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NO PLANAR BILLIARD POSSESSES AN OPEN SET OF QUADRILATERAL TRAJECTORIES

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ABSTRACT. The article is devoted to a particular case of Ivrii's conjecture on periodic orbits of billiards. The general conjecture states that the set of periodic orbits of the billiard in a domain with smooth boundary in the Euclidean space has measure zero. In this article we prove that for any domain with piecewise C^4-smooth boundary in the plane the set of quadrilateral trajectories of the corresponding billiard has measure zero.

CONTENTS

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1. INTRODUCTION

1.1. **Main results.** Given a domain $\Omega \subset \mathbb{R}^m$ with (piecewise) smooth boundary, consider the *billiard dynamical system* which describes the trajectories of a particle (a billiard ball) moving inside this domain. The ball moves along straight lines inside Ω and reflects against the boundary of Ω by the standard reflection law.

Formally, the phase space of this system is the set of pairs (x, v), where $x \in \partial\Omega$ is a point of reflection and v, $\|v\| = 1$ is the velocity of the ball at this point (a unit vector directed towards the interior of the domain Ω). The billiard map sends a pair (x, v) to the pair (x', v'), where x' is the first point along the ray $\{x + tv \mid t \in (0, +\infty)\}$ that belongs to the border $\partial\Omega$ and v' is the speed of the ball after reflection.

This article is devoted to a particular case of the following long-standing problem.

CONJECTURE 1 (V. Ivrii, 1978)**.** *Given a domain in the Euclidean space with sufficiently smooth boundary, the set of periodic orbits of the corresponding billiard has measure zero.*

More precisely, we study the set of pairs (x, v) such that the orbit of (x, v) under the billiard map is finite.

Ivrii's conjecture was proved by D.Vassiliev in the following particular cases:

1. the boundary $\partial\Omega$ is convex and globally regular analytic [20];
2. the boundary $\partial\Omega$ is piecewise smooth and concave, and the result applies to the case, when Ω is a polyhedron [21].

In 1988 V. Petkov and L. Stojanov [13] proved a stronger version of Ivrii's conjecture for a *typical* domain with smooth boundary. Namely, for a typical domain with smooth boundary the set of periodic orbits of a given period is at most finite.

Clearly, it is sufficient to prove that for every k the set Per_k of k-gonal orbits has measure zero. For $k = 2$, this statement is trivial. For triangular trajectories in a planar billiard (i. e., a billiard in $\Omega \subset \mathbb{R}^2$), this statement was proved by M. Rychlik [16]. Part of his proof is computer-assisted and uses MACSYMA program. Later Rychlik's result was proved in a simple and geometric way by L. Stojanov [19]. Later M. Wojtkowski [24] found another simple proof, and Ya. Vorobets [22] generalized the Rychlik's result to higher dimensions. Later Yu. Baryshnikov and V. Zharnitsky [4] have found yet another proof of Rychlik's result.

We will show that the set of quadrilateral periodic orbits of a planar billiard has measure zero.

THEOREM 2. *For every planar billiard with piecewise C^4 smooth boundary, the set* Per_4 *has measure* 0.

In what follows, μ denotes the Lebesgue measure on the billiard phase space (i. e., the set of pairs (x, v) described above).

Obviously, Theorem 2 is implied by the two following theorems.

THEOREM 3. *Suppose that there exists a planar billiard $\Omega \subset \mathbb{R}^2$ with C^4-smooth boundary such that $\mu\,\mathrm{Per}_4 > 0$. Then there exists a planar billiard with piecewise analytic boundary such that the set* Per_4 *has an inner point.*

THEOREM 4. *For any planar billiard with piecewise analytic boundary, the set* Per_4 *has no inner points.*

We shall prove Theorem 3 in Section 2, and Theorem 4 in Section 3. The proof of the former Theorem heavily uses Pfaffian systems theory, namely Birkhoff distribution. As far as we know, Baryshnikov and Zharnitsky [4] were first who used this approach to Ivrii's conjecture.

We also formulate the following conjecture.

CONJECTURE 5. *There exists a function $r = r(k, m)$ such that the following holds. If there exists a billiard in $\mathbb{R}^m$ with piecewise C^r-smooth boundary such that $\mu\,\mathrm{Per}_k > 0$, then there exists a billiard in $\mathbb{R}^m$ with piecewise analytic boundary such that the set* Per_k *has an inner point in the space of all orbits.*

We will deduce this conjecture from Conjecture 25 (see Section 2.3), which is very close to Cartan–Kuranishi–Rashevsky Theorem. Namely, we shall prove the following theorem.

THEOREM 6. *Conjecture 25 implies Conjecture 5.*

REMARK 7. There exists a problem similar to Ivrii's Conjecture in theory of invisible mirror systems. For more details see [1, 14].

1.2. **From Weyl to Ivrii.** Though Conjecture 1 is a pure billiard theory question, it appeared as a geometrical condition in the following PDE problem.

Consider the Dirichlet problem for the Laplace operator Δ in some domain $\Omega \subset \mathbb{R}^m$. The Laplace operator Δ is a negatively-determined self-adjoint operator, therefore its eigenvalues with the Dirichlet boundary condition $u|_{\partial\Omega} = 0$ are negative real numbers $0 \geqslant -\lambda_1^2 \geqslant -\lambda_2^2 \geqslant \cdots \geqslant -\lambda_k^2 \geqslant \dots$ Denote by $N(\lambda)$ the number of the eigenvalues $-\lambda_i^2$ such that $\lambda_i^2 < \lambda^2$, that is,

$$N(\lambda) = k \iff \lambda_k < \lambda \leqslant \lambda_{k+1}.$$

QUESTION. What is the asymptotic behaviour of the function $N(\lambda)$?

H. Weyl [23] proved that $N(\lambda)$ is asymptotically proportional to λ^m, where m is the dimension of Ω.

THEOREM 8 (H. Weyl, [23], 1911)**.** *Let $\Omega \subset \mathbb{R}^m$ be a domain such that $\mu(\Omega) < \infty$ and $\mu(\partial\Omega) = 0$. Then*

$$N(\lambda) = c_0 \mu(\Omega)\lambda^m + o(\lambda^m),$$

where $c_0 = (2\pi)^{-m} w_m$ and w_m stands for the volume of m-dimensional unit ball.

After proving this theorem, Weyl obtained more precise asymptotics of the function $N(\lambda)$ for $\Omega = [a_1, b_1] \times \cdots \times [a_m, b_m]$. It turns out that in this case

$$N(\lambda) = c_0 \mu(\Omega)\lambda^m - c_1 \mu'(\partial\Omega)\lambda^{m-1} + o(\lambda^{m-1}), \tag{1}$$

where $c_1 = \frac{1}{4}(2\pi)^{m-1}\omega_{m-1}$, and μ' is the $(m-1)$-dimensional measure. Weyl conjectured that the same formula holds for every domain $\Omega \subset \mathbb{R}^m$ with sufficiently (piecewise) smooth boundary.

Many mathematicians, including R. Courant [6], B. Levitan [3], V. Avakumovič [2], L. Hörmander [9, 10], J. Duistermaat, V. Guillemin [7], R. Seeley [18] and V. Ivrii contributed to the proof of this conjecture. The best result was achieved by V. Ivrii [11], who proved Weyl's conjecture for domains satisfying an additional geometric condition.

THEOREM 9 (V. Ivrii, [11], 1980)**.** *Let Ω be a domain in $\mathbb{R}^m$ with infinitely smooth boundary. Suppose that in the corresponding billiard the set of periodic orbits has measure zero. Then for Ω, the asymptotic formula* (1) *holds.*

This geometric condition is analogous to the condition that appears in the same problem for Riemannian manifolds without border. In the latter case we should require the set of closed geodesics to have zero measure.

In [8, Theorem 2.1] N. Filonov and Y. Safarov estimated the difference between the function $N(\lambda)$ corresponding to the Dirichlet problem and the function $N(\lambda)$ corresponding to the Neuman problem for domains that do not necessarily satisfy the above non-periodicity condition. They have also obtained an upper bound for the difference between $N(\lambda)$ and the sum of the two asymptotic terms in Weyl's formula (1). Their bounds expressed in terms of measures of the sets of periodic orbits with a fixed number of reflections. If all these measures are equal to zero, their estimate turns into Ivrii's theorem. This bound follows from inequalities (1.2), (1.3) and (2.3) in [8]. The relation between spectral gaps of the square root of the laplacian and periodic orbits was studied by Yu.Safarov [17]. He had shown that an infinite series of spectral gaps with fixed length may occur only in the case, when almost every trajectory is periodic.

The following story was given from Ivrii's presentation for graduate students.

> In 1980 V. Ivrii gave a talk in Ya. Sinai's seminar (Moscow State University) — one of the best seminars on billiards, and he conjectured (see Conjecture 1) that this geometric condition holds for every domain in the Euclidean space with sufficiently smooth boundary. He was told that this conjecture will be proven in a couple of days... in a week... in a month... in a year...
>
> The conjecture still stands!

As we have noted above, the case of triangular orbits was studied by M. Rychlik [16], L. Stojanov [19] and Ya. Vorobets [22]. We study the case of quadrilateral trajectories in planar billiards.

2. REDUCTION TO THE ANALYTIC CASE

2.1. **Pfaffian systems. Definitions.** Let M be an n-dimensional analytic manifold. Let $\mathscr{F}$ be a d-dimensional analytic distribution on M, i. e. $\mathscr{F}(x)$ is a d-dimensional subspace of T_xM for every $x \in M$ and the map $x \mapsto \mathscr{F}(x)$ is analytic.

DEFINITION 10. An l-dimensional surface $U \subset M$ is called *integral* for $\mathscr{F}$ if $T_xU \subset \mathscr{F}(x)$ for every $x \in U$.

A germ $\phi\colon (\mathbb{R}^l, 0) \to M$, $\operatorname{rk} d\phi = l$ is called *integral* for $\mathscr{F}$ if $\operatorname{Im} d\phi(x) \subset \mathscr{F}(\phi(x))$ for $\|x\|$ sufficiently small.

An r-jet $\phi\colon (\mathbb{R}^l, 0) \to M$, $\operatorname{rk} d\phi = l$ is called *integral* for $\mathscr{F}$ if for every 1-form ω that vanishes on $\mathscr{F}$, $\omega(x)|_{\mathscr{F}(x)} = 0$, the reverse image $(\phi^*\omega)(x)$ is zero $(r-1)$-jet.

REMARK 11. If ϕ is an integral r-jet, then for every 1-form ω that vanishes on $\mathscr{F}$ the reverse image $(\phi^* d\omega)(x)$ is zero $(r-2)$-jet. If ϕ is an integral germ, then $\phi^* d\omega = 0$.

DEFINITION 12. An *analytic pfaffian system* is a tuple $(M, \mathscr{F}, l)$, where M is an analytic manifold, $\mathscr{F}$ is an analytic distribution on M and $l \le \dim \mathscr{F}$ is a natural number. The problem is to find all l-dimensional analytic integral surfaces of the distribution $\mathscr{F}$.

We will need to study not only analytic integral surfaces of a distribution, but we will also be interested in finitely-smooth surfaces which are tangent to the distribution at sufficiently many points.

DEFINITION 13. Let $\mathscr{F}$ be an analytic distribution on M, $U \subset M$ be a C^1-smooth submanifold. Let $V \subset U$ be given by $V = \{x \in U \mid T_xU \subset \mathscr{F}(x)\}$, μ be the Lebesgue measure on U. We will say that U is a *pseudo-integral* surface for $\mathscr{F}$ if $\mu(V) > 0$.

2.2. **The distribution corresponding to Ivrii's problem for fixed m and k.** Fix the number of vertices k, $k \geqslant 3$, and the dimension m of Ω, $m = \dim \Omega \ge 2$.

Denote by $A_1, A_2, \dots A_k$ the vertices of a trajectory of the billiard map. Let $A_0 := A_k$, $A_{k+1} := A_1$. We will be interested only in k-gonal trajectories that are non-degenerate in the following sense.

DEFINITION 14. A k-tuple of points $A_1, \dots, A_k \in \mathbb{R}^m$ is called a *non-degenerate k-gon* if

- consequent vertices do not coincide, i. e. $A_i \ne A_{i+1}$ for $i = 1, \dots, k$;
- none of the angles is equal to π, i. e. $\angle A_{i-1}A_iA_{i+1} \ne \pi$ for $i = 1, \dots, k$.

Otherwise this k-tuple is called a *degenerate k-gon*.

Let us explain why it is natural to require a periodic billiard orbit to be a non-degenerate k-gon. If $A_i = A_{i+1}$, then the reflection law at the vertices A_i and A_{i+1} makes no sense. If $\angle A_{i-1}A_iA_{i+1} = \pi$, then the billiard map is not smooth at $(A_{i-1}, \frac{\overrightarrow{A_{i-1}A_i}}{A_{i-1}A_i})$; moreover, if the line $A_{i-1}A_{i+1}$ and the germ of the boundary $\partial\Omega$ at A_i have 2-point contact, then there exists a ray arbitrarily close to $A_{i-1}A_i$ that does not intersect the border $\partial\Omega$ near A_i.

REMARK 15. Suppose that $A_{i-1} \neq A_i$, the border $\partial\Omega$ is C^r-smooth at the points A_{i-1} and A_i, and the ray $A_{i-1}A_i$ is transversal to the border $\partial\Omega$ at the point A_i. Then the germ of the billiard map at the point $(A_{i-1}, \frac{\overrightarrow{A_{i-1}A_i}}{A_{i-1}A_i})$ is C^{r-1}-smooth. If the line $A_{i-1}A_i$ is transversal to the border $\partial\Omega$ at the point A_{i-1} as well, then this germ has C^{r-1}-smooth inverse.

Clearly, the space of all non-degenerate k-gons is an open subset of $\mathbb{R}^{mk}$.

Consider a billiard table $\Omega \subset \mathbb{R}^m$, and take a periodic non-degenerate orbit $A_1 \dots A_k$. The tangent space to the set of k-gons with vertices in $\partial\Omega$ at $A_1 \dots A_k$ is the Cartesian product of tangent hyperplanes $T_{A_i}\partial\Omega$. Due to the reflection law, $T_{A_i}\partial\Omega$ is the exterior bisector of $\angle A_{i-1}A_iA_{i+1}$ for every $i = 1, \dots, k$, hence the $k(m-1)$-dimensional space $\mathscr{F}(A_1, \dots, A_k) = \bigoplus_{i=1}^k T_{A_i}\partial\Omega \subset T_{A_1\dots A_k}\mathbb{R}^{mk}$ is the same for all domains $\Omega \subset \mathbb{R}^m$ such that $A_1 \dots A_k$ is a periodic trajectory for the corresponding billiard.

Thus we obtain a $k(m-1)$-dimensional distribution $\mathscr{F} = \mathscr{F}_{k,m}$ on the space of all non-degenerate k-gons in $\mathbb{R}^m$, which is called *Birkhoff distribution.* This distribution plays the key role in our proof. To formulate precise statements, we will need the following definition.

DEFINITION 16. We will say that an integral r-jet (germ) ϕ of the distribution $\mathscr{F}_{k,m}$ is *non-trivial* if for every $i = 1, \dots, k$ the composition of ϕ with the projection $\pi_i\colon (A_1, \dots, A_k) \mapsto A_i$ has rank $m-1$.

We will say that an integral surface $U \subset \mathbb{R}^{mk}$ is *non-trivial* if the germ of U at almost every point of U is non-trivial.

We will say that $U \subset \mathbb{R}^{mk}$ is a *non-trivial* pseudo-integral surface for $\mathscr{F}$ if the set V from Definition 13 contains a subset $V' \subset V$ of positive Lebesgue measure such that the germ of U at every point $x \in V'$ is non-trivial.

The following two lemmas show that there is a strong connection between billiard tables with “large” set of k-gonal orbits and non-trivial $2(m-1)$-dimensional integral surfaces of the distribution $\mathscr{F}$.

LEMMA 17. *Suppose that there exists a domain $\Omega \subset \mathbb{R}^m$ with C^r-smooth boundary such that $\mu\,\mathrm{Per}_k > 0$. Then the distribution $\mathscr{F}$ possesses a non-trivial pseudo-integral C^{r-1}-smooth surface $U \subset \mathbb{R}^{mk}$.*

Proof. Broadly speaking, it is sufficient to choose U to be the set of billiard trajectories of length k and V to be the set of Lebesgue points of the set Per_k. The formal construction follows.

Let $\widetilde{M}$ be the phase space of the billiard corresponding to Ω. Then $\widetilde{M}$ is a $(2m-2)$-dimensional C^{r-1}-smooth manifold. The billiard map $\mathbf{B}\colon \widetilde{M} \dashrightarrow \widetilde{M}$ is a C^{r-1} map defined almost everywhere on $\widetilde{M}$ (see Remark 15). Moreover, $\operatorname{rk} d\mathbf{B}(x,v) = 2m-2$ at every point (x,v) in the domain of $\mathbf{B}$.

Consider the map $\pi\colon \widetilde{M} \dashrightarrow \mathbb{R}^{mk}$ that sends each pair (A_1, v) to the corresponding billiard trajectory $A_1 \dots A_k$ of length k. Let $(A_1^0, v^0) \in \mathrm{Per}_k$ be a Lebesgue point of the set $\mathrm{Per}_k \subset \widetilde{M}$. Let $A_1^0 A_2^0 \dots A_k^0 = \pi(A_1^0, v^0)$ be the corresponding periodic trajectory. Definitions of Per_k and $\mathbf{B}$ imply that the polygon $A_1^0 A_2^0 \dots A_k^0$ is non-degenerate, and the vertices of this polygon are non-singular points of the boundary $\partial\Omega$.

Let $\widetilde{U} \subset \widetilde{M}$ be a neighborhood of (A_1^0, v^0) such that

- for every $(A_1, v) \in \widetilde{U}$ the polygon $A_1 A_2 \dots A_k = \pi(A_1, v)$ is non-degenerate and the vertices A_i are non-singular points of the boundary $\partial\Omega$;
- the restriction $\pi|_{\widetilde{U}}$ is a diffeomorphism onto its image.

Then the maps $\mathbf{B}^i$, $i = 1, \dots, k$, have rank $2m-2$ at every point of $\widetilde{U}$.

Let $\widetilde{V}$ be the set of Lebesgue points of the set $\mathrm{Per}_k \cap \widetilde{U}$. Let us show that these $U = \pi(\widetilde{U})$ and $V = \pi(\widetilde{V})$ satisfy the assertion of the lemma.

For every $(A_1, v) \in \mathrm{Per}_k \cap \widetilde{U}$ the tangent space to U at $\pi(A_1, v)$ is a subspace of $\mathscr{F}(\pi(A_1, v))$, therefore for every $(A_1, v) \in \widetilde{V}$ the $(r-1)$-jet of U at $\pi(A_1, v)$ is an integral jet for the distribution $\mathscr{F}$.

It remains to verify the non-triviality condition. This condition immediately follows from the equality $\mathrm{rk}\,\mathbf{B}^i = 2m-2$. □

LEMMA 18. *Suppose that there exists an analytic non-trivial integral surface for $\mathscr{F}$. Then there exists a domain $\Omega \subset \mathbb{R}^m$ with piecewise analytic boundary such that the set Per_k has an inner point.*

REMARK 19. Let us show that one cannot omit the non-triviality condition. Indeed, for $m = 2$ and $k = 4$ for every $X, Y \in \mathbb{R}^2$ and $s > XY$ the family

$$\{(X, A_2, Y, A_4) \mid XA_2 + A_2Y = XA_4 + A_4Y = s\}$$

is a (trivial) two-dimensional integral surface of $\mathscr{F}$ that does not correspond to any billiard table.

Proof. Let U be a non-trivial $2(m-1)$-dimensional analytic integral surface of $\mathscr{F}$. Since U is non-trivial, the images of U under the projections π_i (see Definition 16) are $(m-1)$-dimensional analytic submanifolds. Let $\Omega \subset \mathbb{R}^m$ be a domain such that for every i the germ of $\partial\Omega$ at A_i is $\pi_i(U)$. Clearly, every polygon $A_1 \dots A_k \in U$ is an inner point of the set Per_k for Ω. □

2.3. **Cartan prolongations and Cartan–Kuranishi–Rashevsky Theorem.** Cartan [5] defined a prolongation for pfaffian systems on fibrations. To describe this construction in our settings, we will need to slightly modify the definition of a pfaffian system.

DEFINITION 20. A *pfaffian system with transversality conditions* is a pair of a pfaffian system $(M, \mathscr{F}, l)$ and a tuple of distributions $\mathscr{F}_i$ on M. The problem is to find an integral surface U for $(M, \mathscr{F}, l)$ such that for each i the space T_xU is either transversal to $\mathscr{F}_i(x)$, or intersects it on the origin, i.e.

$$\dim(T_xU \cap \mathscr{F}_i(x)) = \max(0, l + \dim \mathscr{F}_i - \dim M).$$

REMARK 21. One can easily show that the non-triviality condition (see Definition 16) is a transversality condition.

Let $(M, \mathscr{F}, l, \{\mathscr{F}_i\})$ be a pfaffian system with transversality conditions. Consider the grassmanian $\mathrm{Gr}_l(TM) = \{(x, E_l) \mid E_l \in \mathrm{Gr}_l(T_x M)\}$. Consider the analytic distribution $\widetilde{\mathscr{F}}$ on $\mathrm{Gr}_l(TM)$ given as follows: $\widetilde{\mathscr{F}}(x, E_l)$ is the pullback of $E_l \subset T_x M$ under the natural projection $\pi\colon \mathrm{Gr}_l(TM) \to M$. Clearly, for every l-dimensional surface $U \subset M$, its lifting $\widetilde{U} = \{(x, T_x U) \mid x \in U\}$ is an integral surface for this distribution.

DEFINITION 22. A subspace $E \subset T_x M$ is called *integral* for $\mathscr{F}$ if for every 1-form ω that vanishes on $\mathscr{F}$ we have $\omega|_E = 0$ (i. e., $E \subset \mathscr{F}(x)$) and $d\omega|_E = 0$.

Denote by $\widetilde{M} \subset \mathrm{Gr}_l(TM)$ the set of pairs (x, E_l) such that $E_l \subset T_x M$ is an integral plane for $(M, \mathscr{F}, l)$ transversal to all $\mathscr{F}_i$. Clearly, $\widetilde{M}$ is a difference of two analytic subsets in $\mathrm{Gr}_l(TM)$, hence $\widetilde{M}$ is a stratified manifold. Generally speaking, the dimension of the intersection $\widetilde{\mathscr{F}}(y) \cap T_y \widetilde{M}$ may depend on $y \in \widetilde{M}$. Let us subdivide strata of $\widetilde{M}$ so that $\dim(\widetilde{\mathscr{F}}(y) \cap T_y M') = \mathrm{const}$ for every stratum $M' \subset \widetilde{M}$. Due to the definition of $\widetilde{\mathscr{F}}$, this condition implies that the restriction of the natural projection $\pi\colon \mathrm{Gr}_l(TM) \to M$ to each stratum has constant rank.

Then for every integral surface $U \subset M$ for $\mathscr{F}$ we have $\widetilde{U} \subset \widetilde{M}$, and $T_{(x,E_l)}\widetilde{U}$ intersects the plane $dx = 0$ on the origin. This motivates the following definition.

DEFINITION 23. Let $(M, \mathscr{F}, l, \{\mathscr{F}_i\})$ be a pfaffian system with transversality conditions. Let M' be a stratum of the stratification described above, let $\mathscr{F}'$ be the restriction of $\widetilde{\mathscr{F}}$ to M', $\mathscr{F}'(y) = \widetilde{\mathscr{F}}(y) \cap T_y M'$. The system $\mathscr{P}' = (M', \mathscr{F}', l, \{dx = 0\})$ is called *first Cartan prolongation* of the original system.

THEOREM 24 (E. Cartan [5], M. Kuranishi [12], P. Rashevsky [15])**.** *Let $\mathscr{P} = (M, \mathscr{F}, l, \{\mathscr{F}_i\})$ be a pfaffian system with transversality conditions. Suppose that $\mathscr{P}$ has no analytic integral surfaces. Consider a sequence of pfaffian systems $\mathscr{P}^{(r)} = (M^{(r)}, \mathscr{F}^{(r)}, l, \{\mathscr{F}_i^{(r)}\})$, such that $\mathscr{P}^{(r+1)}$ is a first prolongation of $\mathscr{P}^{(r)}$, $\mathscr{P}^{(0)} = \mathscr{P}$. Then there exists r_0 depending on $\mathscr{P}$ and on $\{\mathscr{P}^{(i)}\}$ such that $M^{(r_0)} = \varnothing$.*

CONJECTURE 25. *There exists an estimate for r_0 that does not depend on $\mathscr{F}$ and the sequence of prolongations, but depends only on* $\dim M$.

We are pretty sure that this result must be known in pfaffian systems theory for ages, but we have failed to find a reference. Moreover, it seems that this result can be deduced from the detailed analysis of the proof of the main theorem in [15]. Later we will either publish a short report with a reference, or the proof of this result.

2.4. **Reduction to a Cartan–Kuranishi–Rashevsky like theorem.** In this subsection we will prove Theorem 6, i. e. reduce Conjecture 5 to Conjecture 25. The following theorem is the only statement we need to complete the proof.

THEOREM 26. *Let $\mathscr{P} = (M, \mathscr{F}, l, \{\mathscr{F}_i\})$ be an analytic pfaffian system with transversality conditions, $r \geq 1$. Suppose that $\mathscr{P}$ has a C^{r+1}-smooth pseudo-integral surface U. Then there exists an r-th prolongation $\mathscr{P}^{(r)} = (M^{(r)}, \mathscr{F}^{(r)}, l, \{\mathscr{F}_i^{(r)}\})$ of $\mathscr{P}$ such that $M^{(r)} \neq \varnothing$.*

Before proving this theorem, let us deduce Theorem 6 from it and prove an auxiliary lemma.

Proof of Theorem 6. Fix natural numbers k and m. Suppose that there does not exist a billiard with piecewise analytic boundary such that Per_k has an inner point. Due to Lemma 18, the corresponding pfaffian system with transversality conditions has no analytic integral surfaces. Therefore, due to Conjecture 25, there exists $r = r(k,m)$ such that all r-th prolongations of $\mathscr{F}_{k,m}$ are pfaffian systems on the empty set. Due to Theorem 26, $\mathscr{F}_{k,m}$ has no C^{r+1}-smooth pseudo-integral surfaces. Finally, due to Lemma 17, there does not exist a domain $\Omega \subset \mathbb{R}^m$ with piecewise C^{r+2}-smooth boundary such that $\mu \mathrm{Per}_k(\Omega) > 0$. □

The following lemma will be used twice in the proof of Theorem 26.

LEMMA 27. *Let U be a C^2-smooth pseudo-integral surface for an analytic pfaffian system $(M, \mathscr{F}, l)$. Let V be as in Definition 13. Then for every Lebesgue point x of V the tangent plane T_xU is integral for $\mathscr{F}$.*

Proof. Let ω be an analytic 1-form that vanishes on the planes of $\mathscr{F}$. Let $\varphi\colon (R^l, 0) \to (U, x)$ be a C^2-smooth local coordinate system on U. Then the pullback $\varphi^*\omega$ is a C^1-smooth 1-form on a neighborhood of the origin. Since $\omega(y)$ vahishes on $\mathscr{F}(y)$ for every $y \in V$, the pullback $\varphi^*\omega$ vanishes on $\mathbb{R}^l$ at every point of $\varphi^{-1}(V)$. Clearly, the origin is a Lebesgue point of $\varphi^{-1}(V)$, hence $d(\varphi^*\omega)(0) = 0$. Thus $d\omega(x)|_{T_xU} = 0$. □

Proof of Theorem 26. The proof is by induction on r.

Inductive base, $r = 1$. The statement follows immediately from Lemma 27.

Inductive step. Let U be a C^{r+2}-smooth pseudo-integral surface for $\mathscr{F}$. Broadly speaking, the surface $\widetilde{U} = \{(x, T_xU) \mid x \in U\}$ is a C^{r+1}-smooth pseudo-integral surface for the first prolongation of $\mathscr{P}$. This will allow us to apply the inductive assumption to the first prolongation of the original system.

More precisely, put $\widetilde{U} = \{(x, T_xU) \mid x \in U\}$, $\widetilde{V} = \{(x, T_xU) \mid x \in V\}$. Let $\widetilde{M} \subset \mathrm{Gr}_l(TM)$ be the set of integral planes for $\mathscr{P}$.

Due to Lemma 27, $\widetilde{V} \subset \widetilde{M}$. Let $M' \subset \widetilde{M}$ be a stratum of $\widetilde{M}$ such that the intersection $\widetilde{V} \cap M'$ has positive l-dimensional Lebesgue measure. Let $\mathscr{P}' = (M', \mathscr{F}', l, \{\mathscr{F}_i'\})$ be the prolongation of $\mathscr{P}$ to M'. Shrinking V (hence, $\widetilde{V}$) if required, we can and will assume that $\widetilde{V} \subset M'$ and every point of $\widetilde{V}$ is a Lebesgue point of $\widetilde{V}$ (as a subset of $\widetilde{U}$).

Consider a point $p_0 = (x_0, T_{x_0}U) \in \widetilde{V} \subset M'$, and choose a small neighborhood $\widehat{U} \subset \widetilde{U}$ of p_0 such that the orthogonal projection $\sigma\colon \widehat{U} \to M'$ (with respect to some local analytic Riemannian metric) is well-defined. Put $V' = \widehat{U} \cap \widetilde{V}$. Since $V' \subset M'$, $\sigma|_{V'} = \mathrm{id}$. Since p_0 is a Lebesgue point of V', the tangent space $T_{p_0}M'$

includes $T_{p_0}\widehat{U}$, hence the restriction $d\sigma|_{T_{p_0}\widehat{U}}$ is an embedding. Therefore, σ has rank l in a small neighborhood of p_0. Denote by U' the image $\sigma(\widehat{U})$. Then $\widehat{U}\cap U'\supset V'$, and for every point $p\in V'$ we have $T_pU'=T_p\widehat{U}$. Therefore, U' is a C^{r+1}-smooth pseudo-integral surface for $\mathscr{F}'$.

Due to the inductive assumption, there exists an r-th prolongation $\mathscr{P}^{(r+1)}$ of the prolongation of $\mathscr{P}$ to M' such that $M^{(r+1)}\neq\varnothing$. Since $\mathscr{P}^{(r+1)}$ is an $(r+1)$-st prolongation of $\mathscr{P}$, this completes the proof. □

The following theorem allows us to calculate fewer prolongations if we need to ensure that the existence of a smooth pseudo-integral surface imply the existence of an analytic integral surface.

THEOREM 28. *Let $\mathscr{P}=(M,\mathscr{F},l,\{\mathscr{F}_i\})$ be an analytic pfaffian system with transversality conditions, $r\geq 1$. Suppose that $\mathscr{P}$ has a C^{r+1}-smooth pseudo-integral surface U and all non-empty r-th prolongations of $\mathscr{P}$ are l-dimensional distributions. Then $\mathscr{P}$ has an analytic integral surface.*

Proof. The proof is by induction on r. The inductive step is completely analogous to the inductive step in the proof of Theorem 26. Let us prove the inductive base.

Let $r=1$. Let U be a C^2-smooth integral surface for $\mathscr{P}$. Let us introduce V, $\widetilde{U}$, $\widetilde{V}$, $\widetilde{M}$, M', U' and $\mathscr{F}'$ as in the proof of the inductive step of Theorem 26. Without loss of generality, we assume that $\widetilde{U}\subset M'$. One can achieve this by an orthogonal projection, see the proof of Theorem 26.

Consider the natural projection $\pi\colon M'\to M$. Recall that $\mathscr{F}'(x,E_l)$ is the pullback of $E_l\subset T_xM$ under π. Since $\dim\mathscr{F}'(x,E_l)=l=\dim E_l$, the map π is an immersion.

Fix a point $(x_0,T_{x_0}U)\in\widetilde{V}$ and its neighborhood $\widetilde{W}\subset M'$ such that $\pi|_{\widetilde{W}}$ is a smooth embedding. Set $W=\pi(\widetilde{W})$. By definition, shrinking U one can achieve that $U\subset W$. Choose a branch of $\pi^{-1}|_W$ such that $\pi^{-1}(x_0)=(x_0,T_{x_0}U)$. Denote by $\widehat{\mathscr{F}}$ the distribution on W given by $(x,\widehat{\mathscr{F}}(x))=\pi^{-1}(x)$. Clearly, $\widehat{\mathscr{F}}$ is an l-dimensional analytic distribution and U is a pseudo-integral surface for $\widehat{\mathscr{F}}$. The planes of $\widehat{\mathscr{F}}$ are integral planes for $\mathscr{F}$.

Now we can apply the standard procedure from Frobenius Theorem. Since $\widehat{\mathscr{F}}$ is an l-dimensional distribution, for every $x\in W$ either $\widehat{\mathscr{F}}(x)$ is an integral plane for $\widehat{\mathscr{F}}$, or there are no integral planes for $\widehat{\mathscr{F}}$ in T_xW. Denote by W' the submanifold defined by the condition "$\widehat{\mathscr{F}}(x)$ is an integral plane for $\mathscr{F}$". Due to Lemma 27, $V\subset W'$. Let $W''\subset W'$ be the submanifold defined by "$\widehat{\mathscr{F}}(x)\subset T_xW'$", let $W^{(3)}\subset W''$ be given by "$\widehat{\mathscr{F}}(x)\subset T_xW''$", etc., $W^{(\infty)}=\bigcap_i W^{(i)}$.

Recall that $V\subset W'$, thus $T_xU\subset T_xW'$ for every $x\in V$, hence $V\subset W''$, etc. Therefore, $V\subset W^{(\infty)}$. Thus, $W^{(\infty)}$, being an intersection of a decreasing sequence of analytic manifolds $W^{(i)}$, contains a stratum W_0 of maximal dimension of some manifold $W^{(j)}$, and $\dim W_0\geq l$. The restriction $\widehat{\mathscr{F}}|_{W_0}$ satisfies the assumptions of Frobenius Theorem, hence it possesses an analytic integral surface U^{an}. Clearly, this integral surface is an integral surface for $\mathscr{F}$ as well. □

2.5. **Explicit estimate for the required smoothness.** In this section we shall prove Theorem 3. Let $\Omega \subset \mathbb{R}^2$ be a domain with C^4-smooth boundary such that $\mu \mathrm{Per}_4 > 0$. Due to Lemma 17, there exists a non-trivial pseudo-integral C^3-smooth surface $U \subset \mathbb{R}^8$ for $\mathscr{F}_{4,2}(x)$. Let us prove that all second prolongations of $\mathscr{F}_{4,2}(x)$ are 2-dimensional distributions. This will allow us to apply Theorem 28.

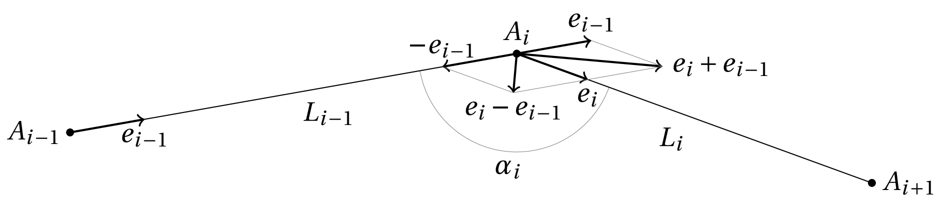


FIGURE 1. Notation for the proof of Theorem 3

Notation. Let L_i be the length of $A_i A_{i+1}$

$$L_i = A_i A_{i+1}.$$

Next, let e_i be the normalized vector $\overrightarrow{A_i A_{i+1}}$; let $e_i^\perp$ be the image of e_i under the rotation through $\pi/2$,

$$e_i = \frac{\overrightarrow{A_i A_{i+1}}}{L_i}; \quad e_i^\perp = R_{\pi/2} e_i.$$

Let α_i be the angle at A_i; let t_i be the tangent of $\alpha_i/2$,

$$\alpha_i = \angle A_{i-1} A_i A_{i+1}; \quad t_i = \tan\left(\frac{\alpha_i}{2}\right).$$

Denote by θ_i and ν_i the following 1-forms,

$$\theta_i = (e_i - e_{i-1}, dA_i); \quad \nu_i = (e_i^\perp, dA_i).$$

Here $(\cdot,\cdot)$ means dot product of a vector and a vector-valued 1-form.

Clearly, $\mathscr{F}_{4,2}$ is defined by $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$. The 1-forms ν_i form a coordinate system on the planes of $\mathscr{F}_{4,2}$.

First prolongation of $\mathscr{F}_{4,2}$. Consider a non-degenerate quadrilateral $A_1 A_2 A_3 A_4$. Let E_4 be the subspace $\mathscr{F}_{4,2}(A_1 A_2 A_3 A_4) \in T_{A_1 A_2 A_3 A_4}\mathbb{R}^8$. Some of the equalities below will hold only if one restricts both sides of the equality to E_4. We shall write $A \overset{\mathscr{F}}{\sim} B$ instead of $A|_{E_4} = B|_{E_4}$.

Since $\theta_i \overset{\mathscr{F}}{\sim} 0$,

$$(e_{i-1}^\perp, dA_i) \overset{\mathscr{F}}{\sim} -(e_i^\perp, dA_i) = -\nu_i; \tag{2}$$

$$(e_i, dA_i) \overset{\mathscr{F}}{\sim} (e_{i-1}, dA_i) \overset{\mathscr{F}}{\sim} t_i (e_i^\perp, dA_i) = t_i \nu_i; \tag{3}$$

$$dL_i = (d\overrightarrow{A_i A_{i+1}}, e_i) = (dA_{i+1}, e_i) - (dA_i, e_i) \overset{\mathscr{F}}{\sim} t_{i+1}\nu_{i+1} - t_i \nu_i \tag{4}$$

Let us compute de_i and $de_i^{\perp}$. Since $\|e_i\| \equiv 1$,

$$de_i = \frac{e_i^{\perp}(d\overrightarrow{A_iA_{i+1}}, e_i^{\perp})}{L_i} = \frac{e_i^{\perp}}{L_i}\left((dA_{i+1}, e_i^{\perp}) - (dA_i, e_i^{\perp})\right) \overset{\mathscr{F}}{\sim} -\frac{e_i^{\perp}}{L_i}(\nu_{i+1} + \nu_i). \tag{5}$$

The last equality holds due to (2). Thus

$$de_i^{\perp} = (de_i)^{\perp} \overset{\mathscr{F}}{\sim} \frac{e_i}{L_i}(\nu_i + \nu_{i+1}). \tag{6}$$

In the sequel we will use the following notation,

$$(\alpha \overset{\wedge}{,} \beta) = \alpha_x \wedge \beta_x + \alpha_y \wedge \beta_y,$$

where $\alpha = (\alpha_x, \alpha_y)$ and $\beta = (\beta_x, \beta_y)$ are vector-valued differential forms.

Now, we will compute $d(e_i, dA_i)$ and $d(e_{i-1}, dA_i)$, then $d\theta_i$.

$$\begin{aligned} d(e_i, dA_i) = (de_i \overset{\wedge}{,} dA_i) &\overset{\mathscr{F}}{\sim} \left(-\frac{e_i^{\perp}}{L_i}(\nu_{i+1} + \nu_i) \overset{\wedge}{,} dA_i\right) \\ &= \frac{(e_i^{\perp}, dA_i)}{L_i} \wedge (\nu_i + \nu_{i+1}) = \frac{\nu_i}{L_i} \wedge (\nu_i + \nu_{i+1}) = \frac{\nu_i \wedge \nu_{i+1}}{L_i}. \end{aligned} \tag{7}$$

$$\begin{aligned} d(e_{i-1}, dA_i) = (de_{i-1} \overset{\wedge}{,} dA_i) &\overset{\mathscr{F}}{\sim} \left(-\frac{e_{i-1}^{\perp}}{L_{i-1}}(\nu_{i-1} + \nu_i) \overset{\wedge}{,} dA_i\right) \\ &= \frac{(e_{i-1}^{\perp}, dA_i)}{L_{i-1}} \wedge (\nu_{i-1} + \nu_i) = \frac{-\nu_i}{L_{i-1}} \wedge (\nu_{i-1} + \nu_i) = \frac{\nu_{i-1} \wedge \nu_i}{L_{i-1}}. \end{aligned} \tag{8}$$

Subtracting (8) from (7), we obtain

$$d\theta_i \overset{\mathscr{F}}{\sim} \frac{\nu_i \wedge \nu_{i+1}}{L_i} - \frac{\nu_{i-1} \wedge \nu_i}{L_{i-1}}. \tag{9}$$

Recall that a 2-dimensional plane $E_2 \subset E_4$ is called integral for $\mathscr{F}_{4,2}$ if $d\theta_i|_{E_2} = 0$ for $i = 1, \dots, 4$. Substituting (9), we get

$$\theta_1|_{E_2} = \theta_2|_{E_2} = \theta_3|_{E_2} = \theta_4|_{E_2} = 0;$$

$$\left.\frac{\nu_1 \wedge \nu_2}{L_1}\right|_{E_2} = \left.\frac{\nu_2 \wedge \nu_3}{L_2}\right|_{E_2} = \left.\frac{\nu_3 \wedge \nu_4}{L_3}\right|_{E_2} = \left.\frac{\nu_4 \wedge \nu_1}{L_4}\right|_{E_2} =: \omega \tag{10}$$

for every integral plane $E_2 \subset T_{A_1A_2A_3A_4}\mathbb{R}^8$.

Consider a non-trivial integral plane E_2. Due to the non-triviality condition, we have $\omega|_{E_2} \neq 0$, thus one can find a basis in E_2 of the form

$$\begin{pmatrix} 0 & L_1 & u_1 & u_2 \\ L_1 & 0 & u_3 & u_4 \end{pmatrix}.$$

Due to (10),

$$\frac{-L_1^2}{L_1} = \frac{L_1u_3}{L_2} = \frac{u_1u_4 - u_2u_3}{L_3} = \frac{u_2L_1}{L_4},$$

hence $u_3 = -L_2$, $u_2 = -L_4$ and $u_1u_4 = L_2L_4 - L_1L_3$. Therefore, E_2 has a basis of the form

$$\begin{pmatrix} 0 & L_1 & \eta & -L_4 \\ L_1 & 0 & -L_2 & \eta' \end{pmatrix} \tag{11}$$

where η and η' are real numbers such that $\eta\eta' = L_2L_4 - L_1L_3$.

The plane E_2 is defined by

$$\begin{aligned} L_1v_3 &= -L_2v_1 + \eta v_2; \\ L_1v_4 &= \eta' v_1 - L_4v_2. \end{aligned} \tag{12}$$

Now we are ready to describe the first Cartan prolongation of $\mathscr{F}_{4,2}$.

Let $M_9 \subset \mathbb{R}^8 \times \mathrm{Gr}_2(\mathbb{R}^8)$ be the phase space of the first prolongation, i. e., the set of pairs $(A_1A_2A_3A_4, E_2)$ such that $A_1A_2A_3A_4$ is a non-degenerate quadrilateral and E_2 is a non-trivial integral plane for $\mathscr{F}_{4,2}$. This set is defined by $\eta\eta' = L_2L_4 - L_1L_3$ in the 10-dimensional submanifold of $R^8 \times \mathrm{Gr}_2(\mathbb{R}^8)$ given by (11). Therefore, $\dim M_9 = 9$.

The first prolongation $\mathscr{F}'$ of $\mathscr{F}_{4,2}$ is given by (12). It is easy to see that $\mathscr{F}'$ is a 3-dimensional distribution on M_9.

Consider the projection of M_9 to $\mathbb{R}^8$. The preimage of each non-degenerate quadrilateral is either a hyperbola $\eta\eta' = L_2L_4 - L_1L_3 \neq 0$, or a pair of lines $\eta\eta' = L_2L_4 - L_1L_3 = 0$. For this reason, we will treat these strata separately.

The stratum $L_1L_3 = L_2L_4$. Denote by $M_7 \subset \mathbb{R}^8$ the 7-dimensional manifold defined by $L_1L_3 = L_2L_4$. Let us find the first prolongation of the restriction of $\mathscr{F}_{4,2}$ to M_7. Clearly, $E_2 \subset T_{A_1A_2A_3A_4}\mathbb{R}^8$ is an integral plane for this restriction if and only if $A_1A_2A_3A_4 \in M_7$, E_2 is an integral plane for $\mathscr{F}_{4,2}$ and $E_2 \subset T_{A_1A_2A_3A_4}M_7$.

Since $L_1L_3 = L_2L_4$, we have $\eta = 0$ or $\eta' = 0$. These cases are analogous to each other, so we will consider only the latter one.

The condition $E_2 \subset T_{A_1A_2A_3A_4}M_7$ means that the form $d(L_2L_4 - L_1L_3)$ vanishes on both basis vectors of E_2. Substituting formulas (4) for dL_i, we get

$$(t_2v_2 - t_1v_1)L_3 + (t_4v_4 - t_3v_3)L_1 = (t_3v_3 - t_2v_2)L_4 + (t_1v_1 - t_4v_4)L_2;$$

Let us evaluate both sides on the basis vectors (11) and substitute $\eta' = 0$. We get

$$\begin{aligned} \eta &= \frac{L_1(t_2 - t_4)(L_3 + L_4)}{t_3(L_4 + L_1)}; \\ t_3 &= t_1. \end{aligned}$$

Therefore, *there is at most one integral plane for $\mathscr{F}_{4,2}|_{M_7}$ passing through each point of* M_7. Namely, there is a unique integral plane if $t_1 = t_3$, and no integral planes otherwise. Thus the first prolongation of the restriction of $\mathscr{F}_{4,2}$ to M_7 is a 2-dimensional distribution. Therefore, all nonempty second prolongations of this restriction are 2-dimensional distributions as well.

Second prolongation of $\mathscr{F}_{4,2}$ over the main stratum $L_1L_3 \neq L_2L_4$. Since $\eta \neq 0$, (12) can be rewritten as

$$\begin{aligned} \eta v_2 &= L_1 v_3 + L_2 v_1; \\ \eta v_4 &= -L_3 v_1 - L_4 v_3. \end{aligned} \tag{13}$$

In order to compute the differentials of both sides of (13), we will need the following "wedge multiplication table" for v_i,

$$\begin{aligned} &v_1 \wedge v_3 \overset{\mathscr{F}'}{\sim} \eta\omega; \quad &&v_1 \wedge v_2 \overset{\mathscr{F}'}{\sim} L_1\omega; \quad &&v_1 \wedge v_4 \overset{\mathscr{F}'}{\sim} -L_4\omega; \\ &v_2 \wedge v_4 \overset{\mathscr{F}'}{\sim} -\eta'\omega; \quad &&v_2 \wedge v_3 \overset{\mathscr{F}'}{\sim} L_2\omega; \quad &&v_3 \wedge v_4 \overset{\mathscr{F}'}{\sim} L_3\omega. \end{aligned} \tag{14}$$

Here $A \overset{\mathscr{F}'}{\sim} B$ means $A|_{E_3} = B|_{E_3}$ for every plane E_3 of $\mathscr{F}'$.

The equalities with $\pm L_i\omega$ on the right-hand side follow from the definition (10) of ω. The first and the fourth equalities follow from (12).

Let us compute dv_i,

$$\begin{aligned} dv_i = d(e_i^\perp, dA_i) = (de_i^\perp \overset{\wedge}{,} dA_i) &\overset{\mathscr{F}}{\sim} \left(\frac{e_i}{L_i}(v_i + v_{i+1}) \overset{\wedge}{,} dA_i\right) \\ = (v_i + v_{i+1}) \wedge \frac{(e_i, dA_i)}{L_i} &\overset{\mathscr{F}}{\sim} (v_i + v_{i+1}) \wedge \frac{t_i v_i}{L_i} = -\frac{t_i v_i \wedge v_{i+1}}{L_i} \overset{\mathscr{F}}{\sim} -t_i\omega. \end{aligned} \tag{15}$$

A plane $E_2' \subset \mathscr{F}'(x)$ is integral for $\mathscr{F}'$ if and only if the exterior derivatives of (13) hold on E_2'. Let us find exterior derivatives of the right hand sides of (13).

$$\begin{aligned} d(L_1v_3 + L_2v_1) &\overset{\mathscr{F}'}{\sim} (t_2v_2 - t_1v_1) \wedge v_3 - L_1t_3\omega + (t_3v_3 - t_2v_2) \wedge v_1 - L_2t_1\omega \\ &\overset{\mathscr{F}'}{\sim} (t_2L_2 - t_1\eta - L_1t_3 - t_3\eta + t_2L_1 - t_1L_2)\omega =: A(L_i; t_i; \eta)\omega \end{aligned}$$

$$\begin{aligned} d(-L_3v_1 - L_4v_3) &\overset{\mathscr{F}'}{\sim} (t_3v_3 - t_4v_4) \wedge v_1 + L_3t_1\omega + (t_4v_4 - t_1v_1) \wedge v_3 + L_4t_3\omega \\ &\overset{\mathscr{F}'}{\sim} (-\eta t_3 - L_4t_4 + L_3t_1 - L_3t_4 - t_1\eta + L_4t_3)\omega =: B(L_i; t_i; \eta)\omega. \end{aligned}$$

Therefore, an integral plane E_2' of $\mathscr{F}'$ is given by

$$\begin{aligned} (d\eta \wedge v_2)|_{E_2'} &= ((A(L_i; t_i; \eta) + \eta t_2)\omega)|_{E_2'}; \\ (d\eta \wedge v_4)|_{E_2'} &= ((B(L_i; t_i; \eta) + \eta t_4)\omega)|_{E_2'}. \end{aligned}$$

Since $(v_2 \wedge v_4)|_{E_2'} \neq 0$, these equations define an unique plane $E_2' \subset \mathscr{F}'(x)$. Therefore, the second prolongation of $\mathscr{F}_{4,2}$ over the main stratum is a 2-dimensional distribution.

Finally, all nonempty second prolongations of $\mathscr{F}_{4,2}$ are 2-dimensional distributions, and $\mathscr{F}_{4,2}$ has a C^3-smooth non-trivial pseudo-integral surface. Due to Theorem 28, $\mathscr{F}_{4,2}$ has an analytic non-trivial integral surface. Thus, due to Lemma 18, there exists a planar billiard with piecewise analytic boundary such that Per_4 has an inner point. This completes the proof of Theorem 3.

3. ANALYTIC CASE

3.1. **Conventions and strategy of the proof.** Recall that our aim is to prove that there does not exist a planar billiard Ω with piecewise analytic boundary such that the set Per_4 has an inner point.

Clearly, the property of being an inner point of the set Per_k is local, i. e. this property depends only on the germs of the boundary $\partial\Omega$ at the vertices of the trajectory. This motivates the following definitions.

DEFINITION 29. Let $\gamma_1, \gamma_2, \dots, \gamma_k \colon \mathbb{R} \to \mathbb{R}^2$ be analytic curves. A k-tuple of points $A_1 A_2 \dots A_k$ is called *a billiard trajectory* for the k-tuple of mirrors $\gamma_1, \dots, \gamma_k$ if $A_i \in \gamma_i$ and the reflection law holds.

We will need to apply this definition for the case when some of the vertices A_i are singular points of the respective mirrors γ_i. Thus we introduce the following convention.

CONVENTION 30. Let $\gamma(t_0)$ be a singular point of an analytic curve γ. We will say that l is the *tangent line* to γ at $\gamma(t_0)$ if

$$l = \lim_{t \to t_0} T_{\gamma(t)}\gamma.$$

In particular, we say that there exists the tangent line at a cusp singular point.

DEFINITION 31. A *k-reflective billiard germ* is a k-tuple of germs of nonconstant analytic maps $\gamma_i \colon (\mathbb{R}, 0) \to (\mathbb{R}^2, A_i)$ such that

- $A_i \neq A_{i+1}$ for $i = 1, \dots, k-1$, $A_k \neq A_1$ (otherwise the reflection law makes no sense);
- the reflection law with respect to the γ_i holds at the A_i, $i = 1, \dots, k$;
- $A_1 A_2 \dots A_k$ is an inner point of the set Per_k of k-gonal billiard orbits.

Clearly, the following statement implies Theorem 4.

THEOREM 32. *There does not exist an analytic* 4*-reflective billiard germ.*

We will prove this theorem instead of Theorem 4. In this subsection we will only give an idea of the proof, and the rest of this section is devoted to the detailed proof.

Assume the converse. Then there exists a 4-reflective analytic billiard germ (a, b, c, d). Let $ABCD = a(0)b(0)c(0)d(0)$ be the corresponding periodic trajectory.

We can extend the mirrors and the families of periodic trajectories analytically. Our strategy will consist in extending the mirrors and the family of periodic trajectories sufficiently far to obtain a contradiction.

Namely, Lemma 41 lists the possible obstructions to analytic extension of a family of 4-periodic trajectories with fixed base vertex $A \in a$. Then Proposition 45, Lemma 50, Proposition 51 and Proposition 60 show that each of these cases holds for at most countable set of base vertices in a. On the other hand, the curve a is uncountable. This contradiction will complete the proof.

3.2. **First observations for k-gonal trajectories.** There are at least three types of objects that one can call a curve: a subset of the plane, a map $\gamma\colon \mathbb{R}\to\mathbb{R}^2$ and a map $\gamma\colon \mathbb{R}\to\mathbb{R}^2$ modulo reparametrization.

CONVENTION 33. Everywhere in this article an analytic curve is a non-constant analytic map $\gamma\colon U\to\mathbb{R}^2$, $U\subset\mathbb{R}$ is an interval, modulo a bianalytic reparametrization. In particular,

- a germ of a curve at a self-intersection point is a germ of *one* of its irreducible branches passing through this point, not a germ of the *union* of these branches;
- a self-intersection point is *not* a singular point provided that all branches are regular curves.

We say that *a curve* $\gamma_1:\mathbb{R}\to\mathbb{R}^2$ *contains a curve* $\gamma_2:\mathbb{R}\to\mathbb{R}^2$, if there exists an analytic mapping $h:\mathbb{R}\to\mathbb{R}$ such that $\gamma_2=\gamma_1\circ h$.

We shall also use the following convention.

CONVENTION 34. If an analytic curve has a limit either in the forward direction, or in the reverse direction, we will attach these limits to the curve and consider them to be singular points of the resulting curve.

As we noted above, we will study analytic extensions of the initial germs. Clearly, these extensions can intersect existing billiard trajectories, so we need to modify the definition of a billiard trajectory.

REMARK 35. In a family of k-periodic billiard trajectories, the vertices of the polygon $A_1\dots A_k$ move in the directions of the exterior bisectors of the angles of this polygon, therefore its perimeter is a constant. We may and will assume that this constant is equal to one, $A_1A_2+\dots+A_{k-1}A_k+A_kA_1=1$.

One of the possible obstructions to the analytic extension of a family of periodic trajectories is *degeneracy* of the limit trajectory. Recall the definition of a non-degenerate k-gon (we just replace m by 2 in Definition 14).

DEFINITION 36. A k-tuple of points $A_1,\dots,A_k\in\mathbb{R}^2$ is called a *non-degenerate k-gon* if

- consequent vertices do not coincide, i. e. $A_i\neq A_{i+1}$ for $i=1,\dots,k$;
- none of the angles is equal to π, i. e. $\angle A_{i-1}A_iA_{i+1}\neq\pi$ for $i=1,\dots,k$.

Otherwise this k-tuple is called a *degenerate k-gon.*

A k-gon such that $A_i=A_{i+1}$ for some i is an obstruction to the extension because the reflection law at A_i makes no sense for such polygons. A k-gon such that $\angle A_{i-1}A_iA_{i+1}=\pi$ is an obstruction to the extension because if, say, the line $A_{i-1}A_{i+1}$ and the mirror γ_i have 2-point contact at A_i, then there exists a ray arbitrarily close to $A_{i-1}A_i$ that does not intersect γ_i near A_i.

Some degenerate and non-degenerate quadrilaterals are shown in Figure 2.

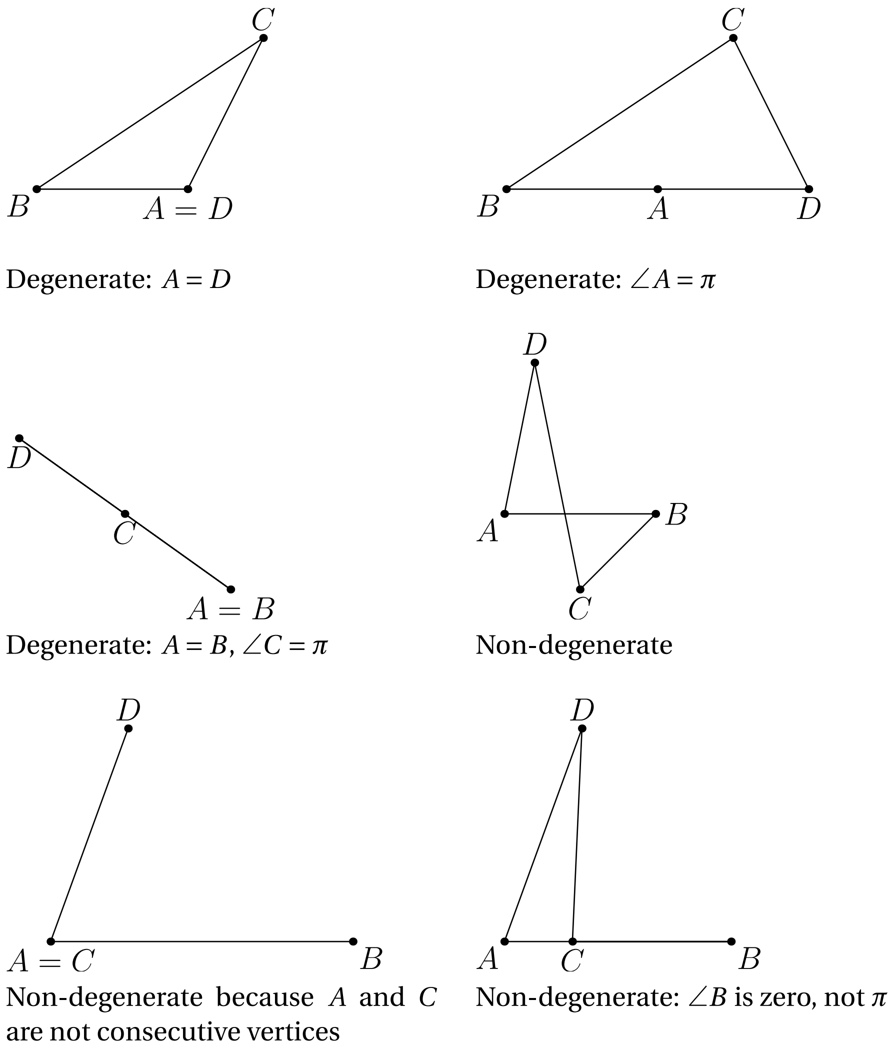


FIGURE 2. Degenerate and non-degenerate quadrilaterals $ABCD$

3.3. **Start of the proof of Theorem 32.** Assume the converse. Then there exists an analytic 4-reflective billiard germ (a, b, c, d). Let us replace these germs by their maximal analytic extensions.

More precisely, given a germ $\gamma\colon (\mathbb{R}, 0) \to \mathbb{R}^2$ we will replace it by a curve $\tilde{\gamma}$ that contains (in the sense of Convention 33) the maximal analytic extension (as a map $\mathbb{R} \to \mathbb{R}^2$) of any analytic reparametrization of γ. This is possible due to the following lemma. This fact should be known for ages but we have not found any reference.

LEMMA 37. *Every germ $\gamma\colon (\mathbb{R}, 0) \to \mathbb{R}^2$ of a real curve admits a unique maximal analytic extension. Namely, there exists an analytic map $\tilde{\gamma}\colon U \to \mathbb{R}^2$, $U \subset \mathbb{R}$ that contains (in the sense of Convention 33) the maximal analytic extension* *of every analytic reparametrization of γ. The map $\tilde{\gamma}$ is unique up to bianalytic reparametrization.*

First, we will prove the local analogue of this lemma.

PROPOSITION 38. *For every germ of curve $\gamma : (\mathbb{R}, 0) \mapsto (\mathbb{R}^2, A)$ there exists a universal germ of curve that contains every analytic parametrization of the given germ γ. If s is the natural parameter of the germ γ, then one can choose $t = s^{\frac{1}{p}}$ as a parameter of the universal germ for appropriate $p \in \mathbb{N}$.*

Proof. It is well-known that every germ of complex analytic curve admits a local injective parametrization (called also local uniformization), unique up to a bianalytic reparametrization. If the curve under consideration is real, then the local uniformization can be chosen real. Moreover, any other analytic parametrization of the germ of curve is the composition of the uniformization and a (not necessarily injective) change of parameter, i.e., the given germ is contained in the uniformization in the sense of Convention 33.

In appropriate coordinates in $\mathbb{R}$ and $\mathbb{R}^2$ such that $A = 0$, the local uniformization of the germ γ has the type $t \mapsto (\gamma_1(t), \gamma_2(t)) = (t^p, ct^q(1 + \phi(t)))$, $0 < p < q$, $c \neq 0$, where ϕ is a germ of analytic function, $\phi(0) = 0$. We have:

$$s(t) = \int_0^t \|\dot{\gamma}(\tau)\| \, d\tau = \int_0^t \sqrt{\dot{\gamma}_1^2(\tau) + \dot{\gamma}_2^2(\tau)} \, d\tau = \int_0^t \tau^{p-1} \chi(\tau) \, d\tau, \tag{16}$$

where $\chi(t) = p + O(t)$ is a germ of analytic function, by the previous formula for γ_j. Thus, the right-hand side in (16) is analytic and equal to $t^p(1 + O(t))$. This completes the proof. □

Proof of Lemma 37. The uniqueness of maximal analytic extension follows from definition. Let us prove its existence.

Consider the unit speed parametrization $\bar{\gamma}_0$ of γ near the origin,

$$\bar{\gamma}_0(s) = \gamma(t(s)), \quad s(t) = \int_0^t \|\dot{\gamma}(\tau)\| \, d\tau,$$

and replace $\bar{\gamma}_0$ by its maximal analytic extension $\bar{\gamma}_1$.

Next we construct a sequence of continuous curves $\bar{\gamma}_i$, $i \in \mathbb{N}$, such that every $\bar{\gamma}_i$ is analytic at all but at most finite set of points, $\|\dot{\bar{\gamma}}_i\| = 1$ at every regular point, and the curve $\bar{\gamma}_{i+1}$ contains $\bar{\gamma}_i$. Namely, if one (or both) of the endpoints of the curve $\bar{\gamma}_i$ is a cusp, then we extend $\bar{\gamma}_i$ beyond this endpoint (resp., both endpoints) till the next singular point. If the curve $\bar{\gamma}_i$ has no limit or tends to infinity in some direction, then we cannot and do not extend it in this direction. The extended curve thus obtained will be denoted $\bar{\gamma}_{i+1}$.

Let $\bar{\gamma}_\infty$ be the union of all these curves $\bar{\gamma}_i$. Recall that a germ of every analytic curve at every singular point in the interior of the domain of the curve is a cusp (we do not count self-intersection points due to Convention 33). Thus $\bar{\gamma}_\infty$ includes (as a set) analytic extension of every analytic reparametrization of γ[1] Now we only need to find an analytic parametrization $\tilde{\gamma}$ of the curve $\bar{\gamma}_\infty$.

[1]A priori, the curve $\bar{\gamma}_\infty$ thus constructed may need an infinite number of extensions and may be an infinite union of the curves $\bar{\gamma}_i$. This means that curve $\bar{\gamma}$ contains an infinite number of

Note that for every parameter value s_i corresponding to a cusp there exists a natural number p_i (given by Proposition 38) such that the curve $\tilde{\gamma}_i\colon t \mapsto \bar{\gamma}_\infty(s_i + t^{p_i})$ is analytic at $t = 0$ and contains every germ of its analytic reparametrization. Thus we can change the analytic structure near each point s_i so that $\bar{\gamma}_\infty$ will become an analytic map from an abstract analytic one-dimensional manifold to the plane. Indeed, it is sufficient to use $\sqrt[p_i]{s - s_i}$ as a new chart near s_i. Any contractible abstract analytic one-dimensional manifold is analytically equivalent to the real line. Hence, there exists a surjective coordinate map s from $\mathbb{R}$ to the definition domain of the curve $\bar{\gamma}_\infty$ equipped with the above structure of analytic manifold. The curve $\tilde{\gamma}\colon \mathbb{R} \to \mathbb{R}^2$, $\tilde{\gamma}(\tau) = \bar{\gamma}_\infty(s(\tau))$ is analytic, and it is a maximal analytic extension of the curve γ. □

We will approach the border of the set Per_4 along the *angle families* $A = \mathrm{const}$, α increases. Formally, fix some initial 4-reflective trajectory $ABCD$. Let us fix the vertex A and start increasing the angle $\alpha = \angle BAD$. Due to the 4-reflectivity of the initial billiard germ, we will obtain a small 1-parametric family $AB_\alpha C_\alpha D_\alpha$ of quadrilateral billiard trajectories. Consider the analytic extension of this family to the maximal possible interval $(\alpha_-, \alpha_+) \subset (0, \pi)$, i. e. we do not try to extend the family beyond $\alpha = 0$ and $\alpha = \pi$.

Clearly, the curves b, c and d contain the curves $\alpha \mapsto B_\alpha$, $\alpha \mapsto C_\alpha$ and $\alpha \mapsto D_\alpha$, respectively.

REMARK 39. The vertices B_α, C_α and D_α can be singular points of the respective curves for some values of $\alpha \in (\alpha_-, \alpha_+)$.

NOTATION. Denote by β_α, γ_α and δ_α the angles $\angle AB_\alpha C_\alpha$, $\angle B_\alpha C_\alpha D_\alpha$ and $\angle C_\alpha D_\alpha A$, respectively. Denote by B_+, C_+, D_+, β_+, γ_+ and δ_+ the limits (if they exist) of B_α, C_α, D_α, β_α, γ_α and δ_α as $\alpha \to \alpha_+$, respectively.

The 4-reflectivity is an analytic condition, hence all trajectories $AB_\alpha C_\alpha D_\alpha$ are 4-reflective. Formally, consider the fourth power of the billiard map, that is, the map of four successive reflections against the border. Since the initial trajectory is 4-reflective, this map is the identity map in some neighbourhood of the initial pair $(A, \frac{\vec{AB}}{AB})$. On the other hand, this map is analytic. Thus its analytic extension along the family of trajectories $AB_\alpha C_\alpha D_\alpha$ is the identity map, hence all trajectories $AB_\alpha C_\alpha D_\alpha$ are 4-reflective.

The following notion will be used in some proofs to consider the similar cases together.

DEFINITION 40. Let $\gamma_1, \gamma_2, \dots, \gamma_k$ be analytic curves. We say that a point X is a *marked point* if it is either a singular point of one of these curves γ_i (including the limits attached to γ_i due to the previous convention), or a self-intersection point of one of the curves γ_i, or an intersection point of two different curves.

---

cusps. In this case it cannot be extended analytically beyond an infinite number of cusps, since every germ of analytic curve either is regular, or has isolated cusp at the base point.

We would like to underline that "two different curves" in this definition means that even if for some $i \neq j$ the curves γ_i and γ_j coincide, we do not mark all the points of γ_i. Thus, the set of marked points is at most countable.

The following lemma provides us the list of possible obstructions to the analytic extension of an angle family.

LEMMA 41. *For any initial quadrilateral one of the following cases holds.*

1. *At least one of the limits* $B_+ = \lim_{\alpha\to\alpha_+} B_\alpha$, $C_+ = \lim_{\alpha\to\alpha_+} C_\alpha$ *and* $D_+ = \lim_{\alpha\to\alpha_+} D_\alpha$ *does not exist.*
2. $AB_+C_+D_+$ *is a degenerate quadrilateral (see Definition 36).*
3. *At least two of the points* B_+, C_+ *and* D_+ *are singular points of the corresponding mirrors*[2].

Proof. Assume the converse, then for some initial quadrilateral

- the limits B_+, C_+ and D_+ exist;
- the quadrilateral $AB_+C_+D_+$ is non-degenerate;
- at most one of the points B_+, C_+, D_+ is a singular point of the corresponding mirror.

Without loss of generality we can assume that B_+ is a regular point of b, and either C_+ or D_+ is a regular point of c or d, respectively. Then we can easily extend the family B_α to some bigger interval. Note that the rays $B_\alpha C_\alpha$ and AD_α are uniquely determined by A, α and B_α. Indeed, the line $B_\alpha C_\alpha$ is the image of the line AB_α under the symmetry with respect to the tangent line to b at B_α, and the ray AD_α is the ray starting from A in the known direction.

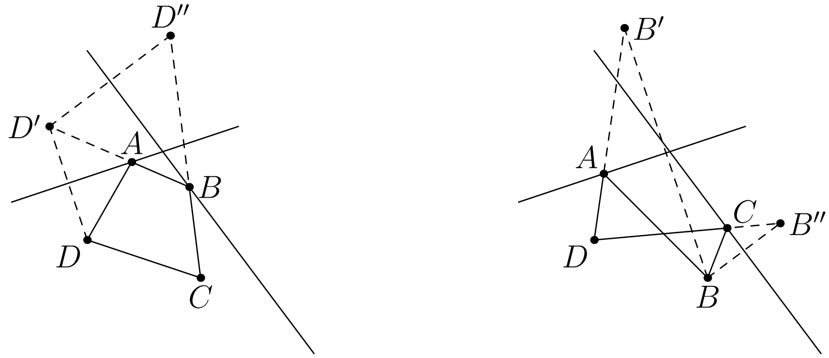
Consider two cases.

Case I. C_+ is a regular point of c, then C_α can be extended to a bigger interval as the intersection point of the ray $B_\alpha C_\alpha$ and the curve c. Hence, we can define the ray $C_\alpha D_\alpha$ for α close enough to α_+ (including the values of α greater than α_+). Therefore we can define D_α as the intersection point of the rays $C_\alpha D_\alpha$ and AD_α. Due to the inequality $\delta_+ \neq \pi$, for α sufficiently close to α_+ this intersection point exists, is unique and analytically depends on α. Finally, we can extend the family $AB_\alpha C_\alpha D_\alpha$ to a bigger interval, which contradicts the assumption that (α_-, α_+) is the maximal interval. Therefore, this case is impossible.

Case II. D_+ is a regular point of d, then D_α can be extended to a bigger interval as the intersection point of the ray AD_α and the curve d. Hence, we can define the ray $D_\alpha C_\alpha$ for α close enough to α_+ (including the values of α greater than α_+). Let us define C_α as the intersection point of the rays $D_\alpha C_\alpha$ and $B_\alpha C_\alpha$. Due to the inequality $\gamma_+ \neq \pi$, this intersection point exists, is unique and analytically depends on α. Finally, we can extend the family $AB_\alpha C_\alpha D_\alpha$ to a bigger interval, which contradicts the assumption that (α_-, α_+) is the maximal interval. Therefore, this case is also impossible.

Finally, both cases are impossible. This completes the proof of the lemma. □

[2] Recall that due to Convention 33 a self-intersection point is not a singular point provided that all the branches passing through this point are regular curves.



(a) The mirrors a and b are straight lines (b) The mirrors a and c are straight lines

FIGURE 3. Two mirrors are straight lines

It is convenient to choose which vertex to fix. In order to avoid renaming of the mirrors in the middle of the proof, we will now rename the mirrors so that the following convention holds.

CONVENTION 42 (Naming convention)**.** We say that a 4-reflective billiard germ (a,b,c,d) with marked mirror a satisfies *the naming convention* if

1. neither a nor c is a line;
2. If a or c is an ellipse, then b or d is a nonsingular curve.

Note that it is possible to rename the mirrors so that the naming convention will hold unless at least two of the mirrors are straight lines. Indeed, if one of the mirrors is a line, let us rename the mirrors so that b is a line, and the naming convention will be satisfied; otherwise, none of the mirrors is a straight line, thus the first condition holds automatically, and it is easy to satisfy the second condition.

LEMMA 43. *At most one of the mirrors a, b, c, d is a straight line.*

The following elegant proof was given by V. Kleptsyn.

Proof. Assume that at least two of the curves a, b, c and d are lines. Let us consider two cases.

Case I. The curves a and b are straight lines (see Fig. 3 (a)). Let us fix a point $D \in d$ and consider a small angle family $A_\delta B_\delta C_\delta D$. Denote by D' the image of the point D under the reflection with respect to the line a. Denote by D'' the image of the point D' under the reflection with respect to the line b. Then for every $C \in c$,

$$DC + CD'' = DC + D'B + BC = DC + DA + AB + BC = 1.$$

Thus c is an ellipse with foci D and D'' *for every* $D \in d$. Therefore all points of the curve d are the foci of the same ellipse which is impossible. Therefore this case is impossible.

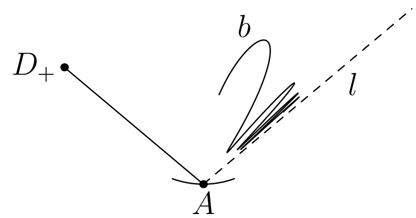


FIGURE 4. Oscillating curve b

Case II. The curves a and c are lines (see Fig. 3 (b)). Let us fix a point $B \in b$ and consider a small angle family $A_\beta B C_\beta D_\beta$. Denote by B' and B'' the images of the point B under the reflection with respect to the lines a and c, respectively. Then for every $D \in d$,

$$B'D + B''D = BA + AD + BC + CD = 1.$$

Thus d is an ellipse with foci B' and B'' for every $B \in b$ which is impossible. Therefore this case is also impossible.

Finally, at most one of the curves a, b, c and d is a line. □

Later we will say "for a generic point $A \in a$" instead of "for a generic point $A \in a$ for every angle family corresponding to this point". In this article we use rather strong notion of genericity.

CONVENTION 44. We say that some property holds *for a generic point* $A \in a$, if it holds for all but at most countable set of points $A \in a$.

The next subsections deal with the cases from Lemma 41 one by one and show that these cases hold for at most countable set of points $A \in a$. Hence there exists a point of the mirror a that satisfies none of these cases, but this contradicts Lemma 41. This contradiction will complete the proof.

3.4. **Existence of the limits.**

In this Subsection we will prove the following proposition.

PROPOSITION 45. *Suppose that the naming convention holds. Then for a generic point $A \in a$ the limits B_+, C_+ and D_+ exist, $B_+ \neq A$ and $D_+ \neq A$.*

In Lemma 46 we will prove that the limits B_+ and D_+ exist and do not coincide with A, and in Lemma 49 we will show that the limit C_+ exists as well.

LEMMA 46. *Suppose that a is not a straight line. Then for a generic point $A \in a$ the limits B_+ and D_+ exist, $B_+ \neq A$ and $D_+ \neq A$.*

Proof. First, let us prove that for a generic point $A \in a$, the limits B_+ and D_+ exist. Due to the symmetry between B and D it is sufficient to show that the limit B_+ exists.

Assume the converse. Then the limit B_+ does not exist for uncountably many points $A \in a$. Take a point $A \in a$ such that the limit B_+ does not exist, see Figure 4.

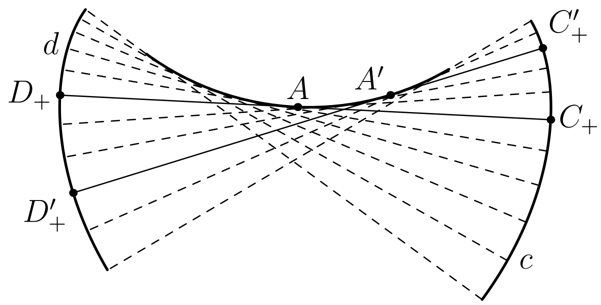


FIGURE 5. A curve and two involutes of this curve

Note that the *line* AB_α is uniquely defined by A and α. Therefore this line tends to some limit position l as $\alpha \to \alpha_+$,

$$l = \lim_{\alpha \to \alpha_+} (\text{line } AB_\alpha).$$

Recall that the perimeter of the quadrilateral $AB_\alpha C_\alpha D_\alpha$ is one, hence B_α belongs to the unit disk centered at A. Therefore, $\operatorname{dist}(B_\alpha, l)$ tends to zero as α tends to α_+.

Since the curve b can oscillate along at most two lines (say, l_1 and l_2), the limit B_+ exists for every point $A \in a$, $A \notin l_1 \cup l_2$. Recall that a is not a line, hence the intersection $a \cap (l_1 \cup l_2)$ is at most countable. Thus, the limit B_+ exists for a generic point A.

Now, let us prove that $B_+ \neq A$ and $D_+ \neq A$. Again, we will only prove that $B_+ \neq A$. Assume the converse, i. e. for uncountably many points $A \in a$ the limit B_+ coincides with A.

Recall that we have attached the limits of the mirror b (if they exist) to the curve b itself, hence $B_+ \in b$. Therefore, the equality $A = B_+$ is possible only if $A \in b$. Note that if $a \neq b$, then the intersection $a \cap b$ is at most countable, thus $A \neq B_+$ for a generic point $A \in a$. Therefore, $a = b$.

Consider the set V of points $A \in a$ such that the limit $B_+(A)$ exists, $B_+(A) = A$ and A neither a marked point nor an inflection point of a. The set of marked points is at most countable, as well as the set of inflection points of a. Therefore, the set V is uncountable.

For $A \in V$, the point B_α tends to A along a regular arc of the mirror a, hence the line AB_α tends to the tangent line to a at A. Therefore, for $A \in V$ the angles α_+ and β_+ must be equal to π, thus the angles γ_+ and δ_+ must be equal to 0. In this case for every $A \in V$ the limits C_+ and D_+ exist and belong to the intersection of $T_A a$ with the mirrors c and d, respectively. Also note that for a generic point $A \in a$ these intersections are regular points of the corresponding curves. Therefore for a generic point $A \in V$ the curves c and d are perpendicular to the tangent line $T_A a$ (reflection law), thus the same holds true for *any* point $A \in a$. Hence the curves c and d are involutes of the mirror a, therefore the curve a is the evolute of c and d (see Figure 5).

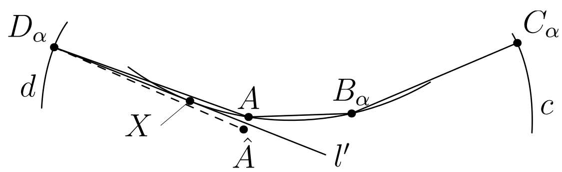


FIGURE 6. Reflection in involute

Note that $A \neq C_+$ and $A \neq D_+$ for $A \in V$. Indeed, the tangent lines to c at C_+ and to d at D_+ are perpendicular to the tangent line to a at A. Therefore the germs (c, C_+) and (d, D_+) cannot coincide with the germ (a, A). Since A is not a marked point, $A \neq C_+$ and $A \neq D_+$.

Consider the trajectory $AB_\alpha C_\alpha D_\alpha$ for $\alpha = \pi - \varepsilon$, $\varepsilon \ll 1$. Let l' be the perpendicular to d at D_α. Let $\hat{A}$ be the image of A under reflection with respect to l'. Since d is an involute for a, the line l' is tangent to a, $l' = T_X a$. Clearly, the segment XB_α intersects the line AD_α and the segment $B_\alpha C_\alpha$ does not intersect this line. Hence, the segment XC_α intersects the line AD_α, and the segment $\hat{A}C_\alpha$ intersects AD_α as well. On the other hand, due to reflection law the ray $D_\alpha \hat{A}$ must coincide with the ray $D_\alpha C_\alpha$, hence the segment $C_\alpha \hat{A}$ does not intersect AD_α. This contradiction completes the proof. □

In order to prove the existence of the limit C_+ we will need the following two easy lemmas.

LEMMA 47. *Suppose that a is not a straight line, and for uncountably many points $A \in a$ the limits B_+ and D_+ exist and are marked points. Then a is an ellipse.*

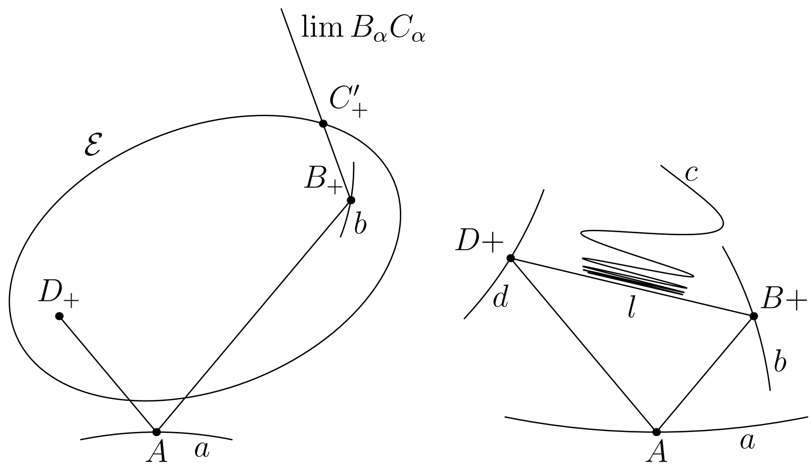
Proof. Note that the function $\phi\colon A \mapsto (B_+, D_+)$ takes countably many values on an uncountable set. Therefore it is a constant on some uncountable subset. Let (B_+^0, D_+^0) be this constant, i. e. $|\phi^{-1}(B_+^0, D_+^0)| > |\mathbb{N}|$. Note that for each point $A \in \phi^{-1}(B_+^0, D_+^0)$ the tangent line $T_A a$ is the exterior bisector of the angle $B_+^0 A D_+^0$. Consider the analytic function $s(A) = AB_+^0 + AD_+^0$. The derivative of this function is equal to zero at uncountably many points, namely at any non-isolated point $A \in \phi^{-1}(B_+^0, D_+^0)$. Hence, $s(A)$ is constant, therefore a is an ellipse or a line. Due to the naming convention, a is not a line, hence a is an ellipse. □

Note that the naming convention implies the assertions of Lemmas 46 and 47.

LEMMA 48. *Suppose that the naming convention holds. Then for a generic point $A \in a$ at least one of the points B_+ and D_+ is a regular point of the corresponding mirror.*

Proof. Assume the converse. Then B_+ and D_+ belong to at most countable set of marked points for uncountably many points $A \in a$. Therefore a is an ellipse but the curves b and d are singular curves. This contradicts our naming convention. □

LEMMA 49. *Suppose that the naming convention holds. Then for a generic point $A \in a$ the limit C_+ exists.*



(a) $1 - s(A, \alpha_+) > B_+D_+$ (b) $1 - s(A, \alpha_+) = B_+D_+$

FIGURE 7. Cases in Lemma 49

Proof. Suppose that the limit C_+ does not exist for uncountably many points $A \in a$.

Denote by $s(A, \alpha)$ the sum $AB_\alpha + AD_\alpha$. Due to Lemma 46, for a generic point $A \in a$ both limits B_+ and D_+ exist, therefore the limit $s(A, \alpha_+)$ of $s(A, \alpha)$ as $\alpha \to \alpha_+$ exists as well. Hence, the limit of the sum $B_\alpha C_\alpha + C_\alpha D_\alpha = 1 - s(A, \alpha)$ also exists and is equal to $1 - s(A, \alpha_+)$, thus the vertex C_α tends to the ellipse $\mathscr{E} = \{X \mid B_+X + XD_+ = 1 - s(A, \alpha_+)\}$.

First consider the case of non-degenerate ellipse $\mathscr{E}$, i. e. $1 - s(A, \alpha_+) > B_+D_+$, see Figure 7 (a). Due to Lemma 48, for a generic point A either B_+ or D_+ is a regular point of the respective mirror. Obviously, it is sufficient to consider the case when B_+ is a regular point of b. In this case both limits $\lim_{\alpha \to \alpha_+} AB_+$ and $\lim_{\alpha \to \alpha_+} T_{B_\alpha} b$ exist, hence the limit of the ray $B_\alpha C_\alpha$ as $\alpha \to \alpha_+$ exists as well. This limit ray intersects the ellipse $\mathscr{E}$ on exactly one point C'_+. Since C_α must tend both to the ray $B_+C'_+$ and to the ellipse $\mathscr{E}$, $\lim_{\alpha \to \alpha_+} C_\alpha = C'_+$. Finally, for a generic point A if $1 - s(A, \alpha_+) > B_+D_+$, then the limit C_+ exists.

Now let us consider the case $1 - s(A, \alpha_+) = B_+D_+$, see Figure 7 (b). In this case C_α must oscillate along the segment B_+D_+. Note that the curve c can oscillate along at most two lines, therefore the line B_+D_+ must be the same (say, l) for uncountably many points A.

Let us prove that none of the mirrors b and d coincide with the line l. As usual, it is sufficient to prove that $b \neq l$. Assume the converse. Recall that due to the naming convention a is not a line, therefore the intersection $a \cap l$ is at most countable. Thus there exist uncountably many points $A \in a$ such that

- the point A does not belong to the line l;
- the line B_+D_+ coincides with the line l;
- the point C_α oscillates along the line l.

Since $A \notin l$, the angle between the lines AB_+ and l is non-zero. Hence, the angle between the line $b = l$ and the reflected ray $B_\alpha C_\alpha$ must tend to the same nonzero number. But in this case the point C_α cannot oscillate along the line l. This contradiction shows that the assumption is false, i. e. none of the mirrors b and d coincide with the line l.

Finally, $b \neq l$ and $d \neq l$, therefore B_+ and D_+ belong to at most countable set of points $(b \cup d) \cap l$ for uncountably many points A. Therefore, B_+ and D_+ do not depend on A for A from some uncountable set. The rest of this paragraph deals only with the points A from this uncountable set. Due to Lemma 47, the curve a is an ellipse. Due to our naming convention, either b or d is either an ellipse or a line.

Without loss of generality we can and will assume that the curve b is an ellipse or a line, thus the limit of $T_B b$ as $B \to B_+$ exists. Note that C_α oscillates along B_+D_+ thus there exists a sequence $\alpha_n \to \alpha_+$ such that the ray $B_{\alpha_n} C_{\alpha_n}$ tends to B_+D_+ as $n \to \infty$. The exterior bisector of the angle $AB_{\alpha_n} C_{\alpha_n}$ is the tangent line to b at the point B_{α_n}, thus the sequence of these bisectors tends to $T_{B_+} b$. Note that both the limit of the sequence of exterior bisectors and the limit of the rays $B_{\alpha_n} C_{\alpha_n}$ do not depend on A, thus the line AB_+ does not depend on A, and the point A must belong to the intersection of this line with the curve a. Therefore, this intersection is uncountable, hence a is a line, which contradicts our naming convention. This contradiction completes the proof. □

3.5. **Case of two singular points.**

The following Lemma reduces the case of two singular points to the case of coinciding limits.

LEMMA 50. *Suppose that the naming convention holds. For a generic point $A \in a$ if two of the points B_+, C_+, D_+ are singular points of the corresponding mirrors, then either $B_+ = C_+$, or $C_+ = D_+$.*

Proof. Assume the converse, then there exist uncountably many points $A \in a$ such that at least two of the points B_+, C_+, D_+ are singular points of the corresponding mirrors, and $B_+ \neq C_+$, $C_+ \neq D_+$.

Due to Lemma 48, for a generic point $A \in a$ either B_+ or D_+ is a regular point of the corresponding mirror, thus either B_+ and C_+, or C_+ and D_+ are singular points of the corresponding mirrors. Due to the symmetry, it is sufficient to consider the former case, B_+ and C_+ are singular points of b and c and $B_+ \neq C_+$.

The set of singular points of an analytic curve is at most countable, thus the set $V(B^0, C^0) = \{A \mid B_+(A) = B^0, C_+(A) = C^0\}$ is uncountable for some two singular points $B^0 \in b$, $C^0 \in c$, $B^0 \neq C^0$. Note that if $A \in V(B^0, C^0) \setminus \{B^0\}$, then $A \neq B_+$ and $B_+ \neq C_+$, hence the limit of the exterior bisector of the angle $AB_\alpha C_\alpha$ as $\alpha \to \alpha_+$ exists. On the other hand, this exterior bisector is the tangent line to b at B_α, thus the limit of the tangent line to b at B as $B \to B_+$ exists.

The line AB_+ is the image of the line B_+C_+ under the reflection with respect to $T_{B_+} b$, hence the line $l = AB_+$ is the same for all points $A \in V(B^0, C^0) \setminus \{B^0\}$. Therefore $V(B^0, C^0) \setminus \{B^0\}$ is a subset of the intersection $l \cap a$ which is at most countable. Thus $V(B^0, C^0)$ is at most countable, which contradicts the statement from the previous paragraph. This contradiction proves the Lemma. □

3.6. **Straight angle case.** The main result of this subsection is the following statement.

PROPOSITION 51. *Suppose that the naming convention holds. For a generic point $A \in a$ if $B_+ \neq C_+$, and $C_+ \neq D_+$, then none of the angles of the quadrilateral $AB_+C_+D_+$ equals π.*

REMARK 52. Recall that for a generic point $A \in a$ the limits B_+, C_+ and D_+ exist and $A \neq B_+$, $A \neq D_+$. The conditions $B_+ \neq C_+$ and $C_+ \neq D_+$ are needed to define the angles of $AB_+C_+D_+$.

We will split the proof of this statement into a few lemmas.

The following three lemmas prove that the angle of measure π cannot appear with another degeneracy for a generic point A. Then we will prove that the straight angle cannot appear without other degeneracies, thus completing the proof of Proposition 51.

LEMMA 53. *Suppose that the naming convention holds. For a generic point $A \in a$ if $B_+ \neq C_+$ and $C_+ \neq D_+$, then at most one of the angles α_+, β_+, γ_+ and δ_+ is equal to π.*

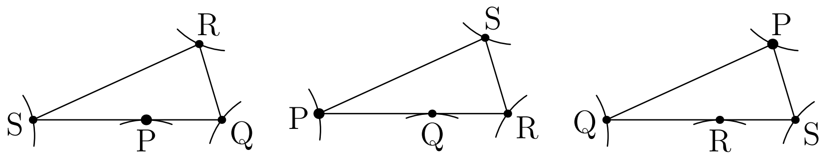
Proof. Suppose that at least two of the angles α_+, β_+, γ_+, δ_+ are equal to π. Then two other angles are equal to 0, and the quadrilateral $AB_+C_+D_+$ is a segment. Note that the angle α always increases, thus $\alpha_+ \neq 0$. Therefore $\alpha_+ = \pi$, hence the line AB_+ is tangent both to a and one of the curves b, c and d. Let p be this other curve, and P be the corresponding vertex.

The set of common tangent lines to two different analytic curves is at most countable, as well as the set of the lines that are tangent to the curve a at two different points (recall that a is not a line). Therefore, $P = A$ and $p = a$. Due to Lemma 46, for a generic point $A \in a$ neither B_+, nor D_+ coincides with A. Hence, $p = c$ and $P = C_+$, i. e. $a = c$ and $A = C_+$.

Using the same arguments as in Lemma 46, one can prove that the mirrors b and d are involutes of the mirror a. Note that for α close enough to π the mirror a has no inflection points between A and C_α. Let l_α be the bisector of the angle $AB_\alpha C_\alpha$. On the one hand, it must intersect the mirror a between the points A and C_α, therefore l_α cannot be tangent to a. On the other hand, it is perpendicular to the involute of a, therefore it must be tangent to a. This contradiction completes the proof. □

LEMMA 54. *Suppose that the naming convention holds. Then there does not exist an uncountable set $V \subset a$ and a point $P \in \mathbb{R}^2$ such that for every $A \in V$ the following conditions hold.*

1. *the limits B_+, C_+ and D_+ exist;*
2. *$A \neq B_+$, $B_+ \neq C_+$, $C_+ \neq D_+$ and $D_+ \neq A$;*
3. *exactly one of the angles of the quadrilateral $AB_+C_+D_+$ equals π;*



Case I: $\angle P = \pi$ Case II: $\angle Q = \pi$ Case III: $\angle R = \pi$

FIGURE 8. One straight angle, one fixed vertex

4. *one of the points A, B_+, C_+, D_+ coincides with P.*

Proof. Assume the converse. Without loss of generality we can and will assume that the same angle of the quadrilateral $AB_+C_+D_+$ equals π for all $A \in V$ and the same vertex coincides with P. Let P, Q, R, S be the vertices of the quadrilateral $AB_+C_+D_+$ enumerated starting from P either in the same or in the opposite cyclic order as A, B_+, C_+, D_+. Denote by p, q, r, s the corresponding mirrors.

Due to the second assumption of the Lemma, the mirrors p, q, r, s have the tangents at the points P, Q, R, S in the sense of Convention 30.

Consider three cases (see Figure 8).

Case I. $\angle P = \pi$. In this case the points S and Q belong to the intersection of the line $T_P p$ with the mirrors s and q, respectively. Note that this intersection is at most countable. Indeed, if either s or q intersects the line $T_P p$ on uncountably many points, then this curve must coincide with $T_P p$, hence either $\angle S = \pi$ or $\angle Q = \pi$ which contradicts Lemma 53. Finally, R also belongs to the countable set of the intersections of two families of lines, namely, the images of the line $T_P p$ under the reflections with respect to the lines $T_Q q$ and $T_S s$. Therefore the set of quadrilaterals $PQRS$ is at most countable. Hence, this case is impossible.

Case II. $\angle Q = \pi$ or $\angle S = \pi$. We will consider only the case $\angle Q = \pi$, because the other case can be reduced to this one by renaming the points. Note that the number of tangent lines to q passing through the point P is at most countable. Therefore the line PQR belongs to at most countable set. Recall that the line RS is the image of the line PR under the reflection with respect to $T_R r$. Note that the curve r cannot coincide with the line PQR. Indeed, otherwise $\angle Q = \angle R = \pi$ which is impossible due to Assumption 3. Therefore the point R belongs to at most countable set, and the line RS belongs to at most countable set as well. Finally, each of the points P, Q, R, S belongs to the union of at most countable set of lines. Therefore, the point A also belongs to the union of at most countable set of lines and due to the naming convention A belongs to at most countable set of points. Thus this case is also impossible.

Case III. $\angle R = \pi$. Let us prove that the set of the possible triangles PQS is discrete. Consider one of the quadrilaterals $PQ_0R_0S_0$ and another quadrilateral $PQRS$ close enough to $PQ_0R_0S_0$. Note that $Q_0R_0S_0$ and QRS are tangent lines to the curve r at close points R and R_0. Therefore the segments QS and Q_0S_0 must intersect each other. On the other hand, the reflection law at vertex P implies

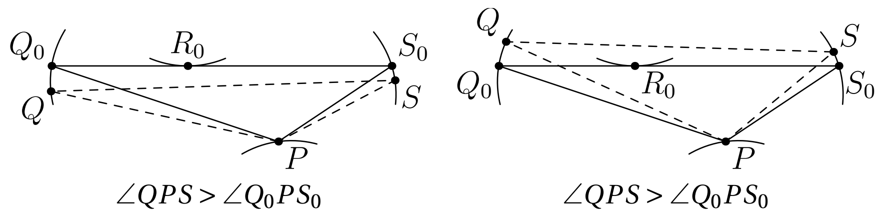


FIGURE 9. Perturbation of a degenerate quadrilateral in Case III

that the directed angles $\angle(\overrightarrow{PS},\overrightarrow{PS_0})$ and $\angle(\overrightarrow{PQ},\overrightarrow{PQ_0})$ have different signs (see Figure 9). Since the line QQ_0 (resp., SS_0) is close to the exterior bisector of the angle $\angle PQ_0S_0$ (resp., $\angle PS_0Q_0$), the points Q and S belong to the same half-plane with respect to the line Q_0S_0, i. e., the segment QS does not intersect the line Q_0S_0. This contradiction proves that Case III is impossible.

Finally, none of the three cases is possible. This proves the lemma. □

LEMMA 55. *Suppose that the naming convention holds. For a generic point $A \in a$ if $B_+ \neq C_+$, $C_+ \neq D_+$ and one of the angles of the quadrilateral $AB_+C_+D_+$ equals π, then none of the vertices of $AB_+C_+D_+$ is a singular point of the respective curve.*

Proof. This lemma is immediately implied by the previous lemma and the fact that the set of singular points of an analytic curve is at most countable. □

So, the previous three lemmas show that the straight angle cannot appear with another degeneracy. The following lemmas prove that the angle of measure π cannot appear alone as well.

LEMMA 56. *Suppose that the naming condition holds. For a generic point $A \in a$ if $B_+ \neq C_+$ and $C_+ \neq D_+$, then none of the angles β_+ and δ_+ equals π.*

Proof. Recall (see Remark 52) that for a generic point $A \in a$ the inequalities $B_+ \neq C_+$ and $C_+ \neq D_+$ imply that the limits β_+, γ_+ and δ_+ exist. Also recall that due to Lemma 55 the points B_+, C_+ and D_+ are regular points of the respective curves.

Assume the converse, i. e. $\beta_+ = \pi$ or $\delta_+ = \pi$ for uncountably many points $A \in a$. Due to the symmetry it is sufficient to consider the case $\beta_+ = \pi$. Note that $\alpha_+ > 0$ thus neither γ_+ nor δ_+ is equal to π (it also follows from Lemma 53). Also note that for a generic point $A \in a$ the curve b and the line AB_+ have only 2-point contact. Consider a trajectory very close to $AB_+C_+D_+$, namely $AB_\alpha C_\alpha D_\alpha$ for $\alpha = \alpha_+ - \varepsilon$, $\varepsilon \ll 1$.

Let us find the order of the length of the segment C_+C_α in two ways, using the path $A \to D \to C$ and using the path $A \to B \to C$.

On the one hand, the angle δ_+ is not equal to π, thus both D_α and the angle of incidence $\delta_\alpha/2$ of AD_α depend smoothly on α at $\alpha = \alpha_+$. Due to the inequality $\gamma \neq \pi$, the point C_α also depends smoothly on α, therefore $C_\alpha C_+ = O(\varepsilon)$.

Recall that $A \neq B_+$ and AB_+ has 2-point contact with b, thus $B_\alpha B_+$ is of the order $\sqrt{\varepsilon}$. Therefore the angle between AB_+ and the tangent line to b at B_α is of the order $\sqrt{\varepsilon}$. Let us compute the angle between $B_\alpha C_\alpha$ and B_+C_+. The angle

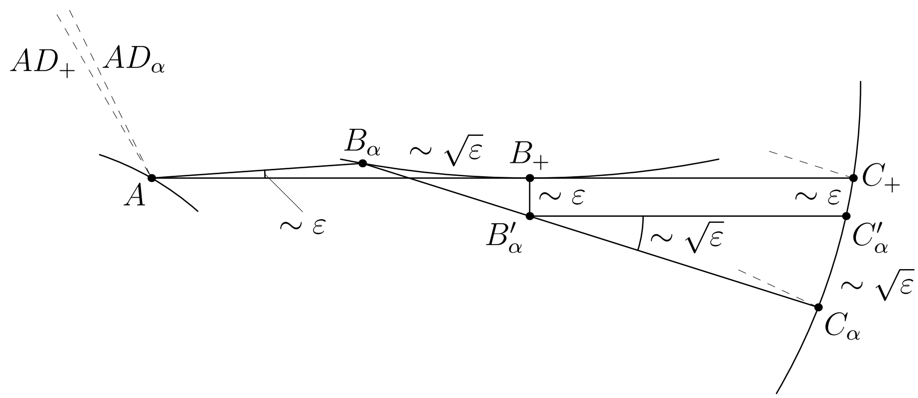


FIGURE 10. A limit trajectory with $\beta_+ = \pi$, and another one close to the limit one

between AB_α and AB_+ is equal to $\varepsilon/2$. Hence, the angle between AB_+ and the image of AB_α under the reflection with respect to the line AB_+ equals $\varepsilon/2$. The line $B_\alpha C_\alpha$ is the image of the same line AB_α under the reflection with respect to the tangent line $T_{B_\alpha} b$. The angle between these two reflecting lines is of the order $\sqrt{\varepsilon}$, therefore the angle between $B_\alpha C_\alpha$ and AB_+ is of the order $\sqrt{\varepsilon}$.

Denote by B'_α the intersection point of the line $B_\alpha C_\alpha$ and the perpendicular $(T_{B_+} b)^\perp$ to b at B_+ (see Figure 10). Note that $B'_\alpha B_+$ is of the order ε. Indeed, the distance between B_α and B_+ is of the order $\sqrt{\varepsilon}$, and the angle between $B_\alpha C_\alpha$ and AB_+ is also of the order $\sqrt{\varepsilon}$, hence the distance between B'_α and the projection of B_α to $(T_{B_+} b)^\perp$ is of the order ε. On the other hand, the distance between this projection and B_+ is also of the order ε. Hence, $B'_\alpha B_+ = O(\varepsilon)$.

Denote by C'_α the intersection point of the mirror c and the line parallel to AC_+ passing through B'_α. The angle between $T_{C_+} c$ and $B_+ C_+$ is non-zero, thus the distance between C_+ and C'_α is of the same order as $B'_\alpha B_+$, i. e. ε. Recall that $B_+ \neq C_+$, therefore the distance between C'_α and C_α is of the same order as the angle between $B_\alpha C_\alpha$ and $B_+ C_+$, i. e. $\sqrt{\varepsilon}$. Thus the distance $C_\alpha C_+$ is of the order $\sqrt{\varepsilon}$.

Finally, we have $C_\alpha C_+ \gtrsim \sqrt{\varepsilon}$ and $C_\alpha C_+ \lesssim \varepsilon$ at the same time which is impossible. Therefore the angle β_+ cannot be equal to π. Recall that due to the symmetry the angle δ_+ cannot be equal to π as well. □

LEMMA 57. *Suppose that the naming convention holds. For a generic point $A \in a$ if $B_+ \neq C_+$ and $C_+ \neq D_+$, then $\alpha_+ \neq \pi$.*

Proof. Assume the converse, i. e. $B_+ \neq C_+$, $C_+ \neq D_+$ and $\alpha_+ = \pi$ for uncountably many points $A \in a$. Recall that due to Lemma 55 for a generic point A the equality $\alpha_+ = \pi$ implies that the vertices of the quadrilateral $AB_+C_+D_+$ are regular points of the corresponding mirrors. Moreover, due to Lemma 53 for a generic point $A \in a$ the equality $\alpha_+ = \pi$ implies that none of the angles β_+, γ_+ and δ_+ is equal to π.

Fix a point A_0 such that all the statements from the previous paragraph hold for A_0. There exists a neighborhood $A_0 \in U \subset a$ and a positive number $\varepsilon > 0$ such that for every point $A \in U$ and any angle $\alpha \in (\pi - \varepsilon, \pi]$ the points $B_\alpha(A)$, $C_\alpha(A)$ and $D_\alpha(A)$ are well-defined regular points of the respective germs of mirrors. Therefore the conditions of the lemma (as well as the genericity conditions from the previous paragraph) hold also for all points $A \in U$. Let us replace U with its subinterval such that the curvature of a is non-zero at all points of U.

Denote by A^s the parametrisation of U by the natural parameter such that the vector $\frac{dA^s}{ds}$ is directed towards the point $B_\pi(A^s)$. Let us show that $C_\pi(A^s)$ does not depend on A^s. To this end consider the families $B^s = B_\pi(A^s)$, $C^s = C_\pi(A^s)$, $D^s = D_\pi(A^s)$, and let us prove that $\frac{dC^s}{dA^s} = 0$. Let k^s be the curvature of the mirror a at a point A^s. We say that k^s is positive if a is locally inside the triangle $B^sC^sD^s$ and negative otherwise. Denote by l^s the line tangent to a at A^s, $l^s = (B^sD^s)$.

Let us compute the derivative $\frac{dC^s}{ds}$ in two ways: using the trajectory $A \to B \to C$, and using the trajectory $A \to D \to C$.

Take a small number ε such that $k^s\varepsilon \geqslant 0$. Note that the angle between the lines l^s and $l^{s+\varepsilon}$ is equal to $k^s\varepsilon + o(\varepsilon)$. Therefore, the angle between the rays $A^{s+\varepsilon}B^{s+\varepsilon}$ and[3] $A^sB_{\pi-2k^s\varepsilon}(A^s)$ is $o(\varepsilon)$ and the distance $\mathrm{dist}(A^s, l^{s+\varepsilon})$ is $o(\varepsilon)$ as well. Hence the length of the segment $B^{s+\varepsilon}B_{\pi-2k^s\varepsilon}$ is $o(\varepsilon)$. Similarly, the angle between the reflected rays $B^{s+\varepsilon}C^{s+\varepsilon}$ and $B_{\pi-2k^s\varepsilon}(A^s)C_{\pi-2k^s\varepsilon}(A^s)$ is $o(\varepsilon)$, and the initial point of the latter ray is $o(\varepsilon)$-close to the former ray. Hence $C^{s+\varepsilon} = C_{\pi-2k^s\varepsilon}(A^s) + o(\varepsilon)$.

On the other hand, the line $A^sD_{\pi-2k^s\varepsilon}(A^s)$ is "nearly parallel" to the line $l^{s-\varepsilon}$, not to the line $l^{s+\varepsilon}$. Therefore applying the same arguments to the path $A \to D \to C$ one can show that $C^{s-\varepsilon} = C_{\pi-2k^s\varepsilon}(A^s) + o(\varepsilon)$. Finally, $C^{s+\varepsilon} = C^{s-\varepsilon} + o(\varepsilon)$ thus $\frac{dC^s}{ds} = 0$ and C^s does not depend on s.

On the other hand, due to Lemma 54 the point $C_\pi(A)$ cannot be the same for uncountably many points $A \in a$. This contradiction proves the lemma. □

LEMMA 58. *Suppose that the naming convention holds. For a generic point $A \in a$ if $B_+ \neq C_+$ and $C_+ \neq D_+$, then $\gamma_+ \neq \pi$.*

Proof. Assume the converse, then for uncountably many points $A \in a$, the point C_+ does not coincide neither with B_+, nor with D_+, and $\gamma_+ = \pi$.

As in the previous lemma, let us choose A_0 such that the limits B_+, C_+, D_+ exist and are regular points of the corresponding mirrors, $A_0 \neq B_+(A_0)$, $B_+(A_0) \neq C_+(A_0)$, $C_+(A_0) \neq D_+(A_0)$, $D_+(A_0) \neq A_0$ and none of the angles α_+, β_+ and δ_+ equals π.

Let us also fix α_0 close to α_+ such that γ_{α_0} is sufficiently close to π, fix a point $C = C_{\alpha_0}$ and start augmenting the angle γ. Obviously, the naming convention will hold for this angular family as well. Note that the points B^γ, A^γ and D^γ will not exit some small neighborhoods of B_+, A_+ and D_+, respectively. Hence, the points A^+, B^+ and D^+ are regular points of the corresponding curves, and $C^+ \neq B^+$, $B^+ \neq A^+$, $A^+ \neq D^+$ and $D^+ \neq C^+$. Therefore, the angle family $A^\gamma B^\gamma C D^\gamma$ extends to the angle $\gamma^+ = \pi$, which is impossible due to Lemma 57. □

[3] The point $B_{\pi-2k^s\varepsilon}(A^s)$ is defined since $k^s\varepsilon \geqslant 0$

Proof of Proposition 51. This proposition follows immediately from Lemmas 56, 57 and 58. □

3.7. **Reduction to the case of coinciding limits.** In this subsection we will summarize the result of the previous subsections into the following proposition.

PROPOSITION 59. *Suppose that the naming convention holds. Then for a generic point $A \in a$ the limits B_+, C_+, D_+ exist and either $B_+ = C_+$, or $C_+ = D_+$.*

Proof. Recall that Lemma 41 states that for *any* point $A \in a$ one of the following cases holds.

1. At least one of the limits $B_+ = \lim_{\alpha\to\alpha_+} B_\alpha$, $C_+ = \lim_{\alpha\to\alpha_+} C_\alpha$ and $D_+ = \lim_{\alpha\to\alpha_+} D_\alpha$ does not exist.
2. $AB_+C_+D_+$ is a degenerate quadrilateral (see Definition 36).
3. At least two of the points B_+, C_+ and D_+ are singular points of the corresponding mirrors[4].

Due to Proposition 45, the first case holds for at most countable set of points $A \in a$. Hence, for a generic point either $AB_+C_+D_+$ is a degenerate quadrilateral, or at least two points among B_+, C_+ and D_+ are singular points of the respective curves.

Due to Lemma 50, for a generic point $A \in a$ the third condition implies $B_+ = C_+$ or $C_+ = D_+$, hence *for a generic point $A \in a$ the quadrilateral $AB_+C_+D_+$ is degenerate.*

Recall that a quadrilateral $AB_+C_+D_+$ is degenerate if either $A = B_+$, or $B_+ = C_+$, or $C_+ = D_+$, or $D_+ = A$, or one of the angles of this quadrilateral equals π. Due to Proposition 45, the equalities $A = B_+$ and $A = D_+$ hold for at most countable set of points $A \in a$. Therefore, for a generic point $A \in a$ either $B_+ = C_+$, or $C_+ = D_+$ or one of the angles of $AB_+C_+D_+$ equals π.

Finally, Proposition 51 states that for a generic point $A \in a$ the latter condition ($\alpha_+ = \pi$ or $\beta_+ = \pi$ or $\gamma_+ = \pi$ or $\delta_+ = \pi$) implies the first one ($B_+ = C_+$ or $C_+ = D_+$). Therefore, for a generic point $A \in a$ either $B_+ = C_+$ or $C_+ = D_+$. □

3.8. **Coinciding limits.** This Subsection is devoted to the following statement.

PROPOSITION 60. *Suppose that the naming convention holds. Then for a generic point $A \in a$ neither $B_+ = C_+$, nor $C_+ = D_+$.*

We will split the proof into a series of lemmas.

First, let us prove that some other degeneracies do not happen at the same time as "$B_+ = C_+$".

LEMMA 61. *Suppose that the naming convention holds and $B_+ = C_+$ for uncountably many points $A \in a$. Then there exists an uncountable set $\Sigma \subset a$ such that for every $A \in \Sigma$*

1. *the limits B_+, C_+ and D_+ exist;*

[4]Recall that due to Convention 33 a self-intersection point is not a singular point provided that all the branches passing through this point are regular curves.

2. *$B_+ = C_+ = X$ is a marked point;*
3. *X is the same point for every $A \in \Sigma$;*
4. *$\triangle AXD_+$ is non-degenerate;*
5. *D_+ is a regular point of d.*

Proof. Due to Proposition 45, the first assertion holds for a generic point $A \in a$.

Let us prove the second assertion. Indeed, otherwise the germ of b at X coincides with the germ of c at X, hence D_+ belongs to the *ray* AB_+, thus $\alpha_+ = 0$ which is impossible.

Let Σ_1 be the set of points $A \in a$ such that the first two assertions hold. Since the set of marked points is at most countable, there exists an uncountable subset $\Sigma_2 \subset \Sigma_1$ such that the first three assertions hold for Σ_2.

Let us prove the fourth assertion. Due to Proposition 45, $A \neq X$ and $A \neq D_+$. Since $\alpha_+ > 0$, it is sufficient to show that $\angle XAD_+ \neq \pi$ for a generic point $A \in a$. Recall that due to the naming convention a is not a line, thus $X \notin T_A a$ for a generic $A \in a$. Therefore, $\triangle AXD_+$ is non-degenerate for a generic $A \in \Sigma_2$. Let Σ_3 be the set of points $A \in \Sigma_2$ such that AXD_+ is non-degenerate.

Let Σ_4 be the set of points $A \in \Sigma_3$ such that D_+ is a regular point of d. Let us show that $\Sigma_3 \setminus \Sigma_4$ is at most countable. Indeed, otherwise D_+ is a marked point, hence D_+ is the same for all A from an uncountable subset $\Sigma' \subset \Sigma_3 \setminus \Sigma_4$. Since AXD_+ is a non-degenerate triangle, the exterior bissector of the angle $AD_\alpha C_\alpha$ tends to the exterior bissector of the angle AD_+X as $\alpha \to \alpha_+$. Thus, $T_{D_+} d$ exists in the sense of Convention 30 and coincides with the exterior bissector of $\angle AD_+X$. Hence Σ' is contained in the intersection of a with the image of the line D_+X under the reflection through the line $T_{D_+} d$. This contradicts the naming convention.

Finally, one can put $\Sigma = \Sigma_4$. □

Next, let us show that X "looks like a transversal intersection of b and c".

LEMMA 62. *Suppose that the naming convention holds and $B_+ = C_+$ for uncountably many points $A \in a$. Let Σ be the set from the previous lemma. Then the limit $\phi(A)$ of the angle between $T_{B_\alpha} b$ and $T_{C_\alpha} c$ exists, is positive, and $\phi|_\Sigma$ is a step function.*

Proof. The composition of reflections with respect to the lines $T_{B_\alpha} b$ and $T_{C_\alpha} c$ tends to the rotation around X through $\pi - \angle AXD_+$. Hence,

$$\lim_{\alpha \to \alpha_+} \angle(T_{B_\alpha} b, T_{C_\alpha} c) = \frac{\pi - \angle AXD_+}{2} =: \phi(A).$$

The right hand side is positive, since the triangle AXD_+ is non-degenerate.

Now let us show that ϕ is a step function. If X is a regular point of c or there exists $T_X c$ in the sense of Convention 30, then the existence of the limit $\phi(A)$ imlies that the tangent line $T_X b$ exists as well. Therefore, $\phi(A) = \angle(T_X b, T_X c)$ does not depend on A.

Suppose that $T_X c$ does not exist. Then $T_X b$ does not exist as well.

Fix two points $A, \widetilde{A} \in \Sigma$ close to each other. Let $\widetilde{A}\widetilde{B}_\kappa\widetilde{C}_\kappa\widetilde{D}_\kappa$ be the angle family corresponding to $\widetilde{A}$. Since neither $T_X b$, nor $T_X c$ exist, the curves $\widetilde{B}_\kappa$ and $\widetilde{C}_\kappa$ are reparametrizations of the curves B_α and C_α, respectively. Hence, $\widetilde{B}_{r(\alpha)} = B_\alpha$ and $\widetilde{C}_{s(\alpha)} = C_\alpha$ for some analytic functions r and s.

We need to show that $\phi(A) = \phi(\widetilde{A})$. Clearly, it is sufficient to show that

$$\liminf_{\substack{\alpha\to\alpha_+\\ \kappa\to\kappa_+}} |\Delta\phi(\alpha,\kappa)| = 0,$$

where

$$\Delta\phi(\alpha,\kappa) = \angle(T_{B_\alpha} b, T_{C_\alpha} c) - \angle(T_{\widetilde{B}_\kappa} b, T_{\widetilde{C}_\kappa} c).$$

Let $\kappa = s(\alpha)$. Then $C_\alpha = \widetilde{C}_\kappa$, hence

$$\Delta\phi(\alpha, s(\alpha)) = \angle(T_{B_\alpha} b, T_{C_\alpha} c) - \angle(T_{\widetilde{B}_{s(\alpha)}} b, T_{\widetilde{C}_{s(\alpha)}} c) = \angle(T_{B_\alpha} b, T_{B_{r^{-1}\circ s(\alpha)}} b).$$

Therefore, it is sufficient to show that

$$\liminf_{\alpha\to\alpha_+} |\angle(T_{B_\alpha} b, T_{B_{t(\alpha)}} b)| = 0, \tag{17}$$

where $t = r^{-1} \circ s$.

Notice that *the line $T_{B_\alpha} b$ cannot pass through A.* Indeed, if $A \in T_{B_{\hat\alpha}} b$, then B_α is not analytic at $\alpha = \hat\alpha$ — a contradiction. Denote by $\psi(\alpha)$ the $\mathbb{R}$-valued azimuth of $T_{B_\alpha} b$. Since $A \notin T_{B_\alpha} b$, the function ψ is bounded in a small neighborhood of α_+. Consider two cases.

Case I. $\sup\{\alpha \mid \psi(\alpha) = \psi(t(\alpha))\} = \alpha_+$. In this case (17) is obvious.

Case II. There exists $\alpha_0 < \alpha_+$ such that $\psi(\alpha) \neq \psi(t(\alpha))$ for $\alpha_0 \le \alpha < \alpha_+$. Without loss of generality, we will assume that $\psi(\alpha) < \psi(t(\alpha))$. Consider a sequence $\alpha_i \to \alpha_+$ such that $\psi(\alpha_i) \to \limsup_{\alpha\to\alpha_+} \psi(\alpha) =: \Psi$. Recall that $\psi(\alpha_i) < \psi(t(\alpha_i))$ and $\limsup_{i\to\infty} \psi(t(\alpha_i)) \le \Psi$. Hence, due to the squeeze lemma, $\psi(t(\alpha_i)) \to \Psi$ as $i \to \infty$. Therefore, $\psi(\alpha_i) - \psi(t(\alpha_i)) \to 0$, which implies (17). □

Proof of Proposition 60. Due to the lemmas above, there exists an uncountable set $\Sigma \subset a$ such that

1. the limits B_+, C_+, D_+ exist for every $A \in \Sigma$;
2. $B_+ = C_+ = X$ is the same point for all $A \in \Sigma$;
3. $\triangle AXD_+$ is non-degenerate, and $\angle AXD_+ = \varphi$ does not depend on $A \in \Sigma$;
4. D_+ is a regular point of d.

The point D_+ is uniquely defined by A, X and φ, i. e. no other information about the curves a, b, c and d is required to find D_+. Indeed, D_+ is the unique point such that $\angle AXD_+ = \varphi$ and $XD_+ + D_+A = 1 - AX$. Hence, *the tangent line $T_A a$ is uniquely defined by A, X and φ*, as is the exterior bissector of the angle $XAD_+(A)$.

Consider the polar coordinate system with the origin at X. The construction described above yields a differential equation $\frac{dr}{d\varphi} = F(r)$, where F is an analytic function $F\colon (0,0.5) \to \mathbb{R}$, $\operatorname{sgn} F = \text{const}$, $\lim_{r\to 0.5^-} F(r) = 0$. The analytic curve a satisfies this equation at an uncountable set of points, thus a is an integral curve for this equation. Therefore, a is a spiral making infinite number of turns around X both as $\varphi \to +\infty$ and as $\varphi \to -\infty$.

The same arguments prove that d is an integral curve for the vector field $\frac{dr}{d\varphi} = -F(r)$. Thus d is the image of a under the reflection through a line passing through X.

The map $\alpha \mapsto C_\alpha$ is not a constant for a generic point $A \in a$. Indeed, if C_α does not depend on α, then b and d are ellipses with foci A and C_α, but d is a spiral. Consider the angle family $A^\gamma B^\gamma C D^\gamma$ with fixed point $C = C_{\alpha_0}$.

Due to Proposition 59, either $A^+ = B^+$, or $A^+ = D^+$.

Case I. Let $A^+ = B^+ = X^*$. Then the curves c and d are spirals around X^*, as in the above discussion. Hence, d is a spiral around two points, X^* and X, thus $X^* = X$ and c is a spiral around X. Therefore, the $\mathbb{R}$-valued azimuth of $T_{C_\alpha} c$ is an unbounded function. This is impossible due to Lemma 62 and the boundness of the azimuth of $T_{B_\alpha} b$, see the proof of the same lemma. Hence, this case is impossible.

Case II. Let $A^+ = D^+ = X^*$. Then the curves b and c are spirals around X^*. Since the azimuth of $T_{B_\alpha} b$ is bounded, $X^* \neq X$.

Therefore, the reflection through XX^* sends a and b to d and c, respectively. Let γ be an angle close to γ^+, so that $A = A^\gamma$ is a generic point of a close to X^*. Consider the angle family $AB_\alpha C_\alpha D_\alpha$ with vertex A fixed at the point constructed above. It is easy to see that for α close enough to α_+, the ray $C_\alpha B_\alpha$ lies between the rays $C_\alpha X$ and $C_\alpha D_\alpha$. Now, let us fix $C = C_\alpha$ at this new position, and consider the new angle family $A^\gamma B^\gamma C D^\gamma$. Since the ray CB^{γ_0} lies between the rays CX and CD^{γ_0}, the vertex B^γ does not leave a small neighborhood of X. Therefore, the value of the angle CX^*B^+ can be made arbitrarily small. On the other hand, this value does not depend on C. This contradiction completes the proof. □

3.9. **Proof of the main theorem.** Now Theorem 32 (and hence Theorem 4) is an easy consequence of Propositions 59 and 60. Indeed, due to Proposition 59 for a generic point A the limits B_+, C_+ and D_+ exist and either $B_+ = C_+$ or $C_+ = D_+$. On the other hand, due to Proposition 60 for a generic point A neither $B_+ = C_+$ nor $C_+ = D_+$. This contradiction completes the proof.

4. FURTHER RESEARCH

In this section we will discuss the case of k-gonal orbits, $k > 4$. We want to use the same strategy, i. e. consider an angle family $A_1 A_2^{\alpha_1} \dots A_k^{\alpha_1}$, $\alpha_1 = \angle A_k A_1 A_2$, and study the limit as the angle α_1 tends to its maximal value $\alpha_1^+ \leqslant \pi$.

4.1. **General case.** The following straightforward generalization of Lemma 41 lists the possible cases for the limit configuration.

LEMMA 63. *Consider a parametric family $A_1 A_2^{\alpha_1} \dots A_k^{\alpha_1}$, where A_1 is a regular point of the corresponding mirror γ_1, $\alpha_1 = \angle A_k A_1 A_2$, $\alpha_1 \in (\alpha_1^-, \alpha_1^+) \subset (0, \pi)$. Then one of the following cases holds.*

1. *At least one of the limits $A_i^+ = \lim_{\alpha_1 \to \alpha_1^+} A_i^{\alpha_1}$ does not exist.*
2. *$A_1 A_2^+ A_3^+ \dots A_k^+$ is a degenerate k-gon (see Definition 36).*

3. *At least two of the points* A_i^+ *are singular points of the corresponding mirrors.*

It seems that this lemma lists the same obstructions as Lemma 41 but actually for $k > 4$ there are much more possible *combinations* of these obstructions. Of course, some of the lemmas developed for the case $k = 4$ can be generalized for $k > 4$, but they do not cover all cases.

Let us list some difficulties that appear only for $k > 4$.

- Some of the limits A_i^+ do not exist.
- At least two of the angles α_i^+ are equal to π.
- One of the angles α_i^+ is equal to π and one of the vertices A_i^+ is a singular point of the respective curve.
- Two consequent vertices coincide, $A_i^+ = A_{i+1}^+$.

There are other cases (say, $A_2^+ = A_3^+$ and one of the angles α_i^+ is equal to π) but we believe that the cases above are the most important.

4.2. **Current status for** $k = 5$**.** As we stated above, the straightforward generalizations of our lemmas do not cover all possible cases even for $k = 5$. The cases that are not covered by these generalizations are sketched in Figures 11 and 12. The vertices known to be marked points are indicated by small empty circles, the vertices known to be regular (non-marked) points are indicated by small black disks, and the points that can be either marked, or non-marked, are indicated by black halfdisks.

One can prove that some of these cases are impossible. For the case of two straight angles, this was proved by V. Kleptsyn. But explaining the ideas required to this proof would take much space, and we still did not proved that *all* of these cases are impossible. Some generalizations used for restricting the list of possible cases will be formulated in the next subsection.

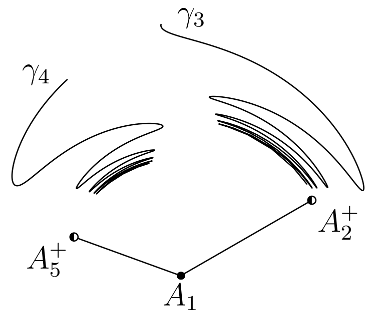
4.3. **Straightforward generalizations.** In this subsection we will formulate some straightforward generalizations of the lemmas used in this article. Since these lemmas do not lead immediately to any remarkable result for $k > 4$, we will not prove them.

LEMMA 64 (cf. Lemma 43)**.** *Let* $\{\gamma_i\}_{i=1}^k$ *be a* k*-reflective billiard germ. Then there are at most* $k-3$ *straight lines among the mirrors* γ_i*.*

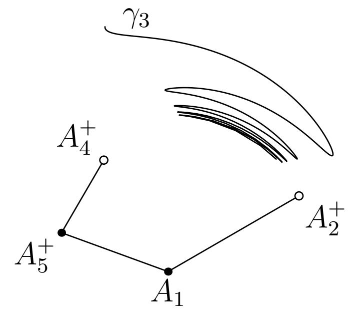
LEMMA 65 (cf. Lemma 46)**.** *Suppose that* γ_1 *is not a straight line. Then for a generic point* $A_1 \in \gamma_1$ *the limits* A_2^+ *and* A_k^+ *exist. If* $k = 5$*, then either* $A_2^+ \neq A_1$*, or* $A_5^+ \neq A_1$*.*

LEMMA 66 (cf. Lemma 49)**.** *Let* p *be a natural number,* $3 \leqslant p \leqslant k-1$*. For a generic point* $A_1 \in \gamma_1$ *the following implication holds. Suppose that the following holds.*

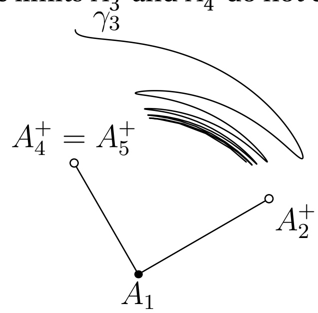
- *All the limits* A_i^+*,* $i \neq p$ *exist.*
- *All the points* A_i^+*,* $i \neq p$ *are regular points of the corresponding mirrors.*
- *For any* $i = 1, \dots, p-2, p+1, \dots, n$ *the limit* A_i^+ *does not coincide with the limit* A_{i+1}^+*.*



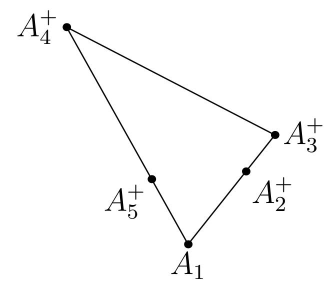
The limits A_3^+ and A_4^+ do not exist



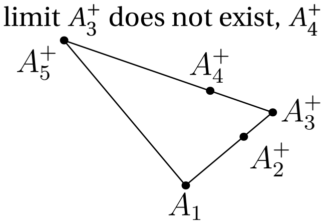
The limit A_3^+ does not exist



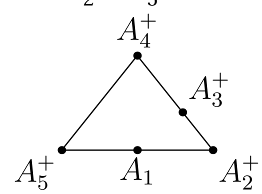
The limit A_3^+ does not exist, $A_4^+ = A_5^+$



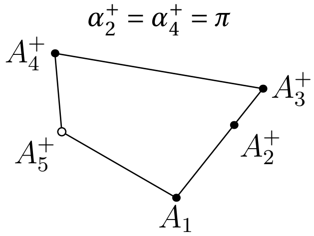
$\alpha_2^+ = \alpha_5^+ = \pi$



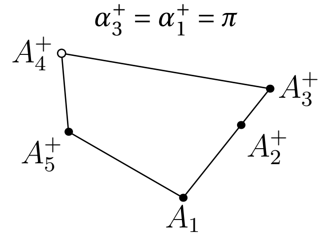
$\alpha_2^+ = \alpha_4^+ = \pi$



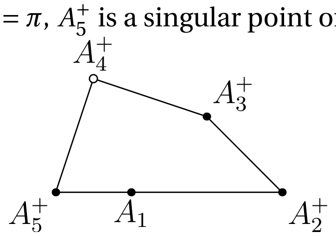
$\alpha_3^+ = \alpha_1^+ = \pi$



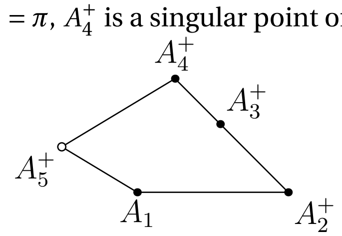
$\alpha_2^+ = \pi$, A_5^+ is a singular point of γ_5



$\alpha_2^+ = \pi$, A_4^+ is a singular point of γ_4



$\alpha_1^+ = \pi$, A_4^+ is a singular point of γ_4



$\alpha_3^+ = \pi$, A_5^+ is a singular point of γ_5

FIGURE 11. "Non-trivial" cases for $k = 5$. Part 1.

- *The perimeter of the* $(k-1)$*-gon* $A_1A_2^+\dots A_{p-1}^+A_{p+1}^+\dots A_k$ *is less than one.*

Then the limit A_p^+ *exists.*

Notice that for $k \geqslant 5$ these two lemmas do not imply existence of all the limits A_i^+.

LEMMA 67 (cf. Lemma 56)**.** *A tangency* $\angle A_1A_2^+A_3^+ = \pi$ *cannot be the only obstruction to the analytic extension of the angle family, i. e. it is impossible that all the following conditions hold:*

- *the limit* A_i^+ *exists for every* $i = 2,\dots,k$*;*
- $A_i^+ \neq A_{i+1}^+$ *for* $i = 2,\dots,k-1$*,* $A_k^+ \neq A_1$*,* $A_1 \neq A_2^+$*;*
- $\angle A_{i-1}^+A_i^+A_{i+1}^+ \neq \pi$ *for* $i = 3,\dots,k-1$*,* $\angle A_{k-1}^+A_k^+A_1 \neq \pi$*,* $\angle A_k^+A_1A_2^+ \neq \pi$*;*
- *each limit* A_i^+ *is a regular point of the corresponding curve;*
- $\angle A_1A_2^+A_3^+ = \pi$.

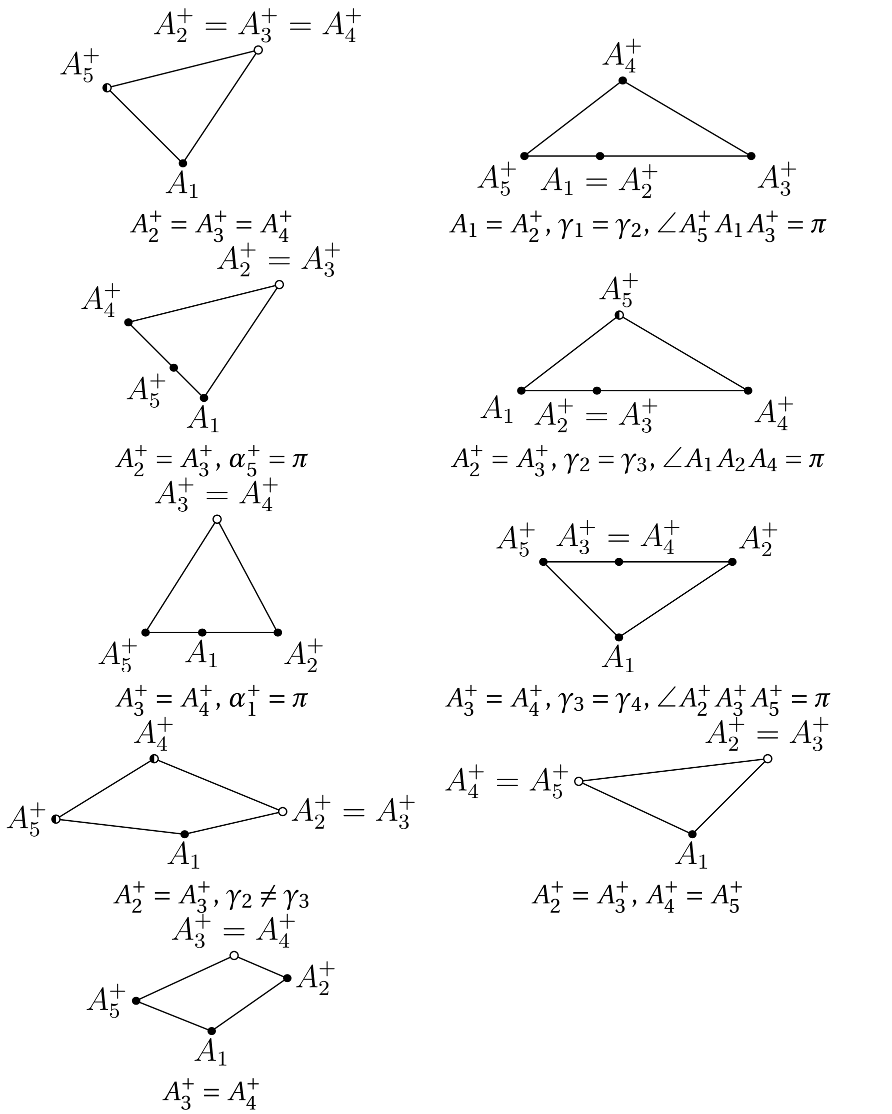


FIGURE 12. "Non-trivial" cases for $k = 5$. Part 2.

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