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Invisible parts of attractors

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Abstract

This paper deals with attractors of generic dynamical systems. We introduce the notion of ε-invisible set, which is an open set of the phase space in which almost all orbits spend on average a fraction of time no greater than ε. For extraordinarily small values of ε (say, smaller than 2^{-100}), these are large neighbourhoods of some parts of the attractors in the phase space which an observer virtually never sees when following a generic orbit.

For any $n \geqslant 100$, we construct a set Q_n in the space of skew products over a solenoid with the fibre a circle having the following properties. Any map from Q_n is a structurally stable diffeomorphism; the Lipschitz constants of the map and its inverse are no greater than L (where L is a universal constant that does not depend on n, say $L < 100$). Moreover, any map from Q_n has a 2^{-n}-invisible part of its attractor, whose size is comparable to that of the whole attractor. The set Q_n is a ball of radius $O(n^{-2})$ in the space of skew products with the C^1 metric. It consists of structurally stable skew products.

Small perturbations of these skew products in the space of all diffeomorphisms still have attractors with the same properties. Thus for all such perturbations, a sizable portion of the attractor is almost never visited by generic orbits and is practically never seen by the observer.

Mathematics Subject Classification: 35B41

1. Introduction

One of the major problems in the theory of dynamical systems is the study of the limit behaviour of orbits. Most orbits tend to invariant sets called attractors. Knowledge of the attractors may indicate the long term behaviour of the orbits.

Yet it appears that large parts of the attractors may be practically invisible. In this paper we describe an open set in the space of dynamical systems whose attractors have a large unobservable part. Precise definitions follow.

1.1. Attractors and ε-invisible open sets

Let X be a *metric measure space*, with a metric d and a finite measure μ. This measure will not necessarily be probabilistic, but we will assume that $\mu(X) \geqslant 1$.

Often, but not always, X will be a compact smooth manifold with or without boundary. In this case the metric is the geodesic distance and the measure is the Riemannian volume.

There are many different, nonequivalent definitions of attractors, some of which are presented below. The following definitions all concern maps $F : X \to X$ which are homeomorphic onto their image.

Definition 1 (Maximal attractor). *An invariant set* $A_{\max}$ *of* F *is called* a maximal attractor in its neighbourhood *provided that there exists a neighbourhood* U *of* $A_{\max}$ *such that*

$$A_{\max} = \bigcap_{n=0}^{\infty} F^n(U).$$

Definition 2 (Milnor attractor [15]). *The* Milnor attractor A_M *of* F *is the minimal invariant closed set that contains the ω-limit sets of almost all points.*

Definition 3 (Statistical attractor [1, 9]). *The* statistical attractor A_{stat} *of* F *is the minimal closed set such that almost all orbits spend an average time of* 1 *in any neighbourhood of* A_{stat}.

Definition 4 (ε-invisible open set). *An open set* $V \subset X$ *is called ε-invisible if the orbits of almost all points visit* V *with average frequency no greater than ε:*

$$\limsup_{N\to\infty} \frac{|\{0 \leqslant k < N | F^k(x) \in V\}|}{N} \leqslant \varepsilon, \qquad \text{for a.e. } x. \tag{1}$$

1.2. Skew products

Skew products may be called *mini Universes of Dynamical Systems*. They are a source of many instructive examples, see for instance [12, 19]. Many properties observed for these products appear to persist as properties of diffeomorphisms for open sets in various spaces of dynamical systems. This heuristic principle was justified in [5, 6, 8]. In this context, an open set of diffeomorphisms with nonhyperbolic invariant measures was found in [7, 14], while other new robust properties of diffeomorphisms were described in [5, 6]. This paper is another application of this heuristic principle.

In this section, X is a Cartesian product $X = B \times M$ with the natural projections $\pi : X \to M$ along B, $p : X \to B$ along M. The set B is the *base*, while M is the *fibre*. Both B and M are metric measure spaces. The distance between two points of X is, by definition, the sum of the distances between their projections onto the base and onto the fibre. The measure on X is the Cartesian product of the measures of the base and of the fibre.

Maps of the form

$$F : B \times M \to B \times M, \qquad F(b, x) = (h(b), f_b(x)) \tag{2}$$

are called *skew products* on X. Denote by C^1_p (p stands for product) the space of all skew products on X, with distance given by

$$d_{C^1_p}(F, \tilde{F}) = \max_B d_{C^1}(f_b^{\pm 1}, \tilde{f}_b^{\pm 1}). \tag{3}$$

Definition 5. *A homeomorphism F of a metric space is called L-moderate if* Lip $F^{\pm 1} \leqslant L$ *(here* Lip *denotes Lipschitz constant).*

We shall consider only L-moderate maps F with $L \leqslant 100$, in order to guarantee that the phenomenon of ε-invisibility is not produced by any extraordinary distortion in the maps F or F^{-1}.

1.3. Skew products over the Smale–Williams solenoid and the main result

Take $R \geqslant 2$, and let $B = B(R)$ denote the solid torus

$$B = S^1_y \times D(R),\ \ S^1_y = \{y \in \mathbb{R}/\mathbb{Z}\}, \qquad D(R) = \{z \in \mathbb{C} | |z| \leqslant R\}.$$

The *solenoid map* is defined as

$$h = h_\lambda : B \to B, \quad (y, z) \mapsto (2y, \mathrm{e}^{2\pi \mathrm{i} y} + \lambda z), \qquad \lambda < 0.1. \tag{4}$$

The exact values of the parameters R and λ are not crucial, since the dynamics of the map h is the same regardless of their particular values.

Let us consider the Cartesian product $X = B \times S^1$, where $S^1 = \mathbb{R}/2\mathbb{Z}$. All skew products in this section are over this Cartesian product, and the map h in the base B will always be the solenoid map. Fix some $L \leqslant 100$, and let $D_L(X)$ (respectively, $C^1_{p,L}(X)$) denote the space of L-moderate smooth maps (respectively, smooth skew products). Our main result on attractors is the following theorem.

Theorem 1 (Main Theorem). *Consider any $n \geqslant 100$. Then there exists a ball Q_n of radius $1/n^2$ in the space $C^1_{p,L}(X)$ with the distance equation (3) having the following property. Any map $\mathcal{G} \in Q_n \cap C^2(X)$ is structurally stable in $D^1(X)$, and has a statistical attractor $A_{\mathrm{stat}} = A_{\mathrm{stat}}(\mathcal{G})$ such that the following hold:*

1. *the projection $\pi(A_{\mathrm{stat}}) \subset S^1$ is a circular arc such that*

$$\left[\frac{6}{n}, 1 - \frac{2}{n}\right] \subset \pi(A_{\mathrm{stat}}) \subset [0, 1]; \tag{5}$$

2. *the set $V = \pi^{-1}(0, \frac{1}{4})$ is ε-invisible for $\mathcal{G}$ with*

$$\varepsilon = 2^{-n}. \tag{6}$$

Moreover, small perturbations of the maps from Q_n in the space $D_L(X)$ of all diffeomorphisms have statistical attractors with the same properties.

Remark 1. It is easy to construct a map with a sizable ε-invisible part of its attractor and with distortion of order ε^{-1} (so with an enormous Lipschitz constant). Indeed, consider an irrational rotation R of a circle. The statistical attractor of R is the whole circle. Take a small arc of length ε and a coordinate change $H : S^1 \to S^1$ that expands this arc to a semicircle U. Then all the orbits of the map $f = H \circ R \circ H^{-1}$ visit the semicircle U with frequency ε. Hence, this large part of the attractor is ε-invisible. However, the map f has a Lipshitz constant of order ε^{-1}. We reject such examples, because they rely on extraordinarily large distortions.

In contrast, in theorem 1 we construct maps on a 'human' scale that produce ε-invisible sets, for extraordinarily small ε. Indeed, our main theorem claims the existence of large ε-invisible sets with ε arbitrarily small, when the Lipschitz constant of the maps in question is uniformly bounded (say, by $L = 100$).

Theorem 1 considers maps defined by 'human-scale parameters of order n'. Intuitively, this means that the maps contain parameters of order n or $1/n$. Formally speaking, this means that these maps are at a distance at least Cn^{-b} from structurally unstable maps, for some $b > 0$. In $C^1_{p,L}(X)$ we prove that we can take $b = 2$, while in $D_L(X)$ we conjecture that we can take $b = 4$.

On the other hand, these maps have large ε-invisible parts of attractors for $\varepsilon = 2^{-n}$. In theorem 1, this part is equal to $A_{\text{stat}} \cap V$, and its size is comparable to the size of the whole attractor. More precisely, the projection $\pi(A_{\text{stat}} \cap V)$ is an arc of about $\frac{1}{4}$ of the total length of the arc $\pi(A_{\text{stat}})$. Roughly speaking, to visualize this part of the attractor, the observer would have to pursue orbits for time intervals of order 2^n. Even for $n = 100$, it is hard to imagine such an experiment.

1.4. Rate of invisibility and distribution of SRB measures

For any compact Riemannian manifold X, with or without boundary, let $D^2_L(X)$ be the space of diffeomorphisms $f : X \to X$ (onto or into) such that $||f||_{C^2} \leqslant L$, and $\text{Lip}(d^2 f^{\pm 1}) \leqslant L$. This space is compact in the C^2 metric by the Arzela–Ascoli theorem.

Suppose that $\text{diam}\, X = 1$. Denote by $S_L(X)$ the set of all structurally stable diffeomorphisms inside $D^2_L(X)$. Let $S^\delta_L(X)$ be the closure of the set of those $f \in S_L(X)$, whose δ-neighbourhood in C^1-metric intersected with $D^2_L(X)$ belongs to $S_L(X)$. In other words, $S^\delta_L(X)$ consists of structurally stable diffeomorphisms that are at least a distance δ away (in the C^1 metric) from structurally unstable maps.

For any structurally stable map $f \in D^2_L(X)$ and any $c \in (0, 1)$, define the c-rate of invisibility $i_{c,X}(f)$ of the statistical attractor of f in the following way. Let $B(p, c) \subset X$ be the ball of radius c centred at $p \in X$. Define

$$i_{c,X}(f) := \inf\{\varepsilon \mid \exists p \in A_{\text{stat}}(f) \text{ such that } B(p, c) \text{ is } \varepsilon\text{-invisible}\}.$$

Due to structural stability, all locally maximal invariant sets of any $f \in S_L(X)$ are hyperbolic. In this case, the union of all maximal attractors in their neighbourhoods coincides with the Milnor, statistical and minimal attractors [3]. Hence, the statistical attractor of f is at the same time the support of the SRB measure μ_∞. From proposition 1, it follows that

$$i_{c,X}(f) = \inf_p \mu_\infty(B(p, c)),$$

as p ranges over the support of the SRB measure. Therefore, the c-rate of invisibility shows just how small the SRB measure of a c-ball centred on the support can actually be.

The function

$$i_{c,X}(\delta) = \min_{S^\delta_L(X)} i_{c,X}(f)$$

is well defined. Indeed, the function $i_{c,X}(f)$ is continuous in f, because of the continuous dependence of the SRB measure on the hyperbolic map in the C^2-topology. It is not difficult to see that $i_{c,X}(\delta) \to 0$ as $\delta \to 0$. The question is:

How rapidly does the rate of invisibility $i_{c,X}(\delta)$ decrease with δ?

The answer depends on X, and is a new characteristic of the space of dynamical systems on X. In contrast, we expect that the decrease in $i_{c,X}(\delta)$ does not depend in an essential way on c.

Similar definitions may be given for vector fields and for other functional spaces inside $D^2_L(X)$. The main examples are skew products and finite-parameter families.

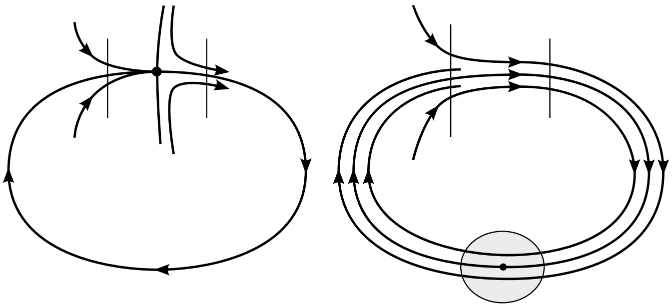

Figure 1. Saddle-node bifurcation on a circle.

Example 1. *Saddle-node bifurcation on a circle. Consider a family of vector fields v_δ on a circle that have a saddle-node singular point for $\delta = 0$ and no other singular points. For postcritical values of the parameter $\delta > 0$, the whole phase space consists of one periodic orbit. The finitely smooth normal form of the family near the singular point is*

$$\dot{x} = \frac{x^2 + \delta}{1 + ax}$$

for some $a \in \mathbb{R}$ [11]. The period of the postcritical orbit is $O(\delta^{-\frac{1}{2}})$. The time spent in any ball of radius c separated from zero (shadowed in figure 1) is $O(c)$. Hence, the rate of invisibility in the family v_δ is $O(\delta^{\frac{1}{2}})$.

Example 2. *Slow-fast systems: the van der Pol equation. Consider the van der Pol family*

$$\dot{x} = y - y^3,$$

$$\dot{y} = \delta(x - a).$$

For $\delta = 0$ this system is structurally unstable. For $\delta > 0$ it has an attracting van der Pol cycle. *The period of this cycle is $O(\delta^{-1})$. The large parts of this cycle close to the orbits $y = c$ of the fast system are $O(\delta)$-invisible. Hence, for the van der Pol family, the rate of invisibility is $O(\delta)$.*

Conjecture 1. *The rate of invisibility $i_{c,S^1}(\delta)$ for diffeomorphisms of a circle equals $O(\delta^{\frac{1}{2}})$. The rate of invisibility for flows on S^2 equals $O(\delta)$.*

Denote by $I_{c,X}(\delta)$ the rate of invisibility for skew products (namely, replace $D_L(X)$ by $C^1_{p,L}(X)$ in the above definitions). Our main theorem shows that for $X = \mathbb{T}^2 \times D$ (where D is a 2-disc),

$$I_{c,X}(\delta) < \exp(-O(\delta^{-\frac{1}{2}})).$$

Thus, the rate of invisibility may decrease as a stretched exponential. We expect that a similar result holds for diffeomorphisms:

$$i_{c,X}(\delta) < \exp(-O(\delta^{-\alpha}))$$

for some $\alpha > 0$. The approach that has been developed in this subsection was suggested by Gorodetski.

The results of [10] give strong evidence to a conjecture that the rate of invisibility decreases with the dimension $k = \dim X$ at least as

$$i_c(\delta) < \exp(-\delta^{-O(k)}).$$

2. Skew products over the Bernoulli shift

In this section we define and study skew products over the Bernoulli shift, which closely mimic the dynamics of skew products over the solenoid.

2.1. Step and mild skew products over the Bernoulli shift

Let Σ^2 be the space of all bi-infinite sequences of 0 and 1, endowed with the standard metric d and $(\frac{1}{2}, \frac{1}{2})$-probability Bernoulli measure P. In other words, if we take $\omega, \omega' \in \Sigma^2$ given by

$$\omega = \dots \omega_{-n} \dots \omega_0 \dots \omega_n \dots$$

$$\omega' = \dots \omega'_{-n} \dots \omega'_0 \dots \omega'_n \dots,$$

then

$$d(\omega, \omega') = 2^{-n}, \text{ where } n = \min\{|k|, \text{such that } \omega_k \neq \omega'_k\}, \tag{7}$$

$$P(\{\omega, \text{ such that } \omega_{i_1} = \alpha_1, ..., \omega_{i_k} = \alpha_k\}) = \frac{1}{2^k}, \tag{8}$$

for any $i_1, ..., i_k \in \mathbb{Z}$ and any $\alpha_1, ..., \alpha_k \in \{0, 1\}$. Let $\sigma : \Sigma^2 \to \Sigma^2$ be the Bernoulli shift

$$\sigma : \omega \mapsto \omega', \ \omega'_n = \omega_{n+1}.$$

A *skew product over the Bernoulli shift* is a map

$$G : \Sigma^2 \times M \to \Sigma^2 \times M, \qquad (\omega, x) \mapsto (\sigma\omega, g_\omega(x)), \tag{9}$$

where the *fibre maps* g_ω are diffeomorphisms of the fibre onto itself. Let $C^1_{p,L}$ denote the space of skew products (9), whose fibre maps g_ω and their inverses g_ω^{-1} have Lipschitz constant no greater than L.

An important class of skew products over the Bernoulli shift consists of the so-called *step skew products*. Given two diffeomorphisms $f_0, f_1 : S^1 \to S^1$, the step skew product over these two diffeomorphisms is

$$F : \Sigma^2 \times M \to \Sigma^2 \times M, \qquad (\omega, x) \mapsto (\sigma\omega, f_{\omega_0}(x)). \tag{10}$$

Thus the fibre maps only depend on the digit ω_0, and not on the whole sequence ω. In contrast to step skew products, general skew products where the fibre maps depend on the whole sequence ω will be called *mild* ones.

2.2. SRB measures and minimal attractors

Consider a metric measure space X. We begin with the definition of the (global) maximal attractor, which is only slightly different from definition 1. Let $\mathcal{G} : X \to X$ be homeomorphic onto its image, but suppose its image is contained strictly in X. The *(global) maximal attractor* of $\mathcal{G}$ is defined as:

$$A_{\max} = \bigcap_{k=0}^{\infty} \mathcal{G}^k(X). \tag{11}$$

Moreover, a measure μ_∞ is called a *good measure* of $\mathcal{G}$ (with respect to the measure μ of X) if it is a limit point of a subsequence of average measures:

$$\mu_\infty = \lim_{n\to\infty} \frac{1}{k_n} \sum_{i=0}^{k_n-1} \mathcal{G}^i_* \mu,$$

in the weak topology, see [4].

The closure of the union of supports of all good measures of $\mathcal{G}$ is called the *minimal attractor*, and it is contained in the statistical attractor (also see [4]). Thus the following inclusions between attractors hold:

$$A_{\min} \subset A_{\text{stat}} \subset A_M \subset A_{\max}. \tag{12}$$

An invariant measure μ_∞ is called an *SRB measure* with respect to μ provided that

$$\int_X \varphi \mathrm{d}\mu_\infty = \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \varphi(\mathcal{G}^i(x)) \tag{13}$$

for almost all $x \in X$ and for all continuous functions $\varphi \in C(X)$ (see [16]). If a good measure is unique and ergodic, then it is an SRB measure.

The connection between an SRB measure and the ε-invisibility property mentioned in definition 4 is the following:

Proposition 1. *Consider X and $\mathcal{G} : X \to X$ as above, and suppose that an SRB measure μ_∞ exists. Then an open set $V \subset X$ is ε-invisible if and only if*

$$\mu_\infty(V) \leqslant \varepsilon.$$

Proof. This proposition immediately follows by letting φ be the characteristic function of V in (13). Of course, the characteristic function is not continuous, but it can be sandwiched between continuous functions arbitrarily tight. □

The classical definitions above traditionally apply to smooth manifolds X, either closed or compact with boundary, for which the measure μ is compatible with the smooth structure (a 'Lebesgue measure'). In the above, we have extended these definitions to general metric measure spaces.

Consider the metric measure space $\Sigma^2 \times S^1$ and let $\pi : \Sigma^2 \times S^1 \to S^1$ be the standard projection. As before, $C^1_{p,L}$ is the space of skew products over the Bernoulli shift with fibre S^1, whose fibre maps and their inverses have Lipschitz constant at most L. The distance on both these spaces is still defined by (3). We will now state the following analogue of the main theorem 1 for (mild) skew products:

Theorem 2. *Consider any $n \geqslant 100$. Then there exists a ball R_n of radius $1/n^2$ in the space $C^1_{p,L}$ with the following property. Any map $G \in R_n$ has a statistical attractor $A_{\text{stat}} = A_{\text{stat}}(G)$ such that the following hold:*

1. *the projection $\pi(A_{\text{stat}}) \subset S^1$ has the property*
$$\pi(A_{\text{stat}}) \subset [0, 1]; \tag{14}$$
2. *the set $V = \pi^{-1}(0, \frac{1}{4})$ is ε-invisible for G with*
$$\varepsilon = 2^{-n}. \tag{15}$$

2.3. North–South skew products

The skew products for which we will verify theorem 2 will be from the open set of so-called North–South skew products, defined below.

Definition 6. *A skew product $G : \Sigma^2 \times S^1 \to \Sigma^2 \times S^1$ is called a North–South skew product if there exist two non-intersecting closed arcs $I, J \subset S^1$ such that all the fibre maps g_ω have* *the following properties:*

1. *Every map g_ω has one attractor and one repeller, both hyperbolic.*
2. *All the attractors of the maps g_ω lie strictly inside I.*
3. *All the repellers of the maps g_ω lie strictly inside J.*
4. *All the maps g_ω bring I into itself and are contracting on I uniformly in ω. Moreover, the maps $g_\omega^{-1}|I$ are expanding.*
5. *All the inverse maps g_ω^{-1} bring J into itself and are contracting on J uniformly in ω. Moreover, the maps $g_\omega|J$ are expanding.*
6. *The maps g_ω and g_ω^{-1} depend continuously on ω in the C^0 topology.*

In the same way, one can define North–South skew products over any map $h : B \to B$ with the fibre a circle.

2.4. Maximal attractors of North–South skew products

Theorem 3. *Let $G : \Sigma^2 \times S^1 \to \Sigma^2 \times S^1$ be a North–South skew product over the Bernoulli shift. Then we have*

(a) The statistical attractor of G is the graph of a continuous function $\gamma = \gamma_G : \Sigma^2 \to I$. The projection $p|A_{\text{stat}} : A_{\text{stat}} \to \Sigma^2$ is a bijection. Under this bijection, $G|A_{\text{stat}}$ becomes conjugated to the Bernoulli shift on Σ^2:

$$\begin{array}{ccc} A_{\text{stat}} & \xrightarrow{G} & A_{\text{stat}} \\ p\big\downarrow & & p\big\downarrow \\ \Sigma^2 & \xrightarrow{\sigma} & \Sigma^2 \end{array} \tag{16}$$

(b) There exists an SRB measure μ_∞ on $\Sigma^2 \times S^1$. This measure is concentrated on A_{stat} and is precisely the pull-back of the Bernoulli measure P on Σ^2 under the bijection $p|A_{\text{stat}} : A_{\text{stat}} \to \Sigma^2$.

Proof. By assumption 4 of definition 6, the map G brings $\Sigma^2 \times I$ strictly inside itself. We can thus consider the global maximal attractor of $G|\Sigma^2 \times I$:

$$A^*_{\max} = \bigcap_{k=1}^{\infty} G^k(\Sigma^2 \times I). \tag{17}$$

We will later prove that

$$A^*_{\max} = A_{\text{stat}}. \tag{18}$$

Proposition 2. *The attractor $A^*_{\max}$ is the graph of a function $\gamma : \Sigma^2 \to I$.*

Proof. This follows from assumption 4 in the definition of North–South skew products. In more detail, a point (ω, x) belongs to $A^*_{\max}$ if and only if (ω, x) belongs to $G^k(\Sigma^2 \times I)$ for all $k \geqslant 1$. This is equivalent to

$$x \in g_{\sigma^{-1}\omega} \circ \cdots \circ g_{\sigma^{-k}\omega}(I) =: I_k(\omega) \tag{19}$$

for all $k \geqslant 1$. By assumption 4, for any fixed ω, the segments $I_k(\omega)$ are nested and shrinking as $k \to \infty$. Hence, in any fibre $\{\omega\} \times S^1$, the maximal attractor $A^*_{\max}$ has exactly one point

$$x(\omega) = \bigcap_{k=1}^{\infty} I_k(\omega).$$

Define the map $\gamma : \Sigma^2 \to I$, $\omega \mapsto x(\omega)$. By this definition, $A^*_{\max}$ is just the graph of γ. □

Proposition 3. *The function γ defined above is continuous.*

Proof. Consider the notation

$$g_{k,\omega} = g_{\sigma^{-1}\omega} \circ g_{\sigma^{-2}\omega} \circ \ldots \circ g_{\sigma^{-k}\omega}.$$

By (19), we have

$$\gamma(\omega) = \bigcap_{k=0}^{\infty} g_{k,\omega}(I).$$

Fix a sequence ω, fix $\delta > 0$ and $m \in \mathbb{N}$. Let ω' be so close to ω that

$$||g_{\sigma^{-k}\omega} - g_{\sigma^{-k}\omega'}|| \leqslant \delta$$

for all $k = 1, \ldots, m$. In the rest of this proof, all norms will refer to the C^0 norm. Write

$$\delta_k = ||g_{k,\omega} - g_{k,\omega'}||.$$

Then, for $k \leqslant m$,

$$\delta_k = ||g_{k-1,\omega} \circ g_{\sigma^{-k}\omega} - g_{k-1,\omega'} \circ g_{\sigma^{-k}\omega'}|| \leqslant T_1 + T_2,$$

where

$$T_1 = ||g_{k-1,\omega} \circ g_{\sigma^{-k}\omega} - g_{k-1,\omega} \circ g_{\sigma^{-k}\omega'}||,$$

$$T_2 = ||g_{k-1,\omega} \circ g_{\sigma^{-k}\omega'} - g_{k-1,\omega'} \circ g_{\sigma^{-k}\omega'}||.$$

Let $l < 1$ be a common contraction coefficient for all the fibre maps $g_\omega|I$. Then we have

$$T_1 \leqslant l^{k-1}\delta,$$

$$T_2 \leqslant \delta_{k-1}.$$

The second inequality holds because the fibre maps bring I into itself and the shift of the argument does not change the C^0 norm. Therefore, we have

$$\delta_k \leqslant \delta_{k-1} + l^{k-1}\delta.$$

Iterating the above inequality gives us

$$\delta_m \leqslant \delta + l\delta + \cdots + l^{m-1}\delta < \frac{\delta}{1-l}.$$

Therefore, the segments $I_m(\omega)$, $I_m(\omega')$ have length no greater than $l^m|I|$ and the distance between their corresponding endpoints is no greater than $\delta/(1-l)$. But this holds for arbitrarily small δ and arbitrarily large m when ω and ω' are close enough. Therefore, (19) implies that $\gamma(\omega)$ and $\gamma(\omega')$ can be made arbitrarily close by making ω, ω' close enough. This precisely proves the continuity of γ. □

2.5. *Statistical attractors of North–South skew products*

Let us now prove (18). The proof relies on the following lemma:

Lemma 1. *For almost all $(\omega, x) \in \Sigma^2 \times S^1$ there exists $k = k(\omega, x) > 0$ such that $G^k(\omega, x) \in \Sigma^2 \times I$.*

Proof. On $S^1 \setminus (I \cup J)$ all the fibre maps g_ω push points away from J and into I. Hence, the orbit of a point (ω, x) will come to $\Sigma^2 \times I$ if and only if there exists k such that

$$G^k(\omega, x) \in \Sigma^2 \times (S^1 \setminus J).$$

This fails to happen only for elements of the set

$$S = \bigcap_{k=0}^{\infty} G^{-k}(\Sigma^2 \times J).$$

We will show that the measure of S is zero. Consider the inverse map

$$G^{-1} : (\omega, x) \mapsto (\sigma^{-1}\omega, g^{-1}_{\sigma^{-1}\omega}(x)).$$

Once again, it is a North–South skew product but the segments I and J now play the opposite roles: J is contracting and I is expanding. By the previous section, the maximal attractor S of $G^{-1}|\Sigma^2 \times J$ is the graph of a continuous function $\gamma^- : \Sigma^2 \to J$. It therefore intersects any fibre $\{\omega\} \times S^1$ at exactly one point. By the Fubini theorem, the measure of S in X is therefore zero. □

The above lemma shows that the ω-limit sets of almost all points in $\Sigma^2 \times S^1$ belong to $A^*_{\max}$. Hence $A^*_{\max}$ is the Milnor attractor of G, and thus contains A_{stat}. We will now prove that $A^*_{\max}$ is precisely equal to A_{stat}.

Consider any good measure μ_∞ of G. For any measurable set $K \subset \Sigma^2$, we have

$$G^{-1}(K \times S^1) = \sigma^{-1}(K) \times S^1$$

and therefore $G_*\mu(K \times S^1) = \mu(\sigma^{-1}(K) \times S^1) = \mu(K \times S^1)$. Iterating this will give us $G^k_*\mu(K \times S^1) = \mu(K \times S^1) = P(K)$ for all k. By the definition of good measure this forces

$$\mu_\infty(K \times S^1) = P(K). \tag{20}$$

But any good measure is supported on A_{stat}, and therefore on $A^*_{\max}$. This and (20) imply that μ_∞ must be the push-forward of P under the isomorphism $(p|A^*_{\max})^{-1}$. In particular, the support of μ_∞ is the whole of $A^*_{\max}$.

By the above, the only possible good measure is μ_∞ given by (20). Its support $A^*_{\max}$ therefore coincides with the minimal attractor $A_{\min}$. Therefore, by (12), we have that

$$A_{\min} = A_{\text{stat}} = A^*_{\max}.$$

This proves statement (a) of theorem 3.

Let us now prove statement (b) of theorem 3. We must now show that $\mu_\infty = (p|A_{\text{stat}})^{-1}_* P$ is an SRB measure (in particular, our proof will imply that μ_∞ is a good measure). To this end, we must show that for almost all $(\omega, x) \in \Sigma^2 \times S^1$ and any continuous function $\varphi \in C(\Sigma^2 \times S^1)$ we have

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \varphi(G^i(\omega, x)) = \int \varphi \, d\mu_\infty. \tag{21}$$

By lemma 1, we may restrict attention to $x \in I$. Then it is easy to note that

$$\text{dist}(G^k(\omega, x), G^k(\omega, \gamma(\omega))) \to 0$$

as $k \to \infty$, uniformly in ω and in x. By the continuity of φ this implies

$$\varphi(G^k(\omega, x)) - \varphi(G^k(\omega, \gamma(\omega))) \to 0.$$

Therefore to prove (21), it is enough to prove it for $x = \gamma(\omega)$, i.e.

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \varphi(G^i(\omega, \gamma(\omega))) = \int \varphi \, d\mu_\infty. \tag{22}$$

Since $p : A_{\rm stat} \to \Sigma^2$ is an isomorphism, the function $\tilde{\varphi} = \varphi \circ (p|A_{\rm stat})^{-1}$ is continuous on Σ^2. Therefore, (22) is equivalent to

$$\lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} \tilde{\varphi}(\sigma^i \omega) = \int_{\Sigma^2} \tilde{\varphi} \mathrm{d}P$$

for almost all ω. This statement is just the ergodicity of σ, which is a well-known result. We have thus proven that μ_∞ is an SRB measure, and this concludes the proof of theorem 3. □

2.6. *Large ε-invisible parts of attractors for skew products over the Bernoulli shift*

In this section we will complete the proof of theorem 2. Recall that we have fixed $n \geqslant 100$. We shall consider a particular North–South step skew product F, whose fibre maps $f_0, f_1 : S^1 \to S^1$ satisfy the properties listed below:

1. The segment I in definition 6 of North–South skew products is [0, 1] and
$$f_0|I = \left(1 - \frac{1}{3n}\right)x + \frac{1}{n^2}, \qquad f_1|I = \frac{1}{4}x + \frac{3}{4}\left(1 - \frac{1}{n}\right). \tag{23}$$
The attractors of f_0 and f_1 are $3/n$ and $1 - (1/n)$, respectively.
2. The segment J in definition 6 of North–South skew products is $[-\frac{2}{3}, -\frac{1}{3}]$ and $f_0|J = f_1|J = (x \mapsto 2x + \frac{1}{2})$.
3. In between the arcs I and J, f_0, f_1 define a 'one way' motion away from J towards I:
$$f_j(x) \geqslant x + \frac{1}{n^2}, \qquad \text{for } x \in \left[-\frac{1}{3}, 0\right]$$
$$f_j(x) \leqslant x - \frac{1}{n^2}, \qquad \text{for } x \in \left[-1, -\frac{2}{3}\right].$$

It is easy to see that any step skew product with these fibre maps is North–South, as in definition 6.

Proposition 4. *Consider the ball R_n of radius $1/n^2$ around F in the space $C^1_{p,L}$ of skew products over the Bernoulli shift. By definition, this ball consists of skew products G such that*

$$d(F, G) = \max_{\omega\in\Sigma^2} d_{C^1}(f^{\pm 1}_\omega, g^{\pm 1}_\omega) \leqslant \frac{1}{n^2}. \tag{24}$$

Then any map G from this ball is a north–south skew product, where the segments I and J from definition 6 are $[0, 1]$ *and* $[-\frac{2}{3}, -\frac{1}{3}]$, *respectively.*

Proof. Consider any $\omega \in \Sigma^2$ with $\omega_0 = 0$. Then the fibre map g_ω of G is $\frac{1}{n^2}$-close to f_0 in C^1. Hence,

$$1 > 1 - \frac{1}{3n} + \frac{1}{n^2} > g'_\omega|I > 1 - \frac{1}{3n} - \frac{1}{n^2} > 1 - \frac{1}{2n}$$

because $n < 100$. Moreover,

$$f_0(0) = \frac{1}{n^2}, \qquad f_0(1) = 1 - \frac{1}{3n} + \frac{1}{n^2}.$$

Hence, $g_\omega(0) > 0$, $g_\omega(1) < 1$. Therefore, $g_\omega(I) \subset I$, g_ω is strictly contracting on I and has a unique attractor $a(\omega)$ on I. Moreover, $f_0(6/n) = (6/n) - (1/n^2)$. Hence, $g_\omega(6/n) < (6/n)$. Therefore, $a(\omega) \in [0, 6/n]$.

In the same way we prove that $g_\omega|J$ has a unique repeller in J, and $g_\omega|J$ is expanding; hence, $g_\omega^{-1}(J) \subset J$ for any $\omega \in \Sigma^2$. Finally, the map g_ω has the property that $g_\omega(x) > x$ for $x \in [-\frac{1}{3}, 0]$ and that $g_\omega(x) < x$ for $x \in [-1, -\frac{2}{3}]$, for any $\omega \in \Sigma^2$.

Consider now ω with $\omega_0 = 1$. Then g_ω is $(1/n^2)$-close in C^1 to f_1. In the same way as above, we prove that g_ω has a unique attractor in I, contracts on I and brings I into itself. All other properties of g_ω are proved exactly as in the case $\omega_0 = 0$. So the maps g_ω satisfy the requirements of definition 6, and thus G is a North–South skew product. □

Proof of theorem 2. Any map $G \in R_n$ is a North–South skew product by proposition 4. Hence, we can apply theorem 3, and this immediately implies statement 1 of theorem 2. Let us now prove statement 2, namely that the set $V = \pi^{-1}(0, \frac{1}{4})$ is ε-invisible for any $G \in R_n$. We need to check that almost every point $(\omega, x) \in \Sigma^2 \times S^1$ visits V with frequency no greater than ε. By lemma 1, it is enough to consider $(\omega, x) \in \Sigma^2 \times I$.

Proposition 5. *Let $k > n$ and $(\omega, x) \in \Sigma^2 \times I$ such that $G^k(\omega, x) \in V$. Then*

$$(\omega_{k-n} \ldots \omega_{k-1}) = (0 \ldots 0).$$

Proof. Let $j \leqslant k-1$ be minimal such that $\omega_{k-j} = 1$. If such a j does not exist or $j > n$, then the proposition is proved (since we assumed $k > n$). Suppose by contraposition that $j \leqslant n$. Then the digit at position zero of the sequence $\sigma^{k-j}\omega$ is 1. Thus the fibre map $g_{\sigma^{k-j}\omega}$ is $\frac{1}{n^2}$ close to f_1, implying

$$1 > g_{\sigma^{k-j}\omega}(0) > f_1(0) - \frac{1}{n^2} > \frac{3}{4} - \frac{1}{n}.$$

Hence,

$$1 > \pi(G^{k-j+1}(\omega, x)) = g_{\sigma^{k-j}\omega}(\pi(G^{k-j}(\omega, x))) > \frac{3}{4} - \frac{1}{n}.$$

Now we claim that for all $x \in [\frac{1}{4}, \frac{3}{4}]$ and $\omega \in \Sigma^2$, we have

$$f_j(x) > x\left(1 - \frac{1}{2n}\right) + \frac{1}{n^2}. \tag{25}$$

This follows readily from (23) by elementary calculations. Together with (24), this implies

$$g_{\tilde{\omega}}(x) > x\left(1 - \frac{1}{2n}\right), \qquad \forall \tilde{\omega} \in \Sigma^2. \tag{26}$$

We can iterate this inequality and get

$$\pi(G^k(\omega, x)) = g_{\sigma^{k-1}\omega} \circ \ldots \circ g_{\sigma^{k-j+1}\omega}(\pi(G^{k-j+1}(\omega, x))) > \left(\frac{3}{4} - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right)^{j-1}$$

$$> \left(\frac{3}{4} - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right)^{n}.$$

One sees that the right-hand side of the above inequality is approximated by $\frac{3}{4\sqrt{e}} > \frac{1}{4}$. A more accurate estimate would indicate that

$$\pi(G^k(\omega, x)) > \left(\frac{3}{4} - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right)^{n} > \frac{1}{4}$$

for $n \geqslant 100$. This contradicts the assumption of the proposition. □

The ergodicity of the Bernoulli shift implies that the subword $(0 \ldots 0)$ (n zeroes) is met in almost all sequences ω with frequency 2^{-n}. This and proposition 5 imply that almost all orbits visit V with frequency at most $\varepsilon = 2^{-n}$. Hence V is ε-invisible indeed, and this concludes the proof of theorem 2. □

3. Skew products over the solenoid

In this section we will prove theorem 1. Our approach will closely mirror the proof of theorem 2 of the previous section.

3.1. The symbolic dynamics and SRB measure for the solenoid map

Let h be the solenoid map (4). Denote by Λ the maximal attractor of this map, which is called the *Smale–Williams solenoid*. Let $\Sigma^2_1 \subset \Sigma^2$ be the set of infinite sequences of 0's and 1's without a tail of 1's infinitely to the right (i.e. sequences which have 0's arbitrarily far to the right). Its metric and measure are inherited from the space Σ^2. Consider the fate map

$$\Phi : \Lambda \to \Sigma^2_1, \qquad \Phi(b) = (\ldots\omega_{-1}\omega_0\omega_1\ldots), \tag{27}$$

where we define $\omega_k = 0$ if $y(h^k(b)) \in [0, \frac{1}{2})$ and $\omega_k = 1$ if $y(h^k(b)) \in [\frac{1}{2}, 1)$. The map Φ is a bijection with a continuous inverse. Moreover, it conjugates the map $h|\Lambda$ with the Bernoulli shift:

$$\begin{array}{ccc} \Lambda & \xrightarrow{h} & \Lambda \\ \Phi\downarrow & & \Phi\downarrow \\ \Sigma^2_1 & \xrightarrow{\sigma} & \Sigma^2_1 \end{array} \tag{28}$$

In addition to the fate map Φ, we can define the 'forward fate map' $\Phi^+(b) = (\omega_0\omega_1\ldots)$, with $\omega_0, \omega_1, \ldots$ described as above. The map $\Phi^+(b)$ is now defined for all b in the solid torus B, and it only depends on $y(b)$. More generally, if $h^{-k}(b)$ exists, then we can define $\Phi^+_{-k}(b) = (\omega_{-k}\ldots\omega_0\omega_1\ldots)$.

It is well known that the SRB measure on Λ is the pullback of the Bernoulli measure on Σ^2_1 under the fate map:

$$\mu_\Lambda = \Phi^* P.$$

3.2. Attractors of North–South skew products over the solenoid

Let $X = B \times S^1$, where B is the solid torus. A North–South skew product over the solenoid will refer to a skew product that satisfies the properties of definition 6 with (Σ^2, ω) replaced by (B, b).

Theorem 4. *Let $\mathcal{G} : X \to X$ be a North–South skew product over the solenoid. Then*

(a) The statistical attractor of $\mathcal{G}$ lies inside $\Lambda \times I$, and is the graph of a continuous map $\gamma : \Lambda \to I$. Under the projection homeomorphism $p : A_{\text{stat}} \to \Lambda$, the restriction $\mathcal{G}|A_{\text{stat}}$ becomes conjugated to the solenoid map on Λ:

$$\begin{array}{ccc} A_{\text{stat}} & \xrightarrow{\mathcal{G}} & A_{\text{stat}} \\ p\downarrow & & p\downarrow \\ \Lambda & \xrightarrow{h} & \Lambda \end{array} \tag{29}$$

(b) There exists an SRB measure μ_∞ on X. This measure is concentrated on A_{stat} and is precisely the pull-back of the Bernoulli measure P on Σ^2_1 under the isomorphism $\Phi \circ p : A_{\text{stat}} \to \Sigma^2_1$.

(c) The skew product $\mathcal{G}$ is structurally stable in $D^1(X)$.

This theorem is proved in the same way as theorem 3 with a single essential difference: we need new arguments to prove the analogue of lemma 1. This will be done in lemma 3 of the next subsection.

3.3. Hyperbolicity

Lemma 2. *Let $\mathcal{G} : X \to X$ be a North–South skew product over the solenoid. Then the invariant sets*

$$A = \bigcap_{k=0}^{\infty} \mathcal{G}^k(B \times I), \qquad S = \bigcap_{k=0}^{\infty} \mathcal{G}^{-k}(\Lambda \times J)$$

are hyperbolic.

Remark 2. The union $A \cup S$ is the non-wandering set of $\mathcal{G}$. The set A is a hyperbolic attractor of index 1, while S is a locally maximal hyperbolic set of index 2.

This lemma is a technical result that will be proved shortly. For now, denote by $W^s S$ the set of all $q \in X$ that attract to S under $\mathcal{G}$:

$$W^s S = \{q \in X | d(\mathcal{G}^k(q), S) \to 0 \text{ as } k \to \infty\}.$$

We claim that set $W^s S$ has measure 0:

$$\text{mes } W^s S = 0. \tag{30}$$

This follows from lemma 2 and Bowen's theorem:

Theorem 5 (Bowen [2]). *Consider a C^2 diffeomorphism of a compact manifold, and a hyperbolic invariant set S of this diffeomorphism which is not a maximal attractor in its neighbourhood. Then the attracting set $W^s S$ has Lebesgue measure 0.*

Now we can prove the following analogue of lemma 1:

Lemma 3. *For almost all $(b, x) \in B \times S^1$, there exists $k = k(b, x)$ such that $\mathcal{G}^k(b, x) \in B \times I$.*

Proof. Note that if the orbit of the point (b, x) eventually escaped $B \times J$, it would be pushed towards $B \times I$, and finally inside $B \times I$. Therefore the statement of the lemma fails only for points whose orbit stays inside $B \times J$ forever, i.e. for points of the set

$$T = \bigcap_{k=0}^{\infty} \mathcal{G}^{-k}(B \times J).$$

But $T \subset W^s S$, because any point whose orbit stays forever in $B \times J$ will be attracted to $\Lambda \times J$ (since B is attracted to Λ), and thus will be attracted to S. This and (30) imply that mes $T = 0$. This concludes the proof of the lemma, and with it the proof of statements (a) and (b) of theorem 4. □

All that remains to prove is lemma 2. Let us recall the definition of hyperbolic sets in the form of the cones condition and then check it for the invariant sets A and S. Here we use [17, 13].

For any $q \in X$ and any subspace $E \subset T_q X$, define the cone with the axes space E and opening α to be the set

$$C(q, E, \alpha) = \{v \in T_q X | \tan \angle(v, E) \leqslant \alpha\}.$$

Suppose that A is an invariant set of a diffeomorphism $f : X \to X$. We say that (A, f) satisfy the *cones condition* if the following holds: there exist two values $\alpha^{\pm}$, two continuous families of cones on A:

$$C^+(q) = C(q, E^+, \alpha^+), \quad C^-(q) = C(q, E^-, \alpha^-), \qquad q \in A$$

and two numbers

$$0 < \lambda < 1 < \mu$$

such that for any $q \in A$ the following relations and inequalities hold:

$$df_q C^+(q) \subset C^+(f(q)), \quad df_q^{-1} C^-(q) \subset C^-(f^{-1}(q)), \tag{31}$$

$$|df_q v| \geqslant \mu |v|, \qquad v \in C^+(q), \tag{32}$$

$$|df_q^{-1} v| \geqslant \lambda^{-1} |v|, \qquad v \in C^-(q).$$

Definition 7. *A compact invariant set A of a diffeomorphism f that satisfies the cones condition above is called* hyperbolic.

Proof of lemma 2. Recall that the coordinates on B are (y, z), and the coordinate on the fibre S^1 is x. The cones condition will be checked in a special metric: we will rescale the coordinates x and z and then use the Euclidian metric in the new coordinates. This trick works because the Jacobian matrix of the skew product over the solenoid is block triangular.

Let $\tilde{x} = \eta^2 x$, $\tilde{z} = \eta z$ be new coordinates. Let $\mathrm{d}s^2 = \mathrm{d}\tilde{x}^2 + \mathrm{d}y^2 + \mathrm{d}\tilde{z}^2$. Then, for $\eta > 0$ small, the matrices $\mathrm{d}\mathcal{G}$ and $\mathrm{d}\mathcal{G}^{-1}$ will be almost diagonal:

$$\mathrm{d}\mathcal{G} = \begin{pmatrix} 2 & & \\ & \lambda & \\ & & g_b' \end{pmatrix} + O(\eta), \tag{33}$$

$$\mathrm{d}\mathcal{G}^{-1} = \begin{pmatrix} \frac{1}{2} & & \\ & \lambda^{-1} & \\ & & \frac{1}{g_b' \circ g_b^{-1}} \end{pmatrix} + O(\eta). \tag{34}$$

Conditions (31) and inequalities (32) are open, so they persist under small perturbations of the operators $\mathrm{d}\mathcal{G}$, $\mathrm{d}\mathcal{G}^{-1}$. Therefore, it is sufficient to check them for the first diagonal terms in (33), (34), and then they will immediately follow for $\mathrm{d}\mathcal{G}$, $\mathrm{d}\mathcal{G}^{-1}$ for η small enough.

Proposition 6. *Consider the following decomposition of a vector space:* $E = E^+ \oplus E^-$. *Let* $\mathcal{A} : E \to E$ *be a block diagonal operator corresponding to this decomposition:*

$$\mathcal{A} = \begin{pmatrix} C & \\ & D \end{pmatrix},$$

with $||C^{-1}|| \leqslant \mu^{-1}$, $||D|| \leqslant \lambda$, $0 < \lambda < 1 < \mu$. *Then the cone*

$$C^+ = (0, E^+, \alpha) = \{(v^+, v^-) \in E \text{ such that } |v^-| \leqslant \alpha |v^+|\}$$

for small α *satisfies the following analogues of (31) and (32):*

$$\mathcal{A}C^+ \subset C^+, \tag{35}$$

$$|\mathcal{A}v| \geqslant \frac{\mu}{\sqrt{1+\alpha^2}} |v| \qquad \forall v \in C^+. \tag{36}$$

For α *small enough, the factor* $\mu/\sqrt{1+\alpha^2}$ *will be greater than 1.*

Proof. The proof is immediate. Let $v = (v^+, v^-)$ be the decomposition corresponding to $E = E^+ \oplus E^-$. Then for any $v \in C^+$,

$$|(\mathcal{A}v)^-| = |Dv^-| \leqslant \lambda|v^-| \leqslant \lambda\alpha|v^+| < \frac{\lambda}{\mu}\alpha|Cv^+| \leqslant \alpha|Cv^+| = \alpha|(\mathcal{A}v)^+|.$$

This proves (35). On the other hand, for any $v \in C^+$,

$$|Av| = |(Cv^+, Dv^-)| \geqslant |Cv^+| \geqslant \mu|v^+| \geqslant \frac{\mu}{\sqrt{1+\alpha^2}}|v|.$$

This proves (36). □

We will now prove that the invariant set A of lemma 2 satisfies the cones condition. Take any $q = (b, x) \in A$. Consider

$$E = T_q X = E^+ \oplus E^-, \qquad E^+ = \mathbb{R}\frac{\partial}{\partial y}, \qquad E^- = \mathbb{C}\frac{\partial}{\partial z} \oplus \mathbb{R}\frac{\partial}{\partial x}.$$

Define

$$C : E^+ \to E^+, \qquad C := \text{diag }(2);$$
$$D : E^- \to E^-, \qquad D = \text{diag }(\lambda, \lambda, g_b'(x)).$$

Since $x \in I$, we have $g_b'(x) < 1$. This splitting and these operators satisfy the assumptions of proposition 6. This implies the statement of lemma 2 for $\mathrm{d}\mathcal{G}$ and C^+ on A.

Now let us show that the set S satisfies the cones condition. Take any $q = (b, x) \in S$, and consider

$$E = T_q X = E^+ \oplus E^-, \qquad E^+ = \mathbb{R}\frac{\partial}{\partial y} + \mathbb{R}\frac{\partial}{\partial x}, \qquad E^- = \mathbb{C}\frac{\partial}{\partial z}.$$

Take

$$C : E^+ \to E^+, \qquad C = \text{diag }(2, g_b'(x)),$$
$$D : E^- \to E^-, \qquad D = \text{diag }(\lambda, \lambda).$$

As $x \subset J$, we have $g_b'(x) > 1$. Hence, this splitting and these operators satisfy the assumptions of proposition 6 again. This implies lemma 2 for $\mathrm{d}\mathcal{G}$ and C^+ on S.

Similar statements for $\mathrm{d}\mathcal{G}^{-1}$ and C^- on A and S are proved in exactly the same way. This concludes the proof of lemma 2. □

To conclude the proof of theorem 4, it remains to establish statement (c): the structural stability of the map $\mathcal{G}$ in $D^1(X)$. According to the criterion of structural stability, we need to check two things:

1. the non-wandering set of $\mathcal{G}$ is hyperbolic and periodic orbits are dense in it (Axiom A);
2. the stable and unstable manifolds of the non-wandering points are transversal.

The first statement is already justified. Indeed, we have shown that the non-wandering set of $\mathcal{G}$ is the union of the surfaces A and S defined in lemma 2, where it is also claimed that this set is hyperbolic. The dynamics on A and S is conjugate to the Bernoulli shift, which is known to have a dense set of periodic points.

It remains to check the transversality of the invariant manifolds. Note that the unstable manifolds of the points $p \in A$ belong to A. Indeed, the complete orbits of these points are well defined and belong to the basin of the attraction of A, even to $B \times I$. Hence, they belong to the maximal attractor of $\mathcal{G}|B \times I$ which is A. The same argument proves that the stable manifolds of the points of S belong to S. This implies that stable and unstable manifolds of the non-wandering points p and q of $\mathcal{G}$ may have a non-empty intersection only if these two points belong simultaneously to A or to S.

Consider the case in which $p, q \in A$; the case $p, q \in S$ is treated in the same way. The desired transversality follows from the cones condition: the tangent planes to stable and unstable manifolds of the non-wandering points in A have complementary dimensions and belong to the stable and unstable cones in the tangent space to X. These cones have zero intersection. This completes the proof of theorem 4.

3.4. Almost step skew products over the solenoid

We now construct an 'almost step' skew product over the solenoid, whose attractor has a large invisible part. Naively, a step skew product over the solenoid would be a diffeomorphism $\mathcal{F}$ as in (2), where the fibre maps f_b depend on the digit $\Phi(b)_0$ only. However, if we set $f_b = f_{\Phi(b)_0}$ for some fixed diffeomorphisms $f_0, f_1 : S^1 \to S^1$, the skew product would be discontinuous at $y(b) \in \{0, \frac{1}{2}\} \subset S^1$. We must fix this discontinuity.

Consider two diffeomorphisms $f_0, f_1 : S^1 \to S^1$, and an isotopy

$$f_t : S^1 \to S^1, t \in [0, 1]$$

between them. If f_0, f_1 are both orientation preserving, then we can (and always will) take $f_t = (1 - t^2) f_0 + t^2 f_1$. The choice of the isotopy f_t above makes this family C^1 in y. In this section, numbers in $[0, 1)$ are written in binary representation. For $y \in [0, 1)$, define

$$f_y := \begin{cases} f_0, & \text{for } y \in [0, 0.011); \\ f_{8y-3}, & \text{for } y \in [0.011, 0.1); \\ f_1, & \text{for } y \in [0.1, 0.111); \\ f_{8-8y}, & \text{for } y \in [0.111, 1). \end{cases} \tag{37}$$

The *almost step skew product* over the solenoid, corresponding to the fibre maps f_0, f_1, is defined as

$$\mathcal{F} : X \to X, \qquad \mathcal{F}(b, x) = (h(b), f_{y(b)}(x)) \tag{38}$$

If f_0 and f_1 satisfy the properties of definition 6, then $\mathcal{F}$ will be a North–South skew product, see figure 2. Since we cannot visualize the four-dimensional phase space, we show in this figure the map

$$\mathcal{F}' : S^1_y \times I \to S^1_y \times I, \qquad (y, x) \mapsto (y, f_y(x)).$$

Remark 3. The main feature of almost step skew products is the following. Consider a word $w = (\omega_0 \ldots \omega_{k+1})$ that contains no cluster 11. Consider a sequence ω with the subword w starting at the zero position. Let $b = \Phi^{-1}(\omega)$. Then

$$f_{h^{k-1}(b)} \circ \ldots \circ f_b = f_{\omega_{k-1}} \circ \ldots \circ f_{\omega_0}. \tag{39}$$

Indeed, the binary expansion of $y(h^i(b))$, for any $0 \leqslant i \leqslant k - 1$, starts with the combination $\omega_i \omega_{i+1} \omega_{i+2}$ which is different from 011 or 111. Hence, by definition, $f_{h^i(b)} = f_{\omega_i}$.

Now fix once and for all $f_0, f_1 : S^1 \to S^1$ to be the two diffeomorphisms of section 2.6, and let $\mathcal{F}$ be the almost step skew product over the solenoid corresponding to these two fibre maps. Consider the ball Q_n of radius $1/n^2$ centred at $\mathcal{F}$ in the space $C^1_{p,L}(X)$ of skew products over the solenoid. By definition, this ball consists of skew products:

$$\mathcal{G} : B \times S^1 \to B \times S^1, \qquad \mathcal{G}(b, x) = (h(b), g_b(x))$$

which satisfy

$$\max_B d_{C^1}(f_b^{\pm 1}, g_b^{\pm 1}) \leqslant \frac{1}{n^2}. \tag{40}$$

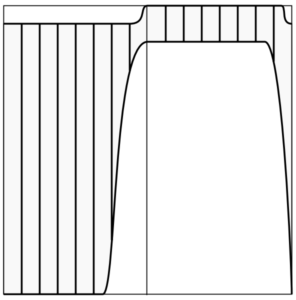

Figure 2. Fibre maps of an almost step skew product.

Note that a linear homotopy between two maps f_0, f_1 consists of North–South skew product diffeomorphisms of the circle. Namely, for any $\tau \in (0, 1)$, the map $f_\tau = (1-\tau)f_0 + \tau f_1$ has a unique attractor in $I = [0, 1]$ and is contracting on I, it has a unique repeller in J and is expanding on J, and has no other fixed points. Hence, $\mathcal{F}$ is a North–South skew product.

Proposition 7. *Let Q_n be the same ball as above. Then any map $\mathcal{G} \in Q_n$ is a North–South skew product over the solenoid map.*

The proof is elementary and very close to that of proposition 4, so we omit it.

Proof of theorem 1. Take any $\mathcal{G} \in Q_n$. By proposition 7, theorem 4 is applicable. Therefore, A_{stat} is homeomorphic to the solenoid Λ, hence connected. Therefore, $\pi(A_{\text{stat}}) \subset S^1$ is connected and therefore an arc. By theorem 4,

$$A_{\text{stat}} = \bigcap_{k=0}^{\infty} \mathcal{G}^k(B \times I). \tag{41}$$

This readily implies the right inclusion $\pi(A_{\text{stat}}) \subset I = [0, 1]$ from (5).

To prove the left inclusion $\pi(A_{\text{stat}}) \supset [\frac{6}{n}, 1-(2/n)]$, note that for any b with $\Phi(b)_0 = 0$, the attractor $a(b)$ of the fibre map g_b is close to that of f_0, and therefore belongs to $(0, 6/n)$. Denote by b_0 the point with the fate $\Phi(b_0) = (\ldots 0 \ldots)$. Then the attractor of g_{b_0} belongs to $(0, (6/n))$.

For any b such that $\Phi(b)_0 = 1$, $\Phi(b)_1 = 0$, the maps g_b, $g_{h(b)}$ are close to f_1, f_0, respectively. For these maps $g_{h(b)} \circ g_b([1-(2/n), 1]) \subset \left[1 - \frac{2}{n}, 1\right]$. Denote by b_{10} the point with the fate $\Phi(b_{10}) = (\ldots 1010 \ldots)$. Then the attractor of map $g_{b_{10}}$ belongs to $(1-(2/n), 1)$.

The point $q_0(\mathcal{G}) := (b_0, a(b_0))$ is a fixed point of $\mathcal{G}|B \times I$. The point $q_{10}(\mathcal{G}) := (b_{10}, a(b_{10}))$ is a periodic point of $\mathcal{G}|B \times I$. By (41), both of these points belong to A_{stat}. Hence, the points $a(b_0)$ and $a(b_{10})$ belong to $\pi(A_{\text{stat}})$. But the attractor A_{stat} is homeomorphic to the solenoid by theorem 4, hence connected. Therefore,

$$\pi(A_{\text{stat}}) \supset [a(b_0), a(b_{10})] \supset \left[\frac{6}{n}, 1 - \frac{2}{n}\right].$$

This proves the left inclusion in (5).

All that is left is to show that the set $V = \pi^{-1}(0, \frac{1}{4})$ is ε-invisible for $\varepsilon = 2^{-n}$. In other words, we must show that the orbits of almost all points $(b, x) \in B \times S^1$ visit V with frequency at most ε. By lemma 3, we may restrict attention to $(b, x) \in B \times I$. Let W be the set of finite words of length $2n$ which do not contain the two-digit sequence 10. These words have the form 0...01...1. The cardinality of W is $2n+1$.

Proposition 8. *Let $k \geqslant 2n$, $(b, x) \in B \times I$ and suppose that $\mathcal{G}^k(b, x) \in V$. If $\omega = \Phi^+(b)$, then*

$$(\omega_{k-2n} \ldots \omega_{k-1}) \in W.$$

Proof. Suppose by contraposition that the conclusion of the Proposition fails. Then let $j \leqslant 2n$ be minimal such that $\omega_{k-j}\omega_{k-j+1} = 10$. Therefore, $g_{h^{k-j}b}$ is $1/n^2$ close to f_1. This implies that

$$\pi(\mathcal{G}^{k-j+1}(b, x)) = g_{h^{k-j}(b)}(\pi(\mathcal{G}^{k-j}(b, x))) > \frac{3}{4} - \frac{1}{n}. \tag{42}$$

Note that inequality (25) persists under linear homotopy. Hence, it holds for any fibre map f_b of $\mathcal{F}$. This implies a similar statement for fibre maps of $\mathcal{G}$:

$$g_{\tilde{b}}(x) > x\left(1 - \frac{1}{2n}\right), \qquad \forall \tilde{b} \in B. \tag{43}$$

Now, applying (43) inductively, we get

$$\pi(\mathcal{G}^k(b, x)) = g_{h^{k-1}b} \circ \ldots \circ g_{h^{k-j+1}b}(\pi(\mathcal{G}^{k-j+1}(b, x))) > \left(\frac{3}{4} - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right)^{j-1}$$

$$> \left(\frac{3}{4} - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right)^{2n}.$$

One sees that the right-hand side of the above inequality is approximated by $(3/4e) > \frac{1}{4}$. A more accurate estimate would indicate that

$$\pi(G^k(\omega, x)) > \left(\frac{3}{4} - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right)^{2n} > \frac{1}{4}$$

for $n \geqslant 100$. This contradicts the assumption of the proposition. □

The ergodicity of the Bernoulli shift implies that subwords in W are met in almost all forward sequences $\omega = (\omega_0\omega_1\omega_2\ldots)$ with frequency 2^{-2n}. But almost all sequences ω correspond under Φ^+ to almost all $b \in B$. Thus we conclude that, for almost all $b \in B$, subwords in W are met in $\Phi^+(b)$ with frequency at most $(2n + 1) \cdot 2^{-2n} < 2^{-n} = \varepsilon$. This and proposition 8 imply that almost all orbits visit V with frequency at most ε, hence V is ε-invisible. □

4. Perturbations

Here we complete the proof of our main result. In the previous subsection, we have proved the assertions of theorem 1 for skew products $\mathcal{G} \in Q_n$. Now we wish to show that the same assertions hold for all nearby diffeomorphisms $\mathcal{H} \in D_L(X)$. In other words, statements 1 and 2 of theorem 1 hold for an open set inside $D_L(X)$ that contains Q_n.

Fix any $\mathcal{G} \in Q_n$, and consider any $\mathcal{H} \in D_L(X)$ which is C^1 close to it. Recall that $I = [0, 1]$. Consider first the maximal attractor of $\mathcal{H}|B \times I$:

$$A^*_{\max}(\mathcal{H}) = \bigcap_{k=0}^{\infty} \mathcal{H}^k(B \times I)$$

This attractor is connected because $B \times I$ is connected. It contains all the complete orbits of $\mathcal{H}$, and in particular it contains fixed points and periodic orbits.

Let $q_0(\mathcal{G})$ and $q_{10}(\mathcal{G})$ be the fixed and periodic points of $\mathcal{G}$ defined in the proof of theorem 1. They are both hyperbolic, and thus persist under small perturbations of $\mathcal{G}$. Hence, the map $\mathcal{H}$ has a fixed point $q_0(\mathcal{H})$ and a periodic point $q_{10}(\mathcal{H})$ close to $q_0(\mathcal{G})$ and $q_{10}(\mathcal{G})$, respectively. If $\mathcal{H}$ is chosen sufficiently close to $\mathcal{G}$, we will have

$$\pi(q_0(\mathcal{H})) \in \left(0, \frac{6}{n}\right), \qquad \pi(q_{10}(\mathcal{H})) \in \left(1 - \frac{1}{2n}, 1\right).$$

Since $q_0(\mathcal{H}), q_{10}(\mathcal{H}) \in \pi(A^*_{\max}(\mathcal{H}))$ and $A^*_{\max}(\mathcal{H})$ is connected, it follows that $A^*_{\max}(\mathcal{H})$ is a circular arc such that

$$\left[\frac{6}{n}, 1 - \frac{2}{n}\right] \subset A^*_{\max}(\mathcal{H}) \subset [0, 1]. \tag{44}$$

By the structural stability of the hyperbolic attractors, $A^*_{\max}(\mathcal{H})$ is hyperbolic. Since $\mathcal{H} \in C^2$, the theorem due to Gorodetski [3] gives

$$A^*_{\max}(\mathcal{H}) = A_{\mathrm{stat}}(\mathcal{H}).$$

Hence, (44) proves conclusion 1 of theorem 1.

As for conclusion 2, let $\mu^{\mathcal{G}}_\infty$ denote the SRB measure for $\mathcal{G}$ (which is described in theorem 4). By statement 2 of theorem 1 and proposition 1, it follows that

$$\mu^{\mathcal{G}}_\infty\left(\pi^{-1}\left(0, \frac{1}{4}\right)\right) \leqslant \varepsilon.$$

The Ruelle theorem on the differentiability of the SRB measure [18] implies that any small perturbation $\mathcal{H}$ of $\mathcal{G}$ has an SRB measure $\mu^{\mathcal{H}}_\infty$, and that this measure depends differentiably on $\mathcal{H}$. In particular, it follows that for $\mathcal{H}$ close enough to $\mathcal{G}$ we will still have

$$\mu^{\mathcal{H}}_\infty\left(\pi^{-1}\left(0, \frac{1}{4}\right)\right) \leqslant \varepsilon.$$

By applying proposition 1 again, it follows that $\pi^{-1}(0, \frac{1}{4})$ is ε-invisible for $\mathcal{H}$.

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