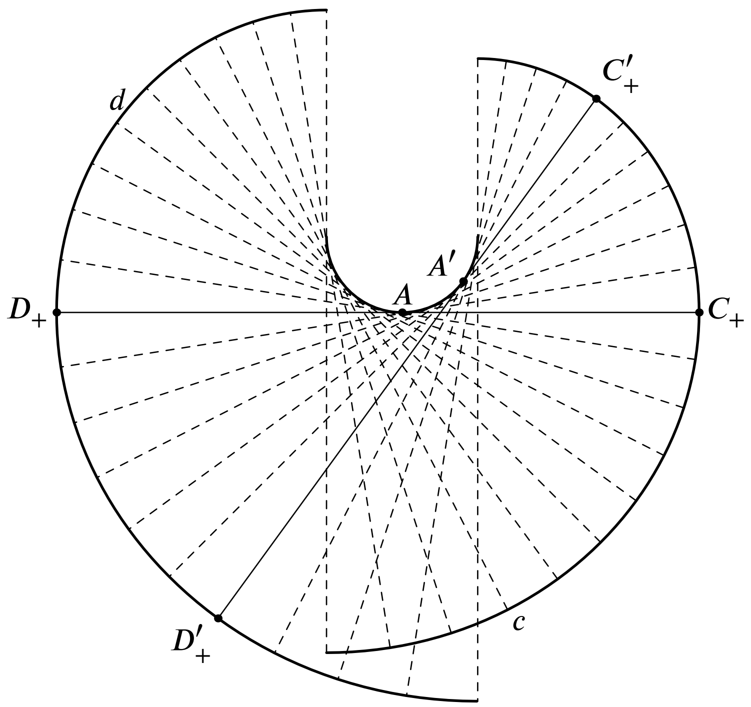


Figure 3.5 A curve and two involutes of this curve

For $A \in V$, the point B_α tends to A along a regular arc of the mirror a, hence the line AB_α tends to the tangent line to a at A. Therefore, for $A \in V$ the angles α_+ and β_+ must be equal to π, thus the angles γ_+ and δ_+ must be equal to 0. In this case for every $A \in V$ the limits C_+ and D_+ exist and belong to the intersection of $T_A a$ with the mirrors c and d, respectively. Also note that for a generic point $A \in a$ these intersections are regular points of the corresponding curves. Therefore for a generic point $A \in V$ the curves c and d are perpendicular to the tangent line $T_A a$ (reflection law), thus the same holds true for *any* point $A \in a$. Hence the curves c and d are involutes of the mirror a, therefore the curve a is the evolute of c and d (see Figure 3.5).

Note that $A \neq C_+$ and $A \neq D_+$ for $A \in V$. Indeed, the tangent lines to c at C_+ and to d at D_+ are perpendicular to the tangent line to a at A. Therefore the germs (c, C_+) and (d, D_+) cannot coincide with the germ (a, A). Since A is not a marked point, $A \neq C_+$ and $A \neq D_+$.

Let us consider the trajectory $AB_\alpha C_\alpha D_\alpha$ for $\alpha = \pi - \varepsilon$, $\varepsilon \ll 1$. Let l' be the perpendicular to d at D_α. Let $\hat{A}$ be the image of A under reflection with respect to l'. Since d is an involute for a, the line l' is tangent to a, $l' = T_X a$. Clearly, the segment XB_α intersects the line AD_α and the segment $B_\alpha C_\alpha$ does not intersect this line. Hence, the segment XC_α intersects the line AD_α, and the segment $\hat{A}C_\alpha$ intersects AD_α as well. On the other hand, due to reflection law the ray $[D_\alpha \hat{A})$ must coincide with the ray $[D_\alpha C_\alpha)$, hence the segment $[C_\alpha \hat{A}]$ does not intersect AD_α. This contradiction completes the proof. ∎

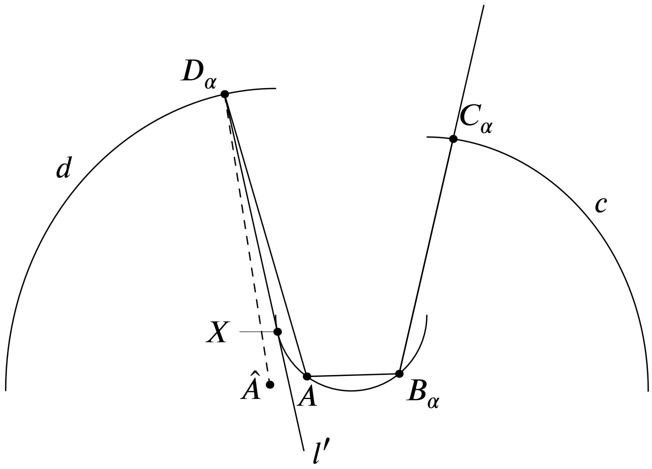


Figure 3.6 Reflection in involute

In order to prove the existence of the limit C_+ we will need the following two easy lemmas.

Lemma 3.3.17 *Suppose that the naming convention holds, and for uncountably many points $A \in a$ the limits B_+ and D_+ exist and are marked points. Then a is an ellipse.*

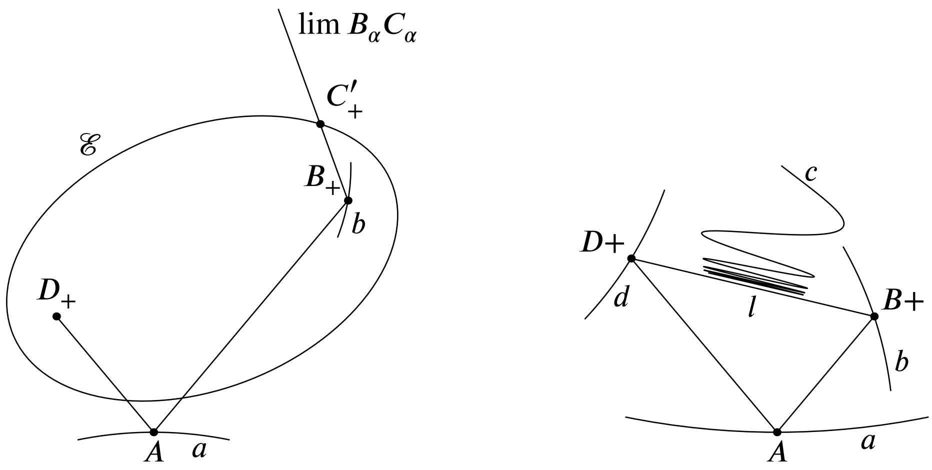
Proof Note that the function $\varphi \colon A \mapsto (B_+, D_+)$ takes countably many values on an uncountable set. Therefore it is a constant on some uncountable subset. Let (B_+^0, D_+^0) be this constant, i. e. $|\varphi^{-1}(B_+^0, D_+^0)| > |\mathbb{N}|$. Note that for each point $A \in \varphi^{-1}(B_+^0, D_+^0)$ the tangent line $T_A a$ is the exterior bisector of the angle $B_+^0 A D_+^0$. Let us consider the analytic function $s(A) = AB_+^0 + AD_+^0$. The derivative of this function is equal to zero at uncountably many points, namely at any non-isolated point $A \in \varphi^{-1}(B_+^0, D_+^0)$. Hence, $s(A)$ is constant, therefore a is an ellipse or a line. Due to the naming convention, a is not a line, hence a is an ellipse. ∎

Lemma 3.3.18 *Suppose that the naming convention holds. Then for a generic point $A \in a$ at least one of the points B_+ and D_+ is a regular point of the corresponding mirror.*

Proof Assume the converse. Then B_+ and D_+ belong to at most countable set of marked points for uncountably many points $A \in a$. Therefore a is an ellipse but the curves b and d are singular curves. This contradicts our naming convention. ∎

Lemma 3.3.19 *Suppose that the naming convention holds. Then for a generic point $A \in a$ the limit C_+ exists.*

Proof Denote by $s(A, \alpha)$ the sum $AB_\alpha + AD_\alpha$. Due to Lemma 3.3.16, for a generic point $A \in a$ both limits B_+ and D_+ exist, therefore the limit $s(A, \alpha_+)$ of $s(A, \alpha)$ as



(a) $1 - s(A, \alpha_+) > B_+D_+$ (b) $1 - s(A, \alpha_+) = B_+D_+$

Figure 3.7 Cases in Lemma 3.3.19

$\alpha \to \alpha_+$ exists as well. Hence, the limit of the sum $B_\alpha C_\alpha + C_\alpha D_\alpha = 1 - s(A, \alpha)$ also exists and is equal to $1 - s(A, \alpha_+)$, thus the vertex C_α tends to the ellipse $\mathscr{E} = \{X \mid B_+X + XD_+ = 1 - s(A, \alpha_+)\}$.

First consider the case of non-degenerate ellipse $\mathscr{E}$, i. e. $1 - s(A, \alpha_+) > B_+D_+$, see Figure 3.7 (a). Due to Lemma 3.3.18, for a generic point A either B_+ or D_+ is a regular point of the respective mirror. Obviously, it is sufficient to consider the case when B_+ is a regular point of b. In this case both limits $\lim\limits_{\alpha\to\alpha_+} AB_+$ and $\lim\limits_{\alpha\to\alpha_+} T_{B_\alpha} b$ exist, hence the limit of the ray $B_\alpha C_\alpha$ as $\alpha \to \alpha_+$ exists as well. This limit ray intersects the ellipse $\mathscr{E}$ by exactly one point C'_+. Since C_α must tend both to the ray $B_+C'_+$ and to the ellipse $\mathscr{E}$, $\lim\limits_{\alpha\to\alpha_+} C_\alpha = C'_+$. Finally, for a generic point A if $1 - s(A, \alpha_+) > B_+D_+$, then the limit C_+ exists.

Now let us consider the case $1 - s(A, \alpha_+) = B_+D_+$, see Figure 3.7 (b).. In this case C_α must oscillate along the segment B_+D_+. Note that the curve c can oscillate along at most two lines, therefore the line B_+D_+ must be the same (say, l) for uncountably many points A.

Let us prove that none of the mirrors b and d coincide with the line l. As usual, it is sufficient to prove that $b \neq l$. Assume the converse. Recall that due to the naming convention a is not a line, therefore the intersection $a \cap l$ is at most countable. Thus there exist uncountably many points $A \in a$ such that

- the point A does not belong to the line l;
- the line B_+D_+ coincides with the line l;
- the curve c oscillates along the line l.

Since $A \notin l$, the angle between the lines AB_+ and l is non-zero. Hence, the angle between the line $b = l$ and the reflected ray $B_\alpha C_\alpha$ must tend to the same nonzero number. But in this case the curve c cannot oscillate along the line l. This contradiction shows that the assumption is false, i. e. none of the mirrors b and d coincide with the line l.

Finally, $b \neq l$ and $d \neq l$, therefore B_+ and D_+ belong to at most countable set of points $(b \cup d) \cap l$ for uncountably many points A. Therefore, B_+ and D_+ do not depend on A for A from some uncountable set. The rest of this paragraph deals only with the points A from this uncountable set. Due to Lemma 3.3.17, the curve a is an ellipse. Due to our naming convention, either b or d is either an ellipse or a line.

Without loss of generality we can and will assume that the curve b is an ellipse or a line, thus the limit of $T_B b$ as $B \to B_+$ exists. Note that C_α oscillates along B_+D_+ thus there exists a sequence $\alpha_n \to \alpha_+$ such that the ray $B_{\alpha_n}C_{\alpha_n}$ tends to B_+D_+ as $n \to \infty$. The exterior bisector of the angle $AB_{\alpha_n}C_{\alpha_n}$ is the tangent line to b at the point B_{α_n}, thus the sequence of these bisectors tends to $T_{B_+}b$. Note that both the limit of the sequence of exterior bisectors and the limit of the rays $B_{\alpha_n}C_{\alpha_n}$ do not depend on A, thus the line AB_+ does not depend on A, and the point A must belong to the intersection of this line with the curve a. Therefore, this intersection is uncountable, hence a is a line, which contradicts our naming convention. This contradiction completes the proof. ∎

3.3.5 Case of two singular points

The following Lemma reduces the case of two singular points to the case of coinciding limits.

Lemma 3.3.20 *Suppose that the naming convention holds. For a generic point $A \in a$ if two of the points B_+, C_+, D_+ are singular points of the corresponding mirrors, then either $B_+ = C_+$, or $C_+ = D_+$.*

Proof Assume the converse, then there exist uncountably many points $A \in a$ such that at least two of the points B_+, C_+, D_+ are singular points of the corresponding mirrors, and $B_+ \neq C_+$, $C_+ \neq D_+$.

Due to Lemma 3.3.18, for a generic point $A \in a$ either B_+ or D_+ is a regular point of the corresponding mirror, thus either B_+ and C_+, or C_+ and D_+ are singular points of the corresponding mirrors. Due to the symmetry, it is sufficient to consider the former case, B_+ and C_+ are singular points of b and c and $B_+ \neq C_+$.

The set of singular points of an analytic curve is at most countable, thus the set $V(B^0, C^0) = \{A \mid B_+(A) = B^0, C_+(A) = C^0\}$ is uncountable for some two singular points $B^0 \in b$, $C^0 \in c$, $B^0 \neq C^0$. Note that if $A \in V(B^0, C^0) \setminus \{B^0\}$, then $A \neq B_+$ and $B_+ \neq C_+$, hence the limit of the exterior bisector of the angle $AB_\alpha C_\alpha$ as $\alpha \to \alpha_+$ exists. On the other hand, this exterior bisector is the tangent line to b at B_α, thus the limit of the tangent line to b at B as $B \to B_+$ exists.

The line AB_+ is the image of the line B_+C_+ under the reflection with respect to $T_{B_+}b$, hence the line $l = AB_+$ is the same for all points $A \in V(B^0, C^0) \setminus \{B^0\}$. Therefore $V(B^0, C^0) \setminus \{B^0\}$ is a subset of the intersection $l \cap a$ which is at most countable. Thus $V(B^0, C^0)$ is at most countable, which contradicts the statement from the previous paragraph. This contradiction proves the Lemma. ∎

3.3.6 Straight angle case

The main result of this subsection is the following statement.

Proposition 3.3.21 *Suppose that the naming convention holds. For a generic point $A \in a$ if $B_+ \neq C_+$, and $C_+ \neq D_+$, then none of the angles of the quadrilateral $AB_+C_+D_+$ equals π.*

Remark 3.3.22 *Recall that for a generic point $A \in a$ the limits B_+, C_+ and D_+ exist and $A \neq B_+$, $A \neq D_+$. The conditions $B_+ \neq C_+$ and $C_+ \neq D_+$ are needed to define the angles of $AB_+C_+D_+$.*

We will split the proof of this statement into a few lemmas.

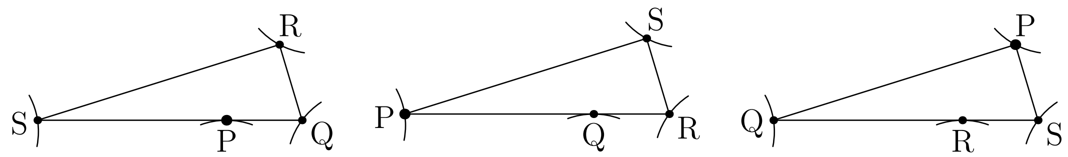
The following three lemmas prove that the angle of measure π cannot appear with another degeneracy for a generic point A. Then we will prove that the straight angle cannot appear without other degeneracies, thus completing the proof of Proposition 3.3.21.

Lemma 3.3.23 *Suppose that the naming convention holds. For a generic point $A \in a$ if $B_+ \neq C_+$ and $C_+ \neq D_+$, then at most one of the angles α_+, β_+, γ_+ and δ_+ is equal to π.*

Proof Suppose that at least two of the angles α_+, β_+, γ_+, δ_+ are equal to π. Then two other angles are equal to 0, and the quadrilateral $AB_+C_+D_+$ is a segment. Note that the angle α always increases, thus $\alpha_+ \neq 0$. Therefore $\alpha_+ = \pi$, hence the line $AB_+C_+D_+$ is tangent both to a and one of the curves b, c and d. Let p be this other curve, and P be the corresponding vertex.

The set of common tangent lines to two different analytic curves is at most countable, as well as the set of the lines that are tangent to the curve a at two different points (recall that a is not a line). Therefore, $P = A$ and $p = a$. Due to Lemma 3.3.16, for a generic point $A \in a$ neither B_+, nor D_+ coincides with A. Hence, $p = c$ and $P = C_+$, i. e. $a = c$ and $A = C_+$.

Using the same arguments as in Lemma 3.3.16, one can prove that the mirrors b and d are involutes of the mirror a. Note that for α close enough to π the mirror a has no



Case I: $\angle P = \pi$ Case II: $\angle Q = \pi$ Case III: $\angle R = \pi$

Figure 3.8 One straight angle, one fixed vertex

inflection points between A and C_α. Let l_α be the bisector of the angle $AB_\alpha C_\alpha$. On the one hand, it must intersect the mirror a between the points A and C_α, therefore l_α cannot be tangent to a. On the other hand, it is perpendicular to the involute of a, therefore it must be tangent to a. This contradiction completes the proof. ∎

Lemma 3.3.24 *Suppose that the naming convention holds. For a generic point $A \in a$ if $B_+ \neq C_+$, $C_+ \neq D_+$ and one of the angles of the quadrilateral $AB_+C_+D_+$ equals π, then none of the vertices of $AB_+C_+D_+$ is a singular point of the respective curve.*

Proof This lemma is immediately implied by the following lemma and the fact that the set of singular points of an analytic curve is at most countable. ∎

Lemma 3.3.25 *Suppose that the naming convention holds. Then there does not exist an uncountable set $V \subset a$ and a point $P \in \mathbb{R}^2$ such that for any $A \in V$ the following conditions hold.*

1. *the limits B_+, C_+ and D_+ exist;*
2. *$A \neq B_+$, $B_+ \neq C_+$, $C_+ \neq D_+$ and $D_+ \neq A$;*
3. *exactly one of the angles of the quadrilateral $AB_+C_+D_+$ equals π;*
4. *one of the points A, B_+, C_+, D_+ coincides with P.*

Proof Assume the converse. Without loss of generality we can and will assume that the same angle of the quadrilateral $AB_+C_+D_+$ equals π for all $A \in V$ and the same vertex coincides with P. Let P, Q, R, S be the vertices of the quadrilateral $AB_+C_+D_+$ enumerated starting from P either in the same or in the opposite cyclic order as A, B_+, C_+, D_+. Denote by p, q, r, s the corresponding mirrors.

Due to the second assumption of the Lemma, the mirrors p, q, r, s have the tangents at the points P, Q, R, S in the sense of Convention 3.3.4.

Consider three cases (see Figure 3.8).

Case I. $\angle P = \pi$. In this case the points S and Q belong to the intersection of the line $T_P p$ with the mirrors s and q, respectively. Note that this intersection is at most countable. Indeed, if either s or q intersects the line $T_P p$ on uncountably many points, then this curve must coincide with $T_P p$, hence either $\angle S = \pi$ or $\angle Q = \pi$ which contradicts Lemma 3.3.23.

Finally, R also belongs to the countable set of the intersections of two families of lines, namely, the images of the line $T_P p$ under the reflections with respect to the lines $T_Q q$ and $T_S s$. Therefore the set of quadrilaterals $PQRS$ is at most countable. Hence, this case is impossible.

Case II. $\angle Q = \pi$ or $\angle S = \pi$. We will consider only the case $\angle Q = \pi$, because the other case can be reduced to this one by renaming the points. Note that the number of tangent lines to q passing through the point P is at most countable. Therefore the line PQR belongs to at most countable set. Recall that the line RS is the image of the line PR under the reflection with respect to $T_R r$. Note that the curve r cannot coincide with a line PQR. Indeed, otherwise $\angle Q = \angle R = \pi$ which is impossible due to Assumption 3. Therefore the point R belongs to at most countable set, and the line RS belongs to at most countable set as well. Finally, each of the points P, Q, R, S belongs to the union of at most countable set of lines. Therefore, the point A also belongs to the union of at most countable set of lines and due to the naming convention A belongs to at most countable set of points. Thus this case is also impossible.

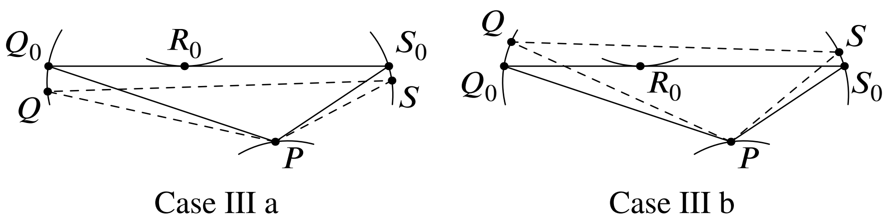


Figure 3.9 Perturbation of a degenerate quadrilateral in Case III

Case III. $\angle R = \pi$, see Figure 3.9. Let us prove that the set of the possible triangles PQS is discrete. Let us consider one of the quadrilaterals $PQ_0R_0S_0$ and another quadrilateral $PQRS$ close enough to $PQ_0R_0S_0$. Note that $Q_0R_0S_0$ and QRS are tangent lines to the curve r at close points R and R_0. Therefore the segments QS and Q_0S_0 must intersect each other. Consider two cases.

Case III a. $\angle QPS > \angle Q_0PS_0$. Since the tangent line $T_P p$ exists, $\angle QPS_0 > \angle Q_0PS_0$ and $\angle Q_0PS > \angle Q_0PS_0$. Since the line QQ_0 (resp., SS_0) is close to the exterior bisector of the angle $\angle PQ_0S_0$ (resp., $\angle PS_0Q_0$), both segments PQ and PS do not intersect the line Q_0S_0. Therefore, the segment QS does not intersect the line Q_0S_0. Thus this case is impossible.

Case III b. $\angle QPS < \angle Q_0PS_0$. Since the tangent line $T_P p$ exists, $\angle QPS_0 < \angle Q_0PS_0$ and $\angle Q_0PS < \angle Q_0PS_0$. Since the line QQ_0 (resp., SS_0) is close to the exterior bisector of the angle $\angle PQ_0S_0$ (resp., $\angle PS_0Q_0$), both segments PQ and PS intersect the line Q_0S_0. Therefore, the segment QS does not intersect the line Q_0S_0. Thus this case is impossible.

Finally, none of the three cases (the last one has two subcases) is possible. This proves the lemma. ∎

So, the previous three lemmas show that the straight angle cannot appear with another degeneracy. The following lemmas prove that the angle of measure π cannot appear alone as well.

Lemma 3.3.26 *Suppose that the naming condition holds. For a generic point $A \in a$ if $B_+ \neq C_+$ and $C_+ \neq D_+$, then none of the angles β_+ and δ_+ equals π.*

Proof Recall (see Remark 3.3.22) that for a generic point $A \in a$ the inequalities $B_+ \neq C_+$ and $C_+ \neq D_+$ imply that the limits β_+, γ_+ and δ_+ exist. Also recall that due to Lemma 3.3.24 the points B_+, C_+ and D_+ are regular points of the respective curves.

Assume the converse, i. e. $\beta_+ = \pi$ or $\delta_+ = \pi$ for uncountably many points $A \in a$. Due to the symmetry it is sufficient to consider the case $\beta_+ = \pi$. Note that $\alpha_+ > 0$ thus neither γ_+ nor δ_+ is equal to π (it also follows from Lemma 3.3.23). Also note that for a generic point $A \in a$ the curve b and the line AB_+ have only 2-point contact. Let us consider a trajectory very close to $AB_+C_+D_+$, namely $AB_\alpha C_\alpha D_\alpha$ for $\alpha = \alpha_+ - \varepsilon$, $\varepsilon \ll 1$.

Let us find the order of the length of the segment C_+C_α in two ways, using the path $A \to D \to C$ and using the path $A \to B \to C$.

On the one hand, the angle δ_+ is not equal to π, thus both D_α and the angle of incidence $\delta_\alpha/2$ of AD_α depend smoothly on α at $\alpha = \alpha_+$. Due to the inequality $\gamma \neq \pi$, the point C_α also depends smoothly on α, therefore $C_\alpha C_+ = O(\varepsilon)$.

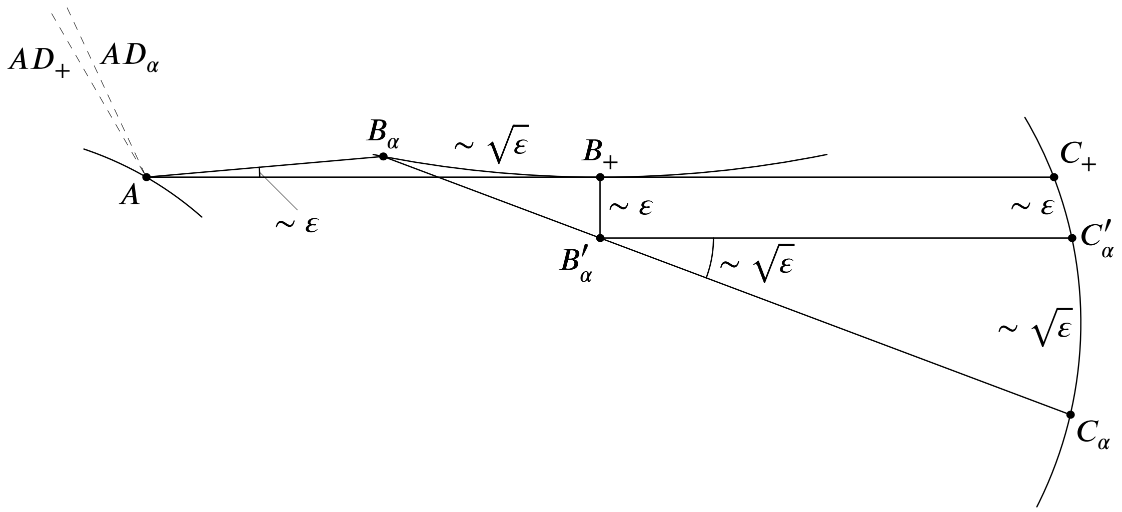


Figure 3.10 A limit trajectory with $\beta_+ = \pi$, and another one close to the limit one

Recall that $A \neq B_+$ and AB_+ has 2-point contact with b, thus $B_\alpha B_+$ is of the order $\sqrt{\varepsilon}$. Therefore the angle between AB_+ and the tangent line to b at B_α is of the order $\sqrt{\varepsilon}$. Let us compute the angle between $B_\alpha C_\alpha$ and B_+C_+. The angle between AB_α and AB_+ is equal to $\varepsilon/2$. Hence, the angle between AB_+ and the image of AB_α under the reflection with respect to the line AB_+ equals $\varepsilon/2$. The line $B_\alpha C_\alpha$ is the image of the same line AB_α

under the reflection with respect to the tangent line $T_{B_\alpha} b$. The angle between these two reflecting lines is of the order $\sqrt{\varepsilon}$, therefore the angle between $B_\alpha C_\alpha$ and AB_+ is of the order $\sqrt{\varepsilon}$.

Denote by B'_α the intersection point of the line $B_\alpha C_\alpha$ and the perpendicular $(T_{B_+} b)^\perp$ to b at B_+. Note that $B'_\alpha B_+$ is of the order ε. Indeed, the distance between B_α and B_+ is of the order $\sqrt{\varepsilon}$, and the angle between $B_\alpha C_\alpha$ and AB_+ is also of the order $\sqrt{\varepsilon}$, hence the distance between B'_α and the projection of B_α to $(T_{B_+} b)^\perp$ is of the order ε. On the other hand, the distance between this projection and B_+ is also of the order ε. Hence, $B'_\alpha B_+ = O(\varepsilon)$.

Denote by C'_α the intersection point of the mirror c and the line parallel to AC_+ passing through B'_α. The angle between $T_{C_+} c$ and $B_+ C_+$ is non-zero, thus the distance between C_+ and C'_α is of the same order as $B'_\alpha B_+$, i. e. ε. Recall that $B_+ \neq C_+$, therefore the distance between C'_α and C_α is of the same order as the angle between $B_\alpha C_\alpha$ and $B_+ C_+$, i. e. $\sqrt{\varepsilon}$. Thus the distance $C_\alpha C_+$ is of the order $\sqrt{\varepsilon}$.

Finally, we have $C_\alpha C_+ \gtrsim \sqrt{\varepsilon}$ and $C_\alpha C_+ \lesssim \varepsilon$ at the same time which is impossible. Therefore the angle β_+ cannot be equal to π. Recall that due to the symmetry the angle δ_+ cannot be equal to π as well. ∎

Lemma 3.3.27 *Suppose that the naming convention holds. For a generic point $A \in a$ if $B_+ \neq C_+$ and $C_+ \neq D_+$, then $\alpha_+ \neq \pi$.*

Proof Assume the converse, i. e. $B_+ \neq C_+$, $C_+ \neq D_+$ and $\alpha_+ = \pi$ for uncountably many points $A \in a$. Recall that due to Lemma 3.3.24 for a generic point A the equality $\alpha_+ = \pi$ implies that the vertices of the quadrilateral $AB_+C_+D_+$ are regular points of the corresponding mirrors. Moreover, due to Lemma 3.3.23 for a generic point $A \in a$ the equality $\alpha_+ = \pi$ implies that none of the angles β_+, γ_+ and δ_+ is equal to π.

Let us fix a point A_0 such that all the statements from the previous paragraph hold for A_0. There exists a neighborhood $A_0 \in U \subset a$ and a positive number $\varepsilon > 0$ such that for any point $A \in U$ and any angle $\alpha \in (\pi - \varepsilon, \pi]$ the points $B_\alpha(A)$, $C_\alpha(A)$ and $D_\alpha(A)$ are well-defined regular points of the respective curves. Therefore the conditions of the lemma (as well as the genericity conditions from the previous paragraph) hold also for all points $A \in U$. Let us replace U with its subinterval such that the curvature of a is non-zero at all points of U.

Denote by A^s the parametrisation of U by the natural parameter such that the vector $\frac{dA^s}{ds}$ is directed towards the point $B_\pi(A^s)$. Let us show that $C_\pi(A^s)$ does not depend on A^s. To this end consider the families $B^s = B_\pi(A^s)$, $C^s = C_\pi(A^s)$, $D^s = D_\pi(A^s)$, and let us prove that $\frac{dC^s}{dA^s} = 0$. Let k^s be the curvature of the mirror a at a point A^s. We say that k^s is positive if a is locally inside the triangle $B^s C^s D^s$ and negative otherwise. Denote by l^s the line tangent to a at A^s, $l^s = (B^s D^s)$.

Let us compute the derivative $\frac{dC^s}{ds}$ in two ways: using the trajectory $A \to B \to C$, and using the trajectory $A \to D \to C$.

Take a small number ε such that $k^s\varepsilon \geqslant 0$. Note that the angle between the lines l^s and $l^{s+\varepsilon}$ is equal to $k^s\varepsilon + o(\varepsilon)$. Therefore, the angle between the rays $A^{s+\varepsilon}B^{s+\varepsilon}$ and[11] $A^sB_{\pi-2k^s\varepsilon}(A^s)$ is $o(\varepsilon)$ and the distance $\operatorname{dist}(A^s, l^{s+\varepsilon})$ is $o(\varepsilon)$ as well. Hence the length of the segment $B^{s+\varepsilon}B_{\pi-2k^s\varepsilon}$ is $o(\varepsilon)$. Similarly, the angle between the reflected rays $B^{s+\varepsilon}C^{s+\varepsilon}$ and $B_{\pi-2k^s\varepsilon}(A^s)C_{\pi-2k^s\varepsilon}(A^s)$ is $o(\varepsilon)$, and the initial point of the latter ray is $o(\varepsilon)$-close to the former ray. Hence $C^{s+\varepsilon} = C_{\pi-2k^s\varepsilon}(A^s) + o(\varepsilon)$.

On the other hand, the line $A^sD_{\pi-2k^s\varepsilon}(A^s)$ is "nearly parallel" to the line $l^{s-\varepsilon}$, not to the line $l^{s+\varepsilon}$. Therefore applying the same arguments to the path $A \to D \to C$ one can show that $C^{s-\varepsilon} = C_{\pi-2k^s\varepsilon}(A^s) + o(\varepsilon)$. Finally, $C^{s+\varepsilon} = C^{s-\varepsilon} + o(\varepsilon)$ thus $\frac{dC^s}{ds} = 0$ and C^s does not depend on s.

On the other hand, due to Lemma 3.3.25 the point $C_\pi(A)$ cannot be the same for uncountably many points $A \in a$. This contradiction proves the lemma. ∎

Lemma 3.3.28 *Suppose that the naming convention holds. For a generic point $A \in a$ if $B_+ \neq C_+$ and $C_+ \neq D_+$, then $\gamma_+ \neq \pi$.*

Proof Assume the converse, then for uncountably many points $A \in a$, the points C_+ does not coincide neither with B_+, nor with D_+, and $\gamma_+ = \pi$.

As in the previous lemma, let us choose A_0 such that the limits B_+, C_+, D_+ exist and are regular points of the corresponding mirrors, $A_0 \neq B_+(A_0)$, $B_+(A_0) \neq C_+(A_0)$, $C_+(A_0) \neq D_+(A_0)$, $D_+(A_0) \neq A_0$ and none of the angles α_+, β_+ and δ_+ equals π.

Let us also fix α_0 close to α_+ such that γ_{α_0} is sufficiently close to π, fix a point $C = C_{\alpha_0}$ and start augmenting the angle γ. Obviously, the naming convention will hold for this angular family as well. Note that the points B^γ, A^γ and D^γ will not exit some small neighborhoods of B_+, A_+ and D_+, respectively. Hence, the points A^+, B^+ and D^+ are regular points of the corresponding curves, and $C^+ \neq B^+$, $B^+ \neq A^+$, $A^+ \neq D^+$ and $D^+ \neq C^+$. Therefore, the angle family $A^\gamma B^\gamma C D^\gamma$ extends to the angle $\gamma^+ = \pi$, which is impossible due to Lemma 3.3.27. ∎

Proof of Proposition 3.3.21 This proposition follows immediately from Lemmas 3.3.26, 3.3.27 and 3.3.28. ∎

3.3.7 Reduction to the case of coinciding limits

In this subsection we will summarize the result of the previous subsections into the following proposition.

[11] The point $B_{\pi-2k^s\varepsilon}(A^s)$ is defined since $k^s\varepsilon \geqslant 0$

Proposition 3.3.29 *Suppose that the naming convention holds. Then for a generic point $A \in a$ the limits B_+, C_+, D_+ exist and either $B_+ = C_+$, or $C_+ = D_+$.*

Proof Recall that Lemma 3.3.11 states that for *any* point $A \in a$ one of the following cases holds.

1. At least one of the limits $B_+ = \lim_{\alpha \to \alpha_+} B_\alpha$, $C_+ = \lim_{\alpha \to \alpha_+} C_\alpha$ and $D_+ = \lim_{\alpha \to \alpha_+} D_\alpha$ does not exist.
2. $AB_+C_+D_+$ is a degenerate quadrilateral (see Definition 3.3.6).
3. At least two of the points B_+, C_+ and D_+ are singular points of the corresponding mirrors.

Due to Proposition 3.3.15, the first case holds for at most countable set of points $A \in a$. Hence, for a generic point either $AB_+C_+D_+$ is a degenerate quadrilateral, or at least two points among B_+, C_+ and D_+ are singular points of the respective curves.

Due to Lemma 3.3.20, for a generic point $A \in a$ the third condition implies $B_+ = C_+$ or $C_+ = D_+$, hence *for a generic point $A \in a$ the quadrilateral $AB_+C_+D_+$ is degenerate.*

Recall that a quadrilateral $AB_+C_+D_+$ is degenerate if either $A = B_+$, or $B_+ = C_+$, or $C_+ = D_+$, or $D_+ = A$, or one of the angles of this quadrilateral equals π. Due to Proposition 3.3.15, the equalities $A = B_+$ and $A = D_+$ hold for at most countable set of points $A \in a$. Therefore, for a generic point $A \in a$ either $B_+ = C_+$, or $C_+ = D_+$ or one of the angles of $AB_+C_+D_+$ equals π.

Finally, Proposition 3.3.21 states that for a generic point $A \in a$ the latter condition ($\alpha_+ = \pi$ or $\beta_+ = \pi$ or $\gamma_+ = \pi$ or $\delta_+ = \pi$) implies the first one ($B_+ = C_+$ or $C_+ = D_+$). Therefore, for a generic point $A \in a$ either $B_+ = C_+$ or $C_+ = D_+$. ∎

3.3.8 Coinciding limits

This Subsection is devoted to the following statement.

Proposition 3.3.30 *For a generic point $A \in a$ neither $B_+ = C_+$ nor $C_+ = D_+$..*

Proof Recall that for a generic point A these limits exist due to Lemmas 3.3.16 and 3.3.19. Due to the symmetry in the naming convention, it is sufficient to show that $B_+ \neq C_+$.

Note that the set of degenerate quadrilaterals splits the set of all quadrilaterals into two connected components: convex quadrilaterals and concave ones. Therefore one connected component of our two-dimensional family of non-degenerate quadrilateral trajectories can contain only trajectories of one of these types, hence these cases can be considered separately.

Convex subcase Let us add one more assumption to the naming convention: $AB + AD \leqslant BC + CD$. It is possible to satisfy this assumption at the initial moment because of the symmetry between A and C in the earlier assumptions. On the other hand, due to convexity of $ABCD$, both AB_α and AD_α decrease on α, therefore the sum $s(A, \alpha) = AB_\alpha + AD_\alpha$ also strictly decreases on α. Hence, the inequality $AB + AD \leqslant BC + CD$ will remain true as α increases. Let us choose an interval $U \subset a$ and an angle α_0 such that $s(A, \alpha_0) < 0.5$ for all $A \in U$. Then for any $A \in U$ and $\alpha \geqslant \alpha_0$, $s(A, \alpha) < s(A, \alpha_0) < 0.5$.

Due to the triangle inequality, for any $A \in U$,

$$B_\alpha C_\alpha \geqslant C_\alpha D_\alpha - B_\alpha D_\alpha = 1 - s(A, \alpha) - B_\alpha C_\alpha - B_\alpha D_\alpha \geqslant 1 - B_\alpha C_\alpha - 2s(A, \alpha),$$

thus

$$B_\alpha C_\alpha \geqslant 0.5 - s(A, \alpha) > 0.5 - s(A, \alpha_0) > 0.$$

Hence, $B_\alpha C_\alpha$ is bounded away from zero, thus $B_+ C_+ > 0$, and $B_+ \neq C_+$.

Concave subcase In this case we cannot add the assumption $AB + AD \leqslant BC + CD$ to the naming convention. Indeed, even if $AB_{\alpha_0} + AD_{\alpha_0} < B_{\alpha_0} C_{\alpha_0} + C_{\alpha_0} D_{\alpha_0}$, the left hand side can increase on α, and the inequality can fail for some $\alpha > \alpha_0$. Therefore we will need other arguments.

Suppose the contrary, $B_+ = C_+$ for uncountably many points $A \in a$.

First, let us prove that for a generic point $A \in a$ the equality $B_+ = C_+$ implies that B_+ is a marked point. Indeed, $B_+ \in b \cap c$, thus either B_+ is a marked point, or $b = c$ and $B_+ = C_+$ is a regular point of this curve. In the latter case AB_+ must be the tangent line to b at the point B_+, thus the line AB_+ must coincide with the line $C_+ D_+$ and the angle α_+ must be equal to zero. On the other hand, $\alpha_+ > \alpha_0 > 0$. This contradiction proves that for a generic point A the equality $B_+ = C_+$ implies that B_+ is a marked point.

Therefore $B_+ = C_+$ is the same point X for uncountably many points A. In the sequel, only points A from this set are studied.

Note that the limits of the tangent lines to the curves b and c as $B \to B_+$ and $C \to C_+$ either both exist or both do not exist. Indeed, the angle between the lines AB_α and $C_\alpha D_\alpha$ is equal to $\pi - 2\angle(T_{B_\alpha} b, T_{C_\alpha} c)$. Hence, the limit of the angle between the tangent lines to b and c at B_α and C_α exists and equals $\frac{1}{2}(\pi - \angle AXD_+)$. Hence either both limits $\lim\limits_{B \to B_+} T_B b$ and $\lim\limits_{C \to C_+} T_C c$ exist, or both of them do not exist.

Let us prove that for a generic point A the equality $B_+ = C_+ = X$ implies that D_+ is a regular point of the curve d. Otherwise for uncountably many points A both points B_+ and D_+ do not depend on A, therefore due to Lemma 3.3.17 the curve a is an ellipse with foci B_+ and D_+. Due to the naming convention, either b or d is also an ellipse or a line. But D_+ is a singular point of the curve d, hence b is an ellipse or a line. Therefore, the limit of the tangent line to b at B_α as $\alpha \to \alpha_+$ exists, as well as the limit of the tangent line to c at C_α. Thus the angle $AXD_+ = \pi - 2\angle(T_{B_+}b, T_{C_+}c)$ does not depend on A. Hence the line AX also does not depend on A, and the mirror a intersects this line on uncountably many points which is impossible. This contradiction proves that for a generic point A the equality $B_+ = C_+ = X$ implies that D_+ is a regular point of the curve d.

Let us reduce the case when both limits $\lim\limits_{B\to B_+} T_B b$ and $\lim\limits_{C\to C_+} T_C c$ do not exist to the case when both limits exist.

Lemma 3.3.31 *Suppose that there exists an analytic* 4*-reflective billiard germ such that*

- *the quadrilateral $AB_\alpha C_\alpha D_\alpha$ is concave;*
- *for uncountably many points $A \in a$ we have $B_+ = C_+$ and both limits $\lim\limits_{B\to B_+} T_B b$ and $\lim\limits_{C\to C_+} T_C c$ do not exist.*

Then there exists another analytic 4*-reflective billiard germ such that for uncountably many points $A \in a$ we have $B_+ = C_+$ and both limits $\lim\limits_{B\to B_+} T_B b$ and $\lim\limits_{C\to C_+} T_C c$ exist.*

Proof Note that for a generic $A \in a$ such that $B_+ = C_+$, the angle α_+ is less than π. Indeed, due to the naming convention a is not a line, hence for a fixed point B_+ a generic line AB_+ is not tangent to a. On the other hand, B_+ belongs to at most countable set. Therefore, for α close enough to α_+ the points B_α and C_α belong to the same half-plane with respect to the line AD_α. Therefore the line AB_α is not parallel to the line $C_\alpha D_\alpha$ (recall that $AB_\alpha C_\alpha D_\alpha$ is a concave quadrilateral), and their intersection point belongs to one of the intervals AB_α or $C_\alpha D_\alpha$.

Let us forget about the curves b and c and consider the space M^3 of all non-degenerate concave quadrilaterals $ABCD$ such that $A \in a$, $D \in d$, $A \neq D$, $AB + BC + CD + DA = 1$, the points B and C belong to the same half-plane with respect to the line AD, and the reflection law holds at the points A and D. Denote by O the intersection point of the lines AB and CD.

A generic point of this space is determined by three parameters: two parameters determine the locations of the points A and D, and the third is the length of AB. Indeed, the point C is uniquely defined by A, B and D unless $B = O$. There are two minor problems with this coordinate system: it doesn't work if $B = O$, and it is hard to express the existence of the limit $\lim\limits_{B\to B_+} T_B b$ in these coordinates.

To resolve these problems, we will consider the following coordinates on M^3:

- the coordinates x_A, y_A, …, x_D, y_D of the points A, B, C and D;

- the lengths AB, BC, CD and DA.

Then our three-dimensional submanifold M^3 of this 12-dimensional space $\mathbb{R}^{12}$ is defined by the following equations:

- $AB^2 = (x_B - x_A)^2 + (y_B - y_A)^2, \ldots, DA^2 = (x_A - x_D)^2 + (y_A - y_D)^2$ (4 equations);
- $A \in a$, $D \in d$ (2 equations);
- the reflection laws at A and D (2 equations);
- $AB + BC + CD + DA = 1$;

and some analytic inequalities ($AB > 0$ etc.).

Denote by ω the 1-form on $\mathbb{R}^{12}$ that checks whether the point B moves along the exterior bisector of the angle ABC,

$$\omega = \left(dB, \frac{\overrightarrow{BA}}{BA} + \frac{\overrightarrow{BC}}{BC} \right) = \left(\frac{x_A - x_B}{AB} + \frac{x_C - x_B}{BC} \right) dx_B + \left(\frac{y_A - y_B}{AB} + \frac{y_C - y_B}{BC} \right) dy_B.$$

The restriction of ω to M^3 defines two-dimensional distribution on M^3. Due to Frobenius Theorem, the distribution defined by ω is integrable on the submanifold $\omega \wedge d\omega = 0$. Obviously, our two-dimensional family of billiard trajectories is a surface tangent to this distribution, hence it is contained in the submanifold $\omega \wedge d\omega = 0$, thus the dimension of the submanifold $\omega \wedge d\omega = 0$ is either 2 or 3.

In the former case, the limit configuration is a singular point of an analytic surface, hence both limits $\lim\limits_{B \to B_+} T_B b$ and $\lim\limits_{C \to C_+} T_C c$ exist, which contradicts the assumption.

Let us consider the latter case, $\omega \wedge d\omega \equiv 0$ on M^3. In this case the distribution $\operatorname{Ker} \omega$ is integrable on M^3, hence each quadrilateral $ABCD \in M^3$ defines the unique integral surface $\sigma \subset M^3$ passing through this quadrilateral. Consider the maps

$$\pi_B, \pi_C \colon \sigma \to \mathbb{R}^2, \quad \pi_B \colon (A, B, C, D) \mapsto B, \quad \pi_C \colon (A, B, C, D) \mapsto C.$$

Due to the definition of the form ω, the images of these maps are 1-dimensional curves $\widetilde{b}$ and $\widetilde{c}$ such that $(a, \widetilde{b}, \widetilde{c}, d)$ is a 4-reflective billiard germ.

Now let us choose a quadrilateral $ABCD \in M^3$ such that CD is small. Formally, let us take a quadrilateral $A^0B^0C^0D^0 \in M^3$ such that A^0 and D^0 are regular points of the curves a and d, respectively. Let O^0 be the intersection point of the rays A^0B^0 and D^0C^0. Recall that $A^0B^0C^0D^0$ is a concave quadrilateral, hence the perimeter of the triangle $A^0O^0D^0$ is less than one. Therefore, for any point C on the segment C^0D^0 one can find the unique point B on the ray A^0B^0 such that the perimeter of the quadrilateral A^0BCD^0 equals one. Let us choose a generic C^1 on the interval C^0D^0 very close to D^0, and find the corresponding point B^1. Note that the points A^0 and B^1 belong to different half-planes with respect to the line C^1D^0 due to the perimeter inequality above.

Now let us consider the mirrors $a, \widetilde{b}, \widetilde{c}, d$ corresponding to the quadrilateral $A^0B^1C^1D^0$. It is easy to see that for a generic point $C^1 \in C^0D^0$ the mirrors $a, \widetilde{b}, \widetilde{c}, d$ satisfy the naming convention.

Let us choose a generic point $C \in \widetilde{c}$ close to C^1 and consider the angle family $A^\gamma B^\gamma C D^\gamma$, $\gamma \in (\gamma_-, \gamma_+)$. Due to the symmetry in the naming convention, this angle family also satisfies the naming convention. Due to Proposition 3.3.29, either $A^+ = B^+$, or $A^+ = D^+$. Note that the angle between the exterior bisector of the angle $D^0C^1B^1$ and the tangent line to d at D^0 tends to some non-zero value as C^1 tends to D^0. For C^1 close enough to D^0 and C close enough to C^1, all the points D^γ are close to D^0, see the similar statement above. The line C^1D^0 separates the vertices A^0 and B^1, see the end of the paragraph before the previous one. Therefore, the line CD^γ separates the vertices A^γ and B^γ as well. Hence, $A^+ \neq B^+$, so $A^+ = D^+$ for a generic point $C \in \widetilde{c}$. Recall that D^γ is close to D^0, hence D^+ is a regular point of the curve d, and the limit $\lim\limits_{D \to D^+} T_D d$ exists. This proves the Lemma.

∎

Finally, there exists a 4-reflective billiard such that the naming convention holds, and for uncountably many points $A \in a$, $B_+ = C_+$ and both limits $\lim\limits_{B \to B_+} T_B b$ and $\lim\limits_{C \to C_+} T_C c$ exist. The point $B_+ = C_+$ is a marked point, therefore it is the same point X for all points A from some uncountable set $V \subset a$.

For $A \in V$, the angle AXD_+ is equal to $\varphi = \pi - 2\angle(T_{B_+}b, T_{C_+}c)$, thus does not depend on A. Therefore, the angle AXD_+ equals φ for all points A from some small neighborhood $U \subset a$. Note that the point D_+ is uniquely defined by the points A, X and the angle φ (i. e. no other information about the curves a, b, c and d is required to find D_+). Indeed, D_+ is the unique point such that $\angle AB_+D_+ = \varphi$ and $B_+D_+ + D_+A = 1 - AB_+$. Hence, the tangent line $T_A a$ depends only on A, X and φ. These tangent lines $T_A a = T_A a(A, X, \varphi)$ form a line field, and a is an integral curve of this line field. Clearly, the line field $T_A a$ is invariant with respect to the rotations around X.

The same arguments prove that d is an integral curve of the image of this line field under the symmetry with respect to a line passing through the point X. It is easy to show that a and d are spirals making infinite number of turns around X.

Note that the map $\alpha \mapsto C_\alpha$ is not a constant for a generic point $A \in a$. Indeed, if C_α does not depend on α, then b and d are ellipses with foci A and C_α. Hence, for any other point A, the function $\alpha \mapsto C_\alpha$ is not a constant. Hence, we can choose $\alpha_0 < \alpha_+$ close enough to α_+ so that C_{α_0} is a generic point of the curve c.

Consider the angle family $A^\gamma B^\gamma C D^\gamma$ with fixed point $C = C_{\alpha_0}$.

Due to Proposition 3.3.29, either $A^+ = B^+$, or $A^+ = D^+$. If C_{α_0} is sufficiently close to X, then B^γ is close to X for all $\gamma_{\alpha_0} < \gamma < \gamma^+$, and the angle between the tangent line to c at C and the tangent line to b at B^γ is close to the angle $\angle(T_{B_+}b, T_{C_+}c)$. Hence, the angle $A^\gamma X D^\gamma$ is close to φ, thus positive, and $A^+ \neq D^+$.

Therefore, $A^+ = B^+$. On the other hand, D^γ cannot pass through the line $T_C c$. Indeed, otherwise at this moment the line CD^γ will coincide with the tangent line $T_C c$, hence the angle γ will be equal to π, which is impossible due to Lemma 3.3.27. Therefore D^γ makes less than one turn around X. The angle $A^\gamma X D^\gamma$ is close to φ, thus A^γ makes less than two turns around X. Therefore, A^γ cannot reach some small neighborhood of X, and $A^+ \neq B^+$ for C sufficiently close to X. This contradiction proves the Proposition.

3.3.9 Proof of the main theorem

Now Theorem 3.1.4 is an easy consequence of Propositions 3.3.29 and 3.3.30. Indeed, due to Proposition 3.3.29 for a generic point A the limits B_+, C_+ and D_+ exist and either $B_+ = C_+$ or $C_+ = D_+$. On the other hand, due to Proposition 3.3.30 for a generic point A neither $B_+ = C_+$ nor $C_+ = D_+$. This contradiction completes the proof.

3.4 Further research

In this section we will discuss the case of k-gonal orbits, $k > 4$. We want to use the same strategy, i. e. consider an angle family $A_1 A_2^{\alpha_1} \ldots A_k^{\alpha_1}$, $\alpha_1 = \angle A_k A_1 A_2$, and study the limit as the angle α_1 tends to its maximal value $\alpha_1^+ \leqslant \pi$.

3.4.1 General case

The following straightforward generalization of Lemma 3.3.11 lists the possible cases for the limit configuration.

Lemma 3.4.1 *Consider a parametric family $A_1 A_2^{\alpha_1} \ldots A_k^{\alpha_1}$, where A_1 is a regular point of the corresponding mirror γ_1, $\alpha_1 = \angle A_k A_1 A_2$, $\alpha_1 \in (\alpha_1^-, \alpha_1^+) \subset (0, \pi)$. Then one of the following cases holds.*

1. *At least one of the limits $A_i^+ = \lim\limits_{\alpha_1 \to \alpha_1^+} A_i^{\alpha_1}$ does not exist.*
2. *$A_1 A_2^+ A_3^+ \ldots A_k^+$ is a degenerate k-gon (see Definition 3.3.6).*
3. *At least two of the points A_i^+ are singular points of the corresponding mirrors.*

It seems that this lemma lists the same obstructions as Lemma 3.3.11 but actually for $k > 4$ there are much more possible *combinations* of these obstructions. Of course, some of the lemmas developed for the case $k = 4$ can be generalized for $k > 4$, but they do not cover all cases.

Let us list some difficulties that appear only for $k > 4$.

- Some of the limits A_i^+ do not exist.
- At least two of the angles α_i are equal to π.
- One of the angles α_i^+ is equal to π and one of the vertices A_i^+ is a singular point of the respective curve.
- Two consequent vertices coincide, $A_i^+ = A_{i+1}^+$.

There are other cases (say, $A_2^+ = A_3^+$ and one of the angles α_i^+ is equal to π) but we believe that the cases above are the most important.

3.4.2 Current status for $k = 5$

As we stated above, the straightforward generalizations of our lemmas do not cover all possible cases even for $k = 5$. The cases that are not covered by these generalizations are sketched in Figure 3.11. The vertices known to be marked points are indicated by small empty circles, the vertices known to be regular (non-marked) points are indicated by small black disks, and the points that can be either marked, or non-marked, are indicated by black halfdisks.

One can prove that some of these cases are impossible. For the case of two straight angles, this was proved by V. Kleptsyn. But explaining the ideas required to this proof would take much space, and we still did not proved that *all* of these cases are impossible. Some generalizations used for restricting the list of possible cases will be formulated in the next subsection.

3.4.3 Straightforward generalizations

In this subsection we will formulate some straightforward generalizations of the lemmas used in this chapter. Since these lemmas do not lead immediately to any remarkable result for $k > 4$, we will not prove them.

Lemma 3.4.2 *(cf. Lemma 3.3.13) Let $\{\gamma_i\}_{i=1}^k$ be a k-reflective billiard germ. Then there are at most $k - 3$ straight lines among the mirrors γ_i.*

Lemma 3.4.3 *(cf. Lemma 3.3.16) Suppose that γ_1 is not a straight line. Then for a generic point $A_1 \in \gamma_1$ the limits A_2^+ and A_k^+ exist. If $k = 5$, then either $A_2^+ \neq A_1$, or $A_5^+ \neq A_1$.*

Lemma 3.4.4 *(cf. Lemma 3.3.19) Let p be a natural number, $3 \leqslant p \leqslant k - 1$. For a generic point $A_1 \in \gamma_1$ the following implication holds. Suppose that the limit A_i^+ exists for any $i \neq p$ and*

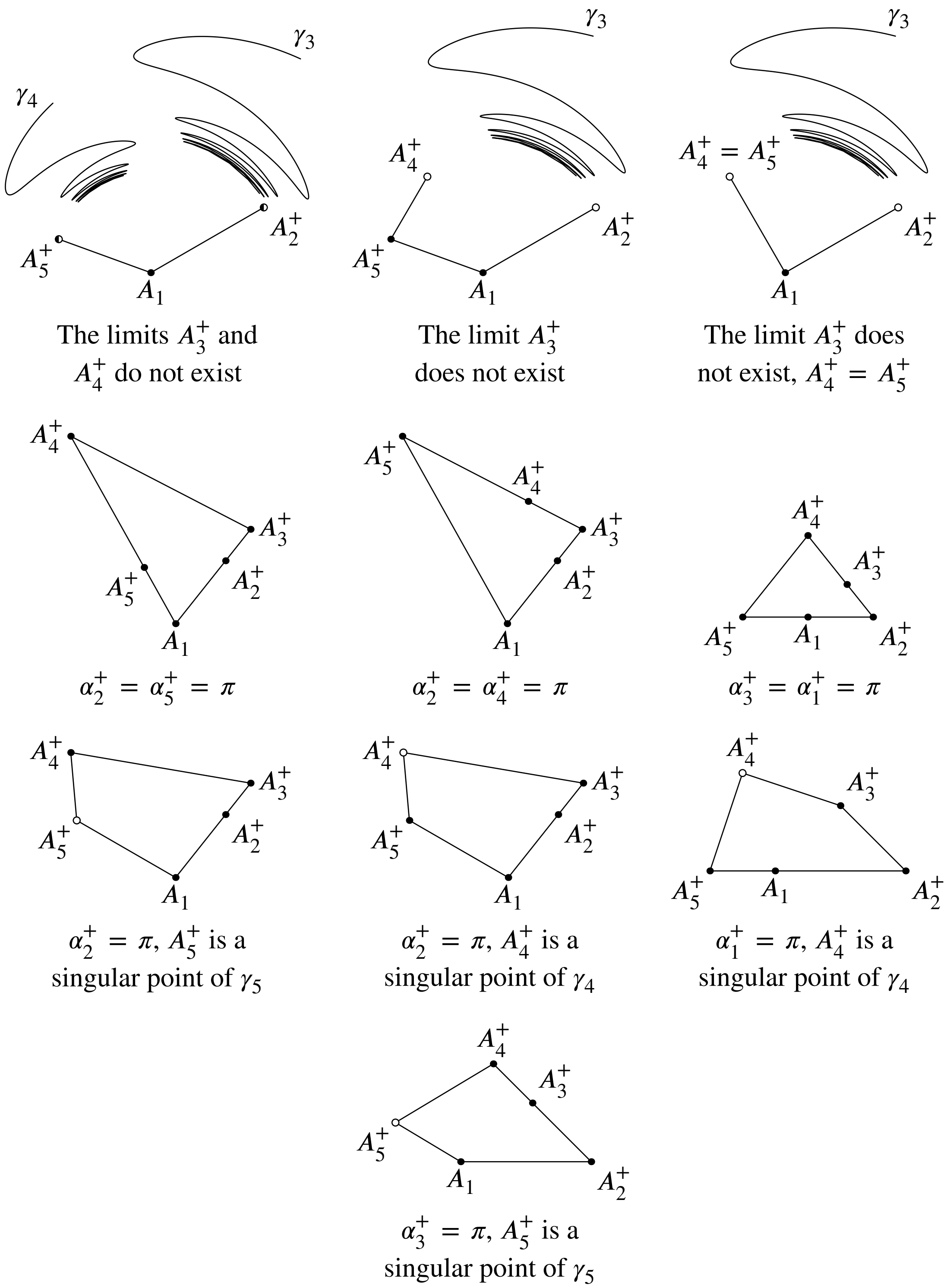


Figure 3.11 'Non-trivial' cases for $k = 5$. Part 1.

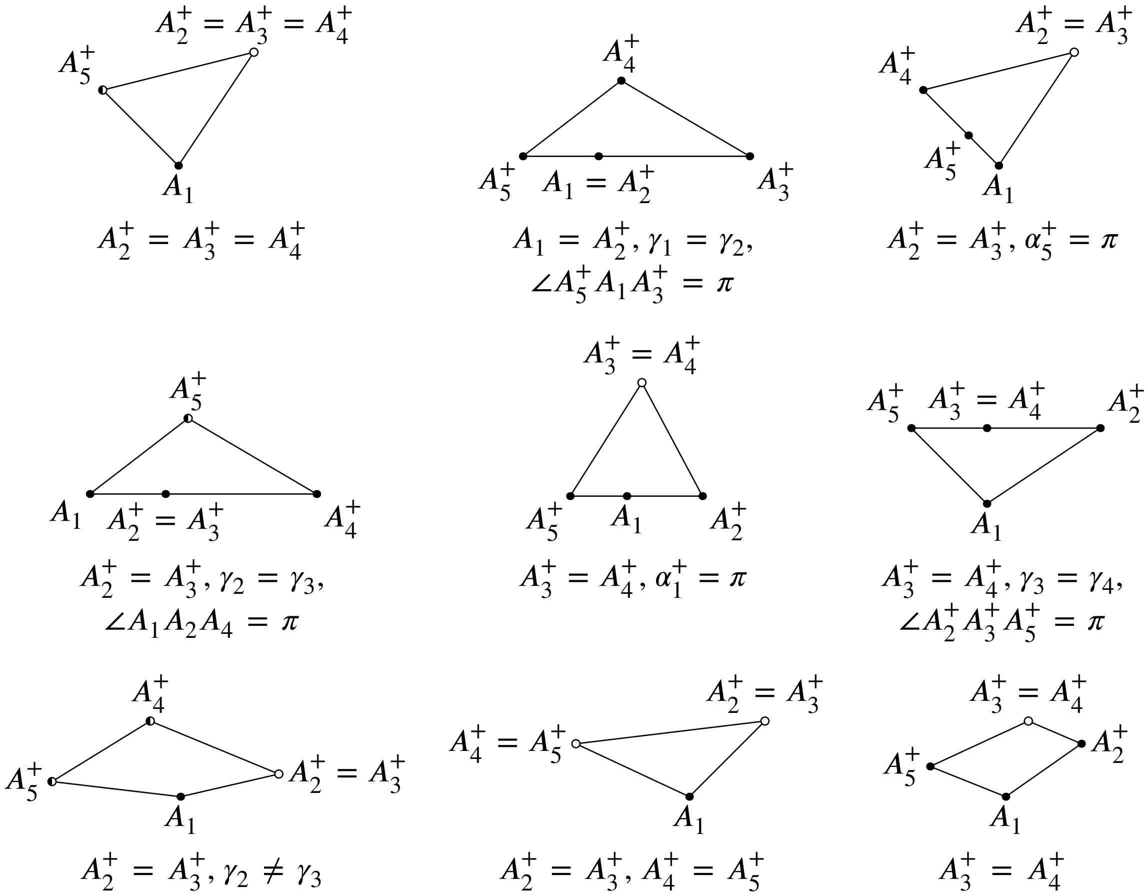


Figure 3.12 'Non-trivial' cases for $k = 5$. Part 2.

$$1 - A_1 A_2^+ - A_2^+ A_3^+ - \ldots - A_{p-2}^+ A_{p-1}^+ - A_{p+1}^+ A_{p+2}^+ - \ldots - A_{k-1}^+ A_k^+ - A_k^+ A_1 > A_{p-1}^+ A_{p+1}^+.$$

Then A_p^+ *exists.*

Notice that for $k \geqslant 5$ these two lemmas do not imply existence of all the limits A_i^+.

Lemma 3.4.5 *(cf. Lemma 3.3.26) A tangency* $\angle A_1 A_2^+ A_3^+ = \pi$ *cannot be the only obstruction to the analytic extension of the angle family, i. e. it is impossible that all the following conditions hold:*

- *the limit* A_i^+ *exists for any* $i = 2, \ldots, k$*;*
- $A_i^+ \neq A_{i+1}^+$ *for* $i = 2, \ldots, k-1$, $A_k^+ \neq A_1$, $A_1 \neq A_2^+$*;*
- $\angle A_{i-1}^+ A_i^+ A_{i+1}^+ \neq \pi$ *for* $i = 3, \ldots, k-1$, $\angle A_{k-1}^+ A_k^+ A_1 \neq \pi$, $\angle A_k^+ A_1 A_2^+ \neq \pi$*;*
- *each limit* A_i^+ *is a regular point of the corresponding curve;*
- $\angle A_1 A_2^+ A_3^+ = \pi$.

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